I. INPUT-OUTPUT FORMALISM FOR DIMENSIONLESS OPERATORS

We consider a single cavity mode interacting with a reservoir. An optical cavity is generally described by a Hamiltonian of the form[1]

$$H = H_{sys} + H_b + H_{int}, \tag{1}$$

where H_{sys} is function of internal-mode operators only, $H_b = \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k$ is the free Hamiltonian of the bath, and $H_{int} = \hbar \sum_k g_k(\hat{b}_k^\dagger \hat{a} + \hat{a}^\dagger \hat{b}_k)$ describes the interaction between bath and cavity field[1]. Heisenberg equation of motion can be obtained for a single mode cavity:

$$i\hbar \frac{d\hat{b}_k}{dt} = [b, H] = \hbar\omega_k \hat{b} + \hbar g_k \hat{a} \tag{2}$$

We can find the reservoir operator $\hat{b}_k$ for $(t > t_0)$ [1]:

$$\hat{b}_k(t) = e^{-i\omega_k(t-t_0)}\hat{b}_k(t_0) - ig_k \int_{t_0}^{t} e^{-i\omega_k(t-t')}\hat{a}(t')dt'. \tag{3}$$

Here the first term is the free evolution of the reservoir mode and the second term is the interaction between harmonic oscillator and reservoir. We can find the reservoir operator $\hat{b}_k$ for $(t_1 > t)$ [1]:

$$\hat{b}_k(t) = e^{-i\omega_k(t-t_1)}\hat{b}_k(t_1) + ig_k \int_{t}^{t_1} e^{-i\omega_k(t-t')}\hat{a}(t')dt'. \tag{4}$$

Heisenberg equation of motion for harmonic oscillator operator $\hat{a}$ is:

$$i\hbar \frac{d\hat{a}}{dt} = [\hat{a}, \hat{H}_{sys}] + \hbar \sum_k g_k \hat{b_k}. \tag{5}$$

Here $\hat{a}$ is the annihilation operator for the harmonic oscillator. By using Eq.(3) the operator can be found for $(t > t_0)$[1]:

$$\begin{aligned} \frac{d\hat{a}}{dt} &= -\frac{i}{\hbar}[a, H_{sys}] - i\sum_k g_k e^{-i\omega_k(t-t_0)}\hat{b}_k(t_0) \\ &- \sum_k g_k^2 \int_{t_0}^{t} e^{-i\omega_k(t-t')}\hat{a}(t')dt'. \end{aligned} \tag{6}$$

In Eq. (8) the integral is $I = -\sum_k g_k^2 \int_{t_0}^{t} e^{-i\omega_k(t-t')}\hat{a}(t')dt' = \eta/2\hat{a}(t)$ where $\eta = 2\pi D(\omega)g(\omega)^2$, $\sum_k \rightarrow 2(L/2\pi)^3 \int d^3k$, and $D(\omega) = L^3\omega^2/(\pi^2c^3)$. Here c is the velocity of light.

$$
\begin{aligned}
I &= -\frac{L^3}{\pi^2 c^3}\int_0^{\infty} d\omega_k \omega_k^2 g_k^2 \int_{t_0}^{t} e^{-i\omega_k(t-t')}\hat{a}(t')dt' \\
&= D(\omega)g^2(\omega)\int_{-\infty}^{\infty} d\omega_k \int_{t_0}^{t} e^{-i\omega_k(t-t')}\hat{a}(t')dt'
\end{aligned}
\tag{7}
$$

By using Eq.(4)

$$
\begin{aligned}
\frac{d\hat{a}}{dt} &= -\frac{i}{\hbar}[a, H_{sys}] - i\sum_k g_k e^{-i\omega_k(t-t_0)}\hat{b}_k(t_0) \\
&+ \sum_k g_k^2 \int_t^{t_1} e^{-i\omega_k(t-t')}\hat{a}(t')dt'.
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\frac{d\hat{a}}{dt} &= -\frac{i}{\hbar}[a, H_{sys}] - \frac{\eta}{2}\hat{a}(t) + g(\omega)\hat{a}_{in}(t) \\
&= -\frac{i}{\hbar}[a, H_{sys}] + \frac{\eta}{2}\hat{a}(t) - g(\omega)\hat{a}_{out}(t).
\end{aligned}
\tag{9}
$$

Evaluating the integrals and by using Eq.(9), we find the input-output relations for two-sided cavity:

$$
\hat{a}_{out}(t) + \hat{a}_{in}(t) = 2\pi D(\omega)g(\omega)\hat{a}(t). \tag{10}
$$

A. Mechanical coupling to vibrations:A generalized problem for harmonic oscillator reservoir

We consider a single harmonic oscillator coupling to vibrations(multi-oscillator heat bath). The Hamiltonian is generally described by [1]

$$
\hat{H} = \hat{H}_m + \hat{H}_{bm}. \tag{11}
$$

Here the energy of harmonic oscillator is described by $\hat{H}_m = p^2/2m_n + 1/2m\omega_m^2 x^2$. m is the mass of the nanomechanical mirror that moving along the OX axis with abscissas x. $\hat{H}_{bm} = \sum_n [p_n^2/2m_n + 1/2m_n\omega_n^2(x - q_n)^2]$ is the total Hamiltonian of the vibrations along the x. The position and momentum are scaled by $\kappa_n = \sqrt{m_n\omega_n^2}$ ($q_n = \tilde{p_n}/\sqrt{m_n\omega_n^2}$ and $p_n = -\tilde{q_n}\sqrt{m_n\omega_n^2}$). The Hamiltonian can be put in the form [1] (Eq.(9.2.21) p.279):

$$
\hat{H}_{bm} = \frac{1}{2}\sum_n [\omega_n^2 \tilde{q_n}^2 + (\tilde{p_n} - \kappa_n x)^2]. \tag{12}
$$

In Eq. (12) commutation relation can be writen as: $[\tilde{q_n}, \tilde{p_n}] = i\hbar$. Heisenberg time evolution of the system for position and momentum operator are:

$$\begin{aligned} \dot{\tilde{q_n}} &= \tilde{p_n} - \kappa_n x; \\ \dot{\tilde{p_n}} &= -\omega_n^2 \tilde{q_n}. \end{aligned} \tag{13}$$

We therefore define dimensionless operators:

$$\begin{aligned} \hat{a_n} &= \frac{\omega_n \tilde{q}_n + i\tilde{p}_n}{\sqrt{2\hbar\omega_n}}; \\ \hat{a_n}^\dagger &= \frac{\omega_n \tilde{q}_n - i\tilde{p}_n}{\sqrt{2\hbar\omega_n}}. \end{aligned} \tag{14}$$

Solving the Heisenberg time evolution of the system, we can obtain time evolution of $a_n(t)$:

$$a_n(t) = e^{-i\omega_n(t-t_0)} a_n(t_0) - \kappa_n \sqrt{\frac{\omega_n}{2\hbar}} \int_{t_0}^{t} e^{-i\omega_n(t-t')} \hat{x}(t') dt'. \tag{15}$$

Similarly time evolution of $\hat{x}$ and $\hat{p}$ can be evaluated by using $i\hbar\dot{\hat{O}} = [\hat{O}, \hat{H}]$:

$$\begin{aligned} \dot{x} &= \frac{p}{m}; \\ \dot{p} &= -m\omega_m^2 x - \sum_n \kappa_n^2 x + \sum_n \kappa_n \tilde{p}_n. \end{aligned} \tag{16}$$

The position operator x and the momentum operator p are scaled by $\bar{x} = x/\sqrt{\hbar/(m\omega_m)}$ and $\bar{p} = p/(\sqrt{\hbar m\omega_m})$ respectively. The time evolution equation for $\bar{p}$ is:

$$\begin{aligned} \dot{\bar{p}} &= -\omega_m(1 - \sum_n \bar{\kappa_n}^2)\bar{x} + \sum_n \frac{\kappa_n}{\sqrt{m\hbar\omega_m}} \frac{i}{\sqrt{2}} \sqrt{\hbar\omega_n} \times \{(\hat{a}_n^\dagger(t_0) e^{i\omega_n(t-t_0)} - \hat{a}_n(t_0) e^{-i\omega_n(t-t_0)}) \\ &- \kappa_n \sqrt{\frac{\omega_n}{2\hbar}} \int_{t_0}^{t} dt' (e^{i\omega_n(t-t')} - e^{-i\omega_n(t-t')}) \bar{x}(t') dt'\}, \end{aligned} \tag{17}$$

In Eq.(17) $(e^{i\omega_n(t-t')} - e^{-i\omega_n(t-t')}) = 2i\sin(\omega_n(t-t'))$ and we will get integration by part($du = \sin(\omega_n(t-t'))$ $v(t') = \bar{x}(t')$)

$$\begin{aligned} \dot{\bar{p}} &= -\omega_m(1 - \sum_n \bar{\kappa_n}^2)\bar{x} + \sum_n g_n^{(m)} \frac{i}{\sqrt{2}} (\hat{a}_n^\dagger(t_0) e^{-i\omega_n(t-t')} - \hat{a}_n^\dagger(t_0) e^{-i\omega_n(t-t')}) \\ &+ \sum_n (g_n^{(m)})^2 [\sin[\omega_n((t-t'))]\bar{x}(t') \mid_{t'=t_0}^{t} - \int_{t_0}^{t} (-1)\cos[\omega_n(t-t')]\dot{\bar{x}}(t') dt'], \end{aligned} \tag{18}$$

In Eq.(18) the integral is $I_2 = \sum_n (g_n^{(m)})^2 [-\sin[\omega_n((t-t_0))]\bar{x}(t_0) - \omega_m/\omega_n \int_{t_0}^{t} \cos[\omega_n(t-t')]\bar{p}(t')dt'$ and $\sum_{r=0}^{d_r} (g_{n,r}^{(m)})^2 \omega_m/\omega_n = G_n(\omega_n)$. The first term in the integrand is zero. So the integral becomes $I_2 = -\int_{t_0}^{t} \bar{p}(t^{'})dt^{'} \sum_n G_n(\omega_n)\cos[\omega_n(t-t^{'})]$. $\rho(\omega)G(\omega)$ is constant (nonzero for only same frequency) and $\int_0^{\infty} d\omega \cos[\omega(t-t^{'}) = \pi\delta(t-t^{'})$. Finally The integral is $I_2 = -(\pi/2)\rho(\omega)G(\omega)\bar{p}(t)$

We derive time evolution equations for $\bar{x}$ and $\bar{p}$:

$$\begin{aligned} \dot{\bar{x}} &= \omega_m \bar{p}; \\ \dot{\bar{p}} &= -\omega_m(1-\sum_n \bar{\kappa_n}^2)\bar{x} + g^{(m)}\hat{\epsilon}_{in} - \gamma_m \bar{p}, \end{aligned} \tag{19}$$

where $\hat{\epsilon}_{in} = \sum i(\hat{a}_n^{\dagger}(t_0)e^{-i\omega_n(t-t^{'})} - \hat{a}_n^{\dagger}(t_0)e^{-i\omega_n(t-t^{'})})/\sqrt{2}$ behaves like momentum. In Eq.(19) $\gamma_m = (\pi/2)\rho(\omega)G(\omega)$, $G(\omega) \cong (g^{(m)})^2$ and $g^{(m)} = \sqrt{(m_n\omega_n)/m\omega_m}\omega_n$

[1] M. O. Scully, and M. S. Zubairy, Quantum Optics (Cambridge Press, Cambridge, 1997).

[2] G. S. Agarwal, Quantum Optics (Cambridge Press, Cambridge, 2013).