Questions — Answers

1. **[20 points]** Consider an economy that produces cookies (c) using labor hours (ℓ). The price of a cookie is p, and the price of an hour of labor is w. The demand and the supply functions for the cookies satisfy

$$q_c^d = \alpha w - \beta p \quad \text{and} \quad q_c^s = \gamma p$$

where $\alpha, \beta, \gamma > 0$ are exogenously given parameters. The demand and the supply functions for labor hours are given as in

$$q_\ell^d = \delta(p - w) \quad \text{and} \quad q_\ell^s = \sigma w$$

where $\delta, \sigma > 0$ are exogenously given parameters.

i. Define a system of linear equations (in matrix form of $Ax = b$) that solves for the equilibrium prices $p^\star$ and $w^\star$.

ii. Is the system homogeneous? Explain.

iii. Under what condition on $(\alpha, \beta, \gamma, \delta, \sigma)$ does the system have a nontrivial (general) solution?

iv. Find the nontrivial (general) solution $w^\star/p^\star$ in terms of $(\alpha, \beta, \gamma, \delta, \sigma)$.

Answer:
The equilibrium is characterized by

$$\alpha w - \beta p = \gamma p \quad \text{and} \quad \delta(p - w) = \sigma w$$

That is, we have

$$\begin{bmatrix} \beta + \gamma & -\alpha \\ \delta & -(\delta + \sigma) \end{bmatrix} \begin{bmatrix} p^\star \\ w^\star \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The system is homogeneous because it has the coefficient vector of $[0, 0]$ on the right-hand side.

For this system to have a a nontrivial (general) solution, the determinant of the 2×2 matrix should be equal to zero. That is, we must have

$$|A| = 0 \Rightarrow -(\beta + \gamma)(\delta + \sigma) + \alpha\delta = 0$$

This implies the following condition on $(\alpha, \beta, \gamma, \delta, \sigma)$:

$$\alpha\delta = (\beta + \gamma)(\delta + \sigma)$$

Since the determinant is equal to zero, one of the equations is redundant. Let's look at the first equation to solve for $w^\star/p^\star$. We have

$$(\beta + \gamma)p^\star = \alpha w^\star \Rightarrow \frac{w^\star}{p^\star} = \frac{\beta + \gamma}{\alpha}$$

If one uses the second equation, the solution is the same as dictated by the restriction $\alpha\delta = (\beta + \gamma)(\delta + \sigma)$ on the parameters.

2. **[10 points]** The profit function of a firm is defined as in

$$\pi(h) \equiv p\sqrt{h} - wh$$

where $h \geq 0$ denotes the demand for labor by the firm, $p > 0$ denotes the exogenously given price of the product, and $w > 0$ denotes the exogenously given price of labor. Solve the firm's problem, i.e., find $h^\star = h(w/p)$.

Answer:
The problem can be formally defined as in

$$\max_{h \geq 0} \; p\sqrt{h} - wh$$

This is a very simple optimization problem. The first-order necessary condition implies that there is unique stationary point:

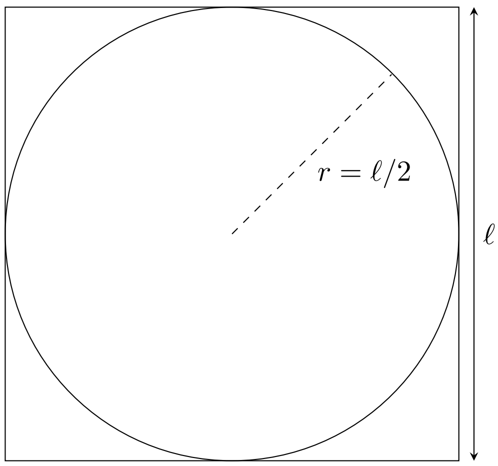
$$h: \ \left(\frac{1}{2}\right) p \left(h^{\star}\right)^{-1/2} - w = 0 \Rightarrow h^{\star} = \left(\frac{1}{4}\right)\left(\frac{w}{p}\right)^{-2}$$

The second-order sufficient condition reads

$$-\left(\frac{1}{4}\right) ph^{-3/2} < 0$$

for any $h \geq 0$. This means the function is strictly concave. Thus, the stationary point is the unique interior global solution of the problem.

3. [20 points] Consider a square where the length of a side is denoted by $\ell > 0$. The circle with the largest area that you can draw inside this square has radius $\ell/2$. Recalling that the area of a circle with radius r is equal to πr^2, formulate (and solve) an optimization problem that indeed proves $r^{\star} = \ell/2$.



a circle inside a square

Answer:
The problem can be stated as in

$$\max_{r} \ \pi r^2 \quad \text{subject to: } r > 0 \quad \text{and} \quad 2r \leq \ell$$

The first constraint originates from the fact that we need to draw a circle. The second one originates from the fact that it should be within the square of length ℓ.

Notice that the only stationary point of πr^2 is $r = 0$ and that it is therefore not the solution given the first constraint. Then, the solution could only be at the other boundary $r = \ell/2$. In other words, since πr^2 is strictly convex and strictly increasing in r, the solution is

$$r^{\star} = \frac{\ell}{2}$$

and the slope of the objective function is strictly positive at the optimum:

$$2\pi r^{\star} = 2\pi\left(\frac{\ell}{2}\right) = 2\pi\ell > 0$$