Homework — Solutions

Question 1. [10 points] The Stability of the Walrasian Equilibrium.

Imagine an economy where there are two goods i and j. The demand functions Q^d and the supply functions Q^s of these goods are given as

$$\begin{aligned} Q_i^d &\equiv \frac{\alpha P_j^{\beta}}{P_i^{\beta}} \\ Q_j^d &\equiv \frac{\gamma P_i}{P_j} \\ Q_i^s &\equiv s(P_i, P_j) \\ Q_j^s &\equiv \frac{\theta P_j}{P_i} \end{aligned}$$

where (P_i, P_j) is a typical price vector, $(\alpha, \beta, \gamma, \theta)$ is a vector of strictly positive constants, and $s(P_i, P_j)$ is a real-valued function.

i. Define the excess demand functions for i and j as functions of prices.
The excess demand functions are

$$\begin{aligned} F^i(P_i, P_j) &\equiv Q_i^d - Q_i^s = \frac{\alpha P_j^{\beta}}{P_i^{\beta}} - s(P_i, P_j) \\ F^j(P_i, P_j) &\equiv Q_j^d - Q_j^s = \frac{\gamma P_i}{P_j} - \frac{\theta P_j}{P_i} \end{aligned}$$

ii. Assume that there exists a unique Walrasian equilibrium, i.e. a unique *ratio* of prices P_i and P_j. Show that this equilibrium is globally (asymptotically) stable (*Hint*: You have to impose a normalization of prices and define an appropriate differential equation).
Since there exists a Walrasian equilibrium, Walras' Law holds at this equilibrium. Thus, if

$$F^j(P_i, P_j) = 0$$

at the Walrasian equilibrium prices, we have

$$F^i(P_i, P_j) = 0$$

We should therefore analyze the stability of the equilibrium using $F^j(P_i, P_j)$; we do not know the explicit form of $F^i(P_i, P_j)$ because of not knowing the explicit form of $s(P_i, P_j)$.

To use $F^j(P_i, P_j)$, the appropriate price ratio is

$$p \equiv \frac{P_j}{P_i}$$

and the differential equation to be used can be defined as

$$\dot{p} = \eta F^j(P_i, P_j) = \eta \left(\frac{\gamma P_i}{P_j} - \frac{\theta P_j}{P_i} \right)$$

where $\eta > 0$ is a constant that determines the speed of adjustment. This equation says that, if there exists excess demand in the market for j ($F^j > 0$) at some prices, then the relative price p with P_j in the numerator should increase. Rewriting, we have

$$\dot{p} = \eta \left(\frac{\gamma}{p} - \theta p \right)$$

and the the Walrasian equilibrium is globally (asymptotically) stable because

$$\frac{\partial \dot{p}}{\partial p} = \eta \left(-\frac{\gamma}{p^2} - \theta \right) = -\eta \left(\frac{\gamma}{p^2} + \theta \right) < 0$$

for all $p > 0$.

iii. What is the equilibrium price ratio equal to?
From $\dot{p} = 0$, we have $\gamma / p^\star - \theta p^\star = 0$ and this implies

$$p^\star = \sqrt{\frac{\gamma}{\theta}}$$

Question 2. [20 points] Class Struggle and Redistribution.

Imagine an economy where there are $N_W > 0$ workers and $N_C > 0$ capitalists. The population $N_W + N_C$ is constant, and an individual cannot change his/her class status.

What differentiate a worker from a capitalist are (i) the ownership of physical capital and (ii) the preference for leisure. Specifically, each capitalist have $k > 0$ machines and he/she does not work. Workers on the other hand have no capital, and each supplies 1 unit of time in a labor market. Notice that all members of a class are identical.

Every individual, whether a worker or a capitalist, has a utility function of the form

$$u \equiv u(c) = \alpha \ln(c) \quad \alpha > 0$$

where c is the consumption of the unique good in the economy.

Capitalists organize the production. Specifically, each capitalist has access to a production technology of the form

$$y \equiv f(k,h) = k^{1-\beta}(xh)^{\beta} \quad \beta \in (0,1)$$

where y is output, x is a measure of productivity, k is the capitalist's stock of physical capital, and h is the labor input that the capitalist hires in the labor market.

Let w denote the unit price of labor, and normalize the price of the consumption good to 1.

Let c_W and c_C respectively denote the equilibrium levels of a typical worker's consumption and a typical capitalist's consumption.

Finally, notice that the model has 4 exogenous variables: (N_W, N_C, k, x).

i. Write down the budget constraint of a worker.
$c_W \leq w$

ii. State the utility maximization problem of a worker formally.
$\max_{c_W} \alpha \ln(c_W)$ subject to $c_W \leq w$

iii. Define the profit function of a capitalist as a function of h. $\pi \equiv k^{1-\beta}(xh)^{\beta} - wh$

iv. Find the profit maximizing level of h.

$$h = \left(\frac{\beta k^{1-\beta} x^{\beta}}{w}\right)^{\frac{1}{1-\beta}}$$

v. Write down the maximum profit function as a function of (k, x, w).

$$\begin{aligned}
\pi &= k^{1-\beta}(xh)^{\beta} - wh \\
\pi &= k^{1-\beta} x^{\beta} \left(\frac{\beta k^{1-\beta} x^{\beta}}{w}\right)^{\frac{\beta}{1-\beta}} - w \left(\frac{\beta k^{1-\beta} x^{\beta}}{w}\right)^{\frac{1}{1-\beta}} \\
\pi &= \beta^{\frac{\beta}{1-\beta}} k \left(\frac{x}{w}\right)^{\frac{\beta}{1-\beta}} - k \beta^{\frac{1}{1-\beta}} \left(\frac{x}{w}\right)^{\frac{\beta}{1-\beta}} \\
\pi &= \left(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}}\right) k \left(\frac{x}{w}\right)^{\frac{\beta}{1-\beta}}
\end{aligned}$$

vi. Find the market clearing level of w.
To find the market clearing level of w, we equate the total supply to the total demand:

$$
\begin{aligned}
N_W &= N_C h \\
N_W &= N_C \left(\frac{\beta k^{1-\beta} x^{\beta}}{w} \right)^{\frac{1}{1-\beta}} \\
\left(\frac{N_W}{N_C} \right)^{1-\beta} &= \frac{\beta k^{1-\beta} x^{\beta}}{w} \\
w &= \frac{\beta k^{1-\beta} x^{\beta}}{\left(\frac{N_W}{N_C} \right)^{1-\beta}} \\
w &= \beta x^{\beta} \left(\frac{k N_C}{N_W} \right)^{1-\beta}
\end{aligned}
$$

vii. Express c_W and c_C as explicit functions of exogenous variables and parameters.
Since $c_W = w$ in equilibrium, we have

$$
c_W = \beta x^{\beta} \left(\frac{k N_C}{N_W} \right)^{1-\beta}
$$

To find c_C, first notice that $c_C = \pi$. To find π, substitute the solution of w into π:

$$
\begin{aligned}
\pi &= \left(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}} \right) k \left(\frac{x}{\beta x^{\beta} \left(\frac{k N_C}{N_W} \right)^{1-\beta}} \right)^{\frac{\beta}{1-\beta}} \\
\pi &= \left(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}} \right) \beta^{-\frac{\beta}{1-\beta}} k \left(\frac{x^{1-\beta}}{\left(\frac{k N_C}{N_W} \right)^{1-\beta}} \right)^{\frac{\beta}{1-\beta}} \\
\pi &= \left(\beta^{\frac{\beta}{1-\beta}} \beta^{-\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}} \beta^{-\frac{\beta}{1-\beta}} \right) k \left(\frac{x^{1-\beta}}{\left(\frac{k N_C}{N_W} \right)^{1-\beta}} \right)^{\frac{\beta}{1-\beta}} \\
\pi &= (1-\beta) x^{\beta} k^{1-\beta} \left(\frac{N_W}{N_C} \right)^{\beta} \\
c_C &= (1-\beta) x^{\beta} k^{1-\beta} \left(\frac{N_W}{N_C} \right)^{\beta}
\end{aligned}
$$

viii. Define a measure i of inequality in this economy as

$$
i(c_W, c_C) \equiv \frac{c_W}{c_C}
$$

and analyze how i changes with exogenous variables and parameters. Provide some economic intuition for k and x not affecting i.

We have

$$
\begin{aligned}
i &= \frac{c_W}{c_C} \\
&= \frac{\beta x^\beta \left(\frac{kN_C}{N_W}\right)^{1-\beta}}{(1-\beta)x^\beta k^{1-\beta}\left(\frac{N_W}{N_C}\right)^\beta} \\
&= \frac{\beta}{1-\beta}\frac{N_C}{N_W}
\end{aligned}
$$

In this model, inequality is affected by the labor elasticity of output and the relative class size. k and x does not affect the level of inequality because the capitalist's machines and the productivity not only increase the capitalist's profit but also the worker's overall productivity and wage.

ix. Under what condition a typical capitalist consumes more than a typical worker?

$$
\begin{aligned}
c_C &> c_W \\
i &< 1 \\
\frac{\beta}{1-\beta}\frac{N_C}{N_W} &< 1 \\
\beta N_C &< (1-\beta)N_W
\end{aligned}
$$

x. Suppose that the condition you have found above is true: the competitive general equilibrium "hurts" the workers in terms of inequality. Suppose that a government comes into play to reduce inequality between capitalists and workers. Specifically, the government wants to implement a redistribution policy such that a tax $\tau \in (0,1)$ on post-production profit is transferred to workers. In this case, the profit of a typical capitalist reads

$$\pi \equiv (1-\tau)f(k,x,h) - wh$$

and the government has a budget constraint of the form

$$N_C \tau f(k,x,h) = N_W r$$

where r denotes the transfer per worker. Thus, a typical worker now earns

$$w + r$$

Does there exist an optimal level of τ that makes i equal to one, i.e. the perfectly egalitarian redistribution policy? If yes, find that optimal level. If no, explain why not (*Hints*: Assume that the government announces τ before the workers and the capitalists act. But be careful; the capitalists will re-optimize because τ

is now an argument in π).
The solution to the extended model satisfies

$$\begin{aligned}
w &= (1-\tau)\beta x^{\beta}k^{1-\beta}\left(\frac{N_C}{N_W}\right)^{1-\beta} \\
\pi &= (1-\tau)(1-\beta)x^{\beta}k^{1-\beta}\left(\frac{N_C}{N_W}\right)^{-\beta} \\
h &= \left(\frac{N_C}{N_W}\right)^{-1} \\
r &= \tau x^{\beta}k^{1-\beta}\left(\frac{N_C}{N_W}\right)^{1-\beta}
\end{aligned}$$

Since we have $c_W = w + r$ and $c_C = \pi$, the level of inequality satisfies

$$\begin{aligned}
i &= \frac{c_W}{c_C} \\
&= \frac{(1-\tau)\beta x^{\beta}k^{1-\beta}\left(\frac{N_C}{N_W}\right)^{1-\beta} + \tau x^{\beta}k^{1-\beta}\left(\frac{N_C}{N_W}\right)^{1-\beta}}{(1-\tau)(1-\beta)x^{\beta}k^{1-\beta}\left(\frac{N_C}{N_W}\right)^{-\beta}} \\
&= \frac{\beta}{1-\beta}\frac{N_C}{N_W} + \frac{1}{1-\beta}\frac{\tau}{1-\tau}\frac{N_C}{N_W}
\end{aligned}$$

As clearly seen, the level of τ that equates i to one is

$$\begin{aligned}
\frac{\beta}{1-\beta}\frac{N_C}{N_W} + \frac{1}{1-\beta}\frac{\tau}{1-\tau}\frac{N_C}{N_W} &= 1 \\
\frac{1}{1-\beta}\frac{N_C}{N_W}\left(\beta + \frac{\tau}{1-\tau}\right) &= 1 \\
\frac{\tau}{1-\tau} &= \frac{1-\beta}{\frac{N_C}{N_W}} - \beta \\
\tau &= \frac{N_W}{N_W+N_C} - \frac{\beta}{1-\beta}\frac{N_C}{N_W+N_C}
\end{aligned}$$

xi. Define the fraction f_W of workers in the population as in

$$f_W \equiv \frac{N_W}{N_W+N_C}$$

How does the perfectly egalitarian tax rate you found above change with f_W? Provide some economic intuition.

Rewriting τ as a function of f_W, we have

$$\begin{aligned}
\tau &= \frac{N_W}{N_W+N_C} - \frac{\beta}{1-\beta}\frac{N_C}{N_W+N_C} \\
\tau &= f_W - \left(\frac{\beta}{1-\beta}\right)(1-f_W) \\
\tau &= \left(\frac{1}{1-\beta}\right)f_W - \frac{\beta}{1-\beta}
\end{aligned}$$

Obviously, we have

$$\frac{\partial \tau}{\partial f_W} = \frac{1}{1-\beta} > 0$$

When the relative share of workers increase, a larger set of workers compete for a smaller set of jobs and their wage decreases. For this reason, the perfectly egalitarian tax rate should be higher.