Midterm Exam — Solutions

Question 1. [15 points] The Market Equilibrium.

Consider a good x that is sold and purchased in a market. The demand function is $Q^d \equiv d(P, Z)$ with $d_P < 0$ and $d_Z < 0$ where Z is an exogenous variable. The supply function is $Q^s \equiv s(P)$ with $s_P > 0$. Suppose that there exists a unique equilibrium in this market. The equilibrium value of the market price is $P^\star$. At this price, the total amount of good x being traded is $Q^\star$. Show that, locally,

$$\frac{\mathrm{d}P^\star}{\mathrm{d}Z} < 0 \quad \text{and} \quad \frac{\mathrm{d}Q^\star}{\mathrm{d}Z} < 0$$

Imposing the equilibrium condition $Q^d = Q^s = Q$, the system reduces into

$$\begin{aligned} Q^\star &= d(P^\star, Z) \\ Q^\star &= s(P^\star) \end{aligned}$$

These define implicit functions

$$\begin{aligned} P^\star &= P(Z) \\ Q^\star &= Q(Z) \end{aligned}$$

To find the marginal effects, take the total differential

$$\begin{aligned} \mathrm{d}Q^\star &= d_P \mathrm{d}P^\star + d_Z \mathrm{d}Z \\ \mathrm{d}Q^\star &= s_P \mathrm{d}P^\star \end{aligned}$$

Eliminating $\mathrm{d}Q$, we have

$$d_P \mathrm{d}P^\star + d_Z \mathrm{d}Z = s_P \mathrm{d}P^\star$$

and this implies

$$\frac{\mathrm{d}P^\star}{\mathrm{d}Z} = \frac{d_Z}{s_P - d_P} < 0$$

Then, using $\mathrm{d}Q^\star = s_P \mathrm{d}P^\star$ and $s_P > 0$, we have

$$\frac{\mathrm{d}Q^\star}{\mathrm{d}Z} = s_P \left(\frac{\mathrm{d}P^\star}{\mathrm{d}Z} \right) = \frac{s_P d_Z}{s_P - d_P} < 0$$

Question 2. [15 points] The Classical Model.

Consider a version of the classical macroeconomic model of aggregate demand and aggregate supply:

$$\begin{aligned} M &\equiv \left(\frac{1}{V} \right) PY \\ Y &\equiv \alpha N \end{aligned}$$

Here, M is the supply of money, V is the velocity of money, P is the aggregate price, Y is the level of output, and N is the total amount of labor supplied. $\alpha > 0$ is a constant parameter representing the productivity of labor. Suppose that $Z \equiv (M, V, N)$ is the vector of exogenous variables satisfying $Z \in \mathbb{R}^3_{++}$.

(a) Find the equilibrium value $P^\star$ of the aggregate price.
In equilibrium, $Y^\star$ is equal to αN. Then, we have

$$P^\star = \frac{MV}{Y^\star} = \frac{MV}{\alpha N}$$

(b) Suppose that N increases by $\mathrm{d}N > 0$ due to population growth. If the monetary authority wants to stabilize P exactly at $P^\star$, what is the required change $\mathrm{d}M$ in the supply of money?
Taking the total differential of $P^\star$ derived above under the assumption that V does not change, we have

$$\begin{aligned} \mathrm{d}P^\star &= \left(\frac{V}{\alpha}\right)\mathrm{d}\left(\frac{M}{N}\right) \\ &= \left(\frac{V}{\alpha}\right)\left(\frac{N\mathrm{d}M - M\mathrm{d}N}{N^2}\right) \end{aligned}$$

Equating this to zero, we have

$$\begin{aligned} 0 &= \left(\frac{V}{\alpha}\right)\left(\frac{N\mathrm{d}M - M\mathrm{d}N}{N^2}\right) \\ 0 &= \frac{N\mathrm{d}M - M\mathrm{d}N}{N^2} \\ 0 &= N\mathrm{d}M - M\mathrm{d}N \\ \mathrm{d}M &= \left(\frac{M}{N}\right)\mathrm{d}N \end{aligned}$$

Question 3. [15 points] Yoda and His Non-Negativity Constraint.
Yoda, a Jedi master with income $w > 0$, wants to maximize his utility

$$u(x) \equiv (\beta - x)^{-1}$$

from good x. β is a preference parameter satisfying $\beta > w$. The price of good x is equal to 1; the budget constraint thus reads $x \leq w$. Show that Yoda would not choose $x^\star = 0$.
Yoda's problem can be written as

$$\max_{x\in\mathbb{R}} (\beta - x)^{-1} \quad \text{subject to: } x \leq w$$

The Lagrange function associated with this problem is

$$L(x,\lambda) \equiv (\beta - x)^{-1} + \lambda(w - x)$$

The first-order necessary conditions read

$$\begin{gathered} -(\beta - x^\star)^{-2}(-1) - \lambda^\star \leq 0, \quad x^\star \geq 0, \quad x^\star\left[-(\beta - x^\star)^{-2}(-1) - \lambda^\star\right] = 0 \\ w - x^\star \geq 0, \quad \lambda^\star \geq 0, \quad \lambda^\star(w - x^\star) = 0 \end{gathered}$$

We want to show that $x^\star \neq 0$. To do this, let's assume $x^\star = 0$ and then find a contradiction. If $x^\star = 0$, we must have

$$\beta^{-2} - \lambda^\star \leq 0 \quad \text{and} \quad \lambda^\star w = 0$$

Since $w > 0$, we have $\lambda^\star = 0$. Then, the first inequality reads

$$\begin{aligned} \beta^{-2} - 0 &\leq 0 \\ \beta^{-2} &\leq 0 \end{aligned}$$

Since $\beta > 0$, this is a contradiction. $x^\star = 0$ does not satisfy the first-order necessary conditions, and it is not optimal.