Consider a static economy. There are two primary resources, land (z) and labor (ℓ), and individuals consume water (w) and bread (b). On the production side,

- some firms produce salt (s) using labor and land,
- some firms produce water using labor and land,
- some firms produce wheat flour (f) using labor and land, and
- some firms produce bread using labor, salt, water, and wheat flour.

The price of a unit of land is p_z, and the price of a unit of labor is p_ℓ. The price of a bread is p_b, the price of a unit of salt is p_s, the price of a unit of water is p_w, and the price of a unit of flour is p_f.

Demographic Structure

The economy is inhabited by a continuum of identical individuals indexed by $h \in [0, 1]$.

Endowments

Each individual h is the owner of a unit of land. Let $e_z(h) = 1$ denote this endowment of land z for the individual h. Each individual is also endowed with a unit of labor. Let $e_\ell(h) = 1$ denote this endowment of labor ℓ for the individual h.

Preferences

Individuals consume bread and water in this economy. The utility function representing the preferences for bread and water is defined as in

$$u\left(c_b(h), c_w(h)\right) \equiv c_b(h)^{1/2} c_w(h)^{1/2}$$

where $c_b(h)$ and $c_w(h)$ respectively denote the consumption demand for bread and water by individual h.

Technologies

There are four production technologies. In what follows, y generically denotes the output, and x's generically denote the production inputs.

Bread Production

The bread industry is inhabited by a continuum of identical firms indexed by $i_b \in [0, 1]$. The Leontiev production function for the representative firm i_b is given as

$$y_b(i_b) = \min\{2x_{b\ell}(i_b), 8x_{bs}(i_b), 2x_{bw}(i_b), 16x_{bf}(i_b)\}$$

where $y_b(i_b)$ is the produced volume of bread and $x_{b\bullet}(i_b)$'s are the used production inputs of labor, salt, water, and flour.

Salt Production

The salt industry is inhabited by a continuum of identical firms indexed by $i_s \in [0, 1]$. Salt production requires labor and land, and the technological relationship is represented by a Cobb-Douglas function in the form of

$$y_s(i_s) = x_{s\ell}(i_s)^{1/4} x_{sz}(i_s)^{3/4}$$

for the representative firm i_s.

Water Production

The water industry is inhabited by a continuum of identical firms as well. Let $i_w \in [0,1]$ index these firms. For the representative firm, the Cobb-Douglas production function for water takes labor and land as inputs as in

$$y_w(i_w) = x_{w\ell}(i_w)^{1/4} x_{wz}(i_w)^{3/4}.$$

Flour Production

Finally, the flour industry is also inhabited by a continuum of identical firms indexed by $i_f \in [0,1]$, and a Cobb-Douglas technology is specified as in

$$y_f(i_f) = x_{f\ell}(i_f)^{3/4} x_{fz}(i_f)^{1/4}$$

Equilibrium

Your task here is to construct a general equilibrium of this economy. What we are looking in particular is a symmetric general equilibrium in the sense that all individuals have the same consumption bundle $c_b(h) = c_b$ and $c_w(h) = c_w$. Similarly, since all firms in a given industry are identical, firms in a given industry will have the same demand for the inputs and the same volume of output in equilibrium.

Questions — Answers

1. **[1 point]** Find the total supplies of land and labor.
Answer:

$$S^z = \int_0^1 e_z(h)dh = \int_0^1 1dh = 1 \quad \text{and} \quad S^\ell = \int_0^1 e_\ell(h)dh = \int_0^1 1dh = 1$$

2. **[2 points]** Suppose that there is no borrowing and that the individuals spend all of their income on consumption. Write down the budget constraint for individual h.
Answer:
The total expenditure on water and bread should be equal to total income from land and labor. Then, given that individual h has 1 unit of land 1 unit of labor, we have

$$p_b c_b(h) + p_w c_w(h) = p_z + p_\ell.$$

3. **[2 points]** Solve individual h's utility maximization problem. That is, find c_b and c_w that maximize u subject to the budget constraint (*Hint*: Do you notice at this stage that the optimal values should be functions of some prices only?)
Answer:
The specified Cobb-Douglas utility function attains its unique maximum when the total income is divided equally between the expenditure on bread and the expenditure on water:

$$c_b(h) = c_b = \frac{p_z + p_\ell}{2p_b} \quad \text{and} \quad c_w(h) = c_w = \frac{p_z + p_\ell}{2p_w}$$

4. **[5 points]** Let CD^b and CD^w denote the total consumption demand for bread and water. Express these as functions of some prices.
Answer:

$$\begin{aligned} CD^b &= \int_0^1 c_b(h)dh = \int_0^1 \left(\frac{p_z + p_\ell}{2p_b}\right) dh = \frac{p_z + p_\ell}{2p_b} \\ CD^w &= \int_0^1 c_w(h)dh = \int_0^1 \left(\frac{p_z + p_\ell}{2p_w}\right) dh = \frac{p_z + p_\ell}{2p_w} \end{aligned}$$

5. [**5 points**] The next task is to solve the firms' problems. Let's start with the salt industry. The representative firm i_s minimizes its total cost of $p_z x_{sz}(i_s) + p_\ell x_{s\ell}(i_s)$ given an arbitrary production target of $y_s(i_s) > 0$ (*Why?*). Formally, the firm solves

$$\min_{x_{sz}(i_s), x_{s\ell}(i_s)} \quad p_z x_{sz}(i_s) + p_\ell x_{s\ell}(i_s)$$
$$\text{subject to:} \quad y_s(i_s) = x_{s\ell}(i_s)^{1/4} x_{sz}(i_s)^{3/4}.$$

Find, then, the interior solutions of $x_{sz}(i_s)$ and $x_{s\ell}(i_s)$.

Answer:

Notice that the above problem is identical to

$$\max_{x_{sz}(i_s)} \; -p_z x_{sz}(i_s) - p_\ell \left(\frac{y_s(i_s)^4}{x_{sz}(i_s)^3} \right)$$

The first-order necessary condition for an interior solution reads

$$x_{sz}(i_s): \quad -p_z + 3 p_\ell y_s(i_s)^4 x_{sz}(i_s)^{-4} = 0$$

and implies the solution of $x_{sz}(i_s)$ as in

$$x_{sz}(i_s) = 3^{1/4} \left(\frac{p_\ell}{p_z} \right)^{1/4} y_s(i_s).$$

Noting that the second-order sufficient condition of

$$-12 p_\ell y_s(i_s)^4 x_{sz}(i_s)^{-5} < 0$$

is satisfied for any $x_{sz}(i_s) > 0$, this indeed is the solution to the problem. Finally, using the production function, we can find the optimal characterization of $x_{s\ell}(i_s)$ as in

$$x_{s\ell}(i_s) = 3^{-3/4} \left(\frac{p_z}{p_\ell} \right)^{3/4} y_s(i_s).$$

6. [**No point**] Now, formulate and solve the representative firms' problems in water and flour industries given $y_w(i_w) > 0$ and $y_f(i_f) > 0$ (*Hint*: These two problems for firms i_w and i_f are basically the same problem with firm i_s's problem).

Answer:

The solutions read

$$\begin{aligned}
x_{wz}(i_w) &= 3^{1/4} \left(\frac{p_\ell}{p_z} \right)^{1/4} y_w(i_w) \\
x_{w\ell}(i_w) &= 3^{-3/4} \left(\frac{p_z}{p_\ell} \right)^{3/4} y_w(i_w) \\
x_{fz}(i_f) &= 3^{-3/4} \left(\frac{p_\ell}{p_z} \right)^{3/4} y_f(i_f) \\
x_{f\ell}(i_f) &= 3^{1/4} \left(\frac{p_z}{p_\ell} \right)^{1/4} y_f(i_f)
\end{aligned}$$

7. [**5 points**] Solve the representative firm's problem in the bread industry given an arbitrary production target of $y_b(i_b) > 0$, find $(x_{b\ell}(i_b), x_{bs}(i_b), x_{bw}(i_b), x_{bf}(i_b))$ as functions of $y_b(i_b)$.

Answer:

The solutions simply follow from perfect complementarity implied by the Leontiev production function:

$$\begin{aligned} x_{b\ell}(i_b) &= \left(\frac{1}{2}\right) y_b(i_b) \\ x_{bs}(i_b) &= \left(\frac{1}{8}\right) y_b(i_b) \\ x_{bw}(i_b) &= \left(\frac{1}{2}\right) y_b(i_b) \\ x_{bf}(i_b) &= \left(\frac{1}{16}\right) y_b(i_b) \end{aligned}$$

8. [**4 points**] Having solved all the decision problems of the model economy, let's find some industry-wide and economy-wide aggregates under symmetry. In what follows, suppose that the production targets are symmetric within industries (*Why*?):

$$y_b(i_b) = y_b,\ y_s(i_s) = y_s,\ y_w(i_w) = y_w,\ \text{and } y_f(i_f) = y_f.$$

First, find the total demands for labor and land.

Answer:

The total demand for labor originates from bread, salt, water and flour production:

$$D^\ell = \int_0^1 x_{b\ell}(i_b)di_b + \int_0^1 x_{s\ell}(i_s)di_s + \int_0^1 x_{w\ell}(i_w)di_w + \int_0^1 x_{f\ell}(i_f)di_f$$

Then, using the solutions obtained above and the symmetry with respect to the production targets, we have

$$D^\ell = \int_0^1 \left(\frac{1}{2}\right) y_b di_b + \int_0^1 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} y_s di_s + \int_0^1 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} y_w di_w + \int_0^1 3^{1/4}\left(\frac{p_z}{p_\ell}\right)^{1/4} y_f di_f$$

$$D^\ell = \left(\frac{1}{2}\right) y_b + 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} y_s + 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} y_w + 3^{1/4}\left(\frac{p_z}{p_\ell}\right)^{1/4} y_f$$

The same reasoning applied to the demand for land implies

$$D^z = 3^{1/4}\left(\frac{p_\ell}{p_z}\right)^{1/4} y_s + 3^{1/4}\left(\frac{p_\ell}{p_z}\right)^{1/4} y_w + 3^{-3/4}\left(\frac{p_\ell}{p_z}\right)^{3/4} y_f$$

9. [**4 points**] Let's now find the total demands for bread, water, salt, and flour.

Answer:

The total demand for bread is simply equal to its total consumption demand CD^b, and we have

$$D^b = \frac{p_z + p_\ell}{2p_b}$$

The total demand for water is equal to the consumption demand CD^w plus the total intermediate input demand from the bread industry. Thus, we have

$$D^w = \frac{p_z + p_\ell}{2p_w} + \int_0^1 x_{bw}(i_b)di_b = \frac{p_z + p_\ell}{2p_w} + \left(\frac{1}{2}\right) y_b$$

Applying the similar reasoning, the total demand for salt and flour can be written as

$$\begin{aligned} D^s &= \int_0^1 x_{bs}(i_b)di_b = \left(\frac{1}{8}\right) y_b \\ D^f &= \int_0^1 x_{bf}(i_b)di_b = \left(\frac{1}{16}\right) y_b \end{aligned}$$

10. [**8 points**] You should now be ready to state the market-clearing conditions for primary factors and produced commodities. Specifically, equate the total demand and the total supply of land, labor, bread, water, salt and flour to obtain six equations.
Answer:

$$
\begin{aligned}
D^z = S^z \quad &\Rightarrow \quad 3^{1/4}\left(\frac{p_\ell}{p_z}\right)^{1/4} y_s + 3^{1/4}\left(\frac{p_\ell}{p_z}\right)^{1/4} y_w + 3^{-3/4}\left(\frac{p_\ell}{p_z}\right)^{3/4} y_f = 1 \\
D^\ell = S^\ell \quad &\Rightarrow \quad \left(\frac{1}{2}\right) y_b + 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} y_s + 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} y_w + 3^{1/4}\left(\frac{p_z}{p_\ell}\right)^{1/4} y_f = 1 \\
D^b = S^b \quad &\Rightarrow \quad \frac{p_z + p_\ell}{2p_b} = y_b \\
D^w = S^w \quad &\Rightarrow \quad \frac{p_z + p_\ell}{2p_w} + \left(\frac{1}{2}\right) y_b = y_w \\
D^s = S^s \quad &\Rightarrow \quad \left(\frac{1}{8}\right) y_b = y_s \\
D^f = S^f \quad &\Rightarrow \quad \left(\frac{1}{16}\right) y_b = y_f
\end{aligned}
$$

11. [**4 points**] I hope you realize that the model economy you are studying has 10 unknowns. These are

$$p_\ell, p_z, p_b, p_w, p_s, p_f, y_b, y_w, y_s, \text{ and } y_f$$

The remaining four equations you need will come from the zero-profit conditions of firms in four industries. To this end, define the average cost for the representative firms in bread, water, salt and flour industries, and equate the average cost to price in each particular industry to have four new equations.
Answer:
The average cost in the bread industry (for any firm) is

$$ac_b \equiv \frac{C_b}{y_b} = \frac{p_\ell x_{b\ell} + p_w x_{bw} + p_s x_{bs} + p_f x_{bf}}{y_b} = \left(\frac{1}{2}\right) p_\ell + \left(\frac{1}{8}\right) p_s + \left(\frac{1}{2}\right) p_w + \left(\frac{1}{16}\right) p_f$$

The average cost in the salt industry (for any firm) is

$$ac_s \equiv \frac{C_s}{y_s} = \frac{p_\ell x_{s\ell} + p_z x_{sz}}{y_s} = p_\ell 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} + p_z 3^{1/4}\left(\frac{p_\ell}{p_z}\right)^{1/4}$$

The average cost in the water industry (for any firm) is

$$ac_w \equiv \frac{C_w}{y_w} = \frac{p_\ell x_{w\ell} + p_z x_{wz}}{y_b} = p_\ell 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} + p_z 3^{1/4}\left(\frac{p_\ell}{p_z}\right)^{1/4}$$

The average cost in the flour industry (for any firm) is

$$ac_f \equiv \frac{C_f}{y_f} = \frac{p_\ell x_{f\ell} + p_z x_{fz}}{y_f} = p_\ell 3^{1/4}\left(\frac{p_z}{p_\ell}\right)^{1/4} + p_z 3^{-3/4}\left(\frac{p_\ell}{p_z}\right)^{3/4}$$

Equating price and the average cost for all these industries implies

$$
\begin{aligned}
p_b = ac_b \quad &\Rightarrow \quad p_b = \left(\frac{1}{2}\right) p_\ell + \left(\frac{1}{8}\right) p_s + \left(\frac{1}{2}\right) p_w + \left(\frac{1}{16}\right) p_f \\
p_s = ac_s \quad &\Rightarrow \quad p_s = p_\ell 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} + p_z 3^{1/4}\left(\frac{p_\ell}{p_z}\right)^{1/4} \\
p_w = ac_w \quad &\Rightarrow \quad p_w = p_\ell 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} + p_z 3^{1/4}\left(\frac{p_\ell}{p_z}\right)^{1/4} \\
p_f = ac_f \quad &\Rightarrow \quad p_f = p_\ell 3^{1/4}\left(\frac{p_z}{p_\ell}\right)^{1/4} + p_z 3^{-3/4}\left(\frac{p_\ell}{p_z}\right)^{3/4}
\end{aligned}
$$

12. [**No point**] Rewrite the 10 equilibrium-defining equations you have found above by numbering them from (1) to (10).
Answer:

$$(1) \qquad 1 = 3^{1/4}\left(\frac{p_\ell}{p_z}\right)^{1/4} y_s + 3^{1/4}\left(\frac{p_\ell}{p_z}\right)^{1/4} y_w + 3^{-3/4}\left(\frac{p_\ell}{p_z}\right)^{3/4} y_f$$

$$(2) \qquad 1 = \left(\frac{1}{2}\right) y_b + 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} y_s + 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} y_w + 3^{1/4}\left(\frac{p_z}{p_\ell}\right)^{1/4} y_f$$

$$(3) \qquad y_b = \frac{p_z + p_\ell}{2p_b}$$

$$(4) \qquad y_w = \frac{p_z + p_\ell}{2p_w} + \left(\frac{1}{2}\right) y_b$$

$$(5) \qquad y_s = \left(\frac{1}{8}\right) y_b$$

$$(6) \qquad y_f = \left(\frac{1}{16}\right) y_b$$

$$(7) \qquad p_b = \left(\frac{1}{2}\right) p_\ell + \left(\frac{1}{8}\right) p_s + \left(\frac{1}{2}\right) p_w + \left(\frac{1}{16}\right) p_f$$

$$(8) \qquad p_s = p_\ell 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} + p_z 3^{1/4}\left(\frac{p_\ell}{p_z}\right)^{1/4}$$

$$(9) \qquad p_w = p_\ell 3^{-3/4}\left(\frac{p_z}{p_\ell}\right)^{3/4} + p_z 3^{1/4}\left(\frac{p_\ell}{p_z}\right)^{1/4}$$

$$(10) \qquad p_f = p_\ell 3^{1/4}\left(\frac{p_z}{p_\ell}\right)^{1/4} + p_z 3^{-3/4}\left(\frac{p_\ell}{p_z}\right)^{3/4}$$

13. [**2 points**] Under the normalization $p_\ell = 1$, I have solved the model numerically to obtain the following results (with strictly positive real-valued prices):

$$\begin{aligned}
p_b &= 2.4302281438976811883220904502575 \\
p_f &= 2.0701944979692303023104386860643 \\
p_s &= 2.8813455804393668710843008518056 \\
p_w &= 2.8813455804393668710843008518056 \\
p_z &= 1.9371729625282293913277899365341 \\
y_b &= 0.6042998411288030648802097112607 \\
y_f &= 0.0377687400705501915550131069538 \\
y_s &= 0.0755374801411003831100262139076 \\
y_w &= 0.8118376481434636710286371349590
\end{aligned}$$

Pick any two equations, and check that these equations are satisfied at the equilibrium values presented above.
Answer:
I have simply checked (5) and (6). They are satisfied (as all the others).

14. [**4 points**] Is the alternative normalization of $p_\ell = p_z$ a legitimate one? Why or why not? What do you think of the alternative normalization $p_s = p_w$? Discuss briefly.
Answer:
$p_\ell = p_z$ is not a legitimate one because it is too restrictive unless we observe such an equality in the real-world data. $p_s = p_w$ is also not useful or appropriate because it does not introduce new information about prices; the model already tells us that we should have $p_s = p_w$ in equilibrium.

15. [**4 points**] Your final task is to formulate the social planner's problem. Suppose that there exists a benevolent social planner whose preferences are represented by the utility function $u(c_b, c_w)$ defined above. The social planner would take all the allocation decisions without letting free markets to operate. For simplicity, let only one production plant operate in an industry. Can you solve the social planner's problem, or characterize the solution in any meaningful way? (*Hint*: The social planner is restricted only by the resource constraints

$$x_{b\ell} + x_{s\ell} + x_{w\ell} + x_{f\ell} = 1 \quad \text{and} \quad x_{sz} + x_{wz} + x_{fz} = 1,$$

the production functions, and some other constraints such as $y_w = c_w + x_{bw}$ and $y_f = x_{bf}$.)

Answer:

$$\begin{aligned}
\max \quad & c_b^{1/2} c_w^{1/2} \\
\text{subject to:} \quad & x_{b\ell} + x_{s\ell} + x_{w\ell} + x_{f\ell} = 1 \\
& x_{sz} + x_{wz} + x_{fz} = 1 \\
& y_b = 2x_{b\ell} = 8x_{bs} = 2x_{bw} = 16x_{bf} \\
& y_s = x_{s\ell}^{1/4} x_{sz}^{3/4} \\
& y_w = x_{w\ell}^{1/4} x_{wz}^{3/4} \\
& y_f = x_{f\ell}^{3/4} x_{fz}^{1/4} \\
& c_b = y_b \\
& y_w = c_w + x_{bw} \\
& y_s = x_{bs} \\
& y_f = x_{bf}
\end{aligned}$$