Input - Output Relations

A) Optical Coupling to cavity outside

$$\hat{H} = \hat{H}_{sys} + \hat{H}_{bath} + \hat{H}_{int}$$

$$\hat{H}_{bath} = \sum_k \hbar\omega_k \hat{b}_k^{\dagger} \hat{b}_k \qquad \hat{H}_{int} = \hbar \sum_k g_k \cdot (\hat{b}_k^{\dagger} \hat{a} + \hat{a}^{\dagger} \hat{b}_k)$$

$$i\hbar \dot{\hat{b}}_k = [\hat{b}_k, \hat{H}] = \hbar\omega_k \hat{b}_k + \hbar g_k \hat{a}$$

$$\Rightarrow \dot{\hat{b}}_k = -i\omega_k \hat{b}_k - i g_k \hat{a}$$

$$\Rightarrow \hat{b}_k(t) = e^{-i\omega_k(t-t_0)} \cdot \hat{b}_k(t_0) - i g_k \int_{t_0}^{t} dt' \cdot e^{-i\omega_k(t-t')} \cdot \hat{a}(t') \quad \text{for } t > t_0 \qquad \text{--- (4a)}$$

$$\Rightarrow \hat{b}_k(t) = e^{-i\omega_k(t-t_1)} \cdot \hat{b}_k(t_1) + i g_k \int_{t}^{t_1} dt' \cdot e^{-i\omega_k(t-t')} \cdot \hat{a}(t') \quad \text{for } t < t_1 \qquad \text{--- (4b)}$$

$$i\hbar \dot{\hat{a}}(t) = [\hat{a}, \hat{H}] = [\hat{a}, \hat{H}_{sys}] + \hbar \sum_k g_k \hat{b}_k$$

$$\dot{\hat{a}}(t) = -\frac{i}{\hbar} [\hat{a}, \hat{H}_{sys}] - i \sum_k g_k \hat{b}_k$$

$$\xrightarrow{(4a)} \dot{\hat{a}}(t) = -\frac{i}{\hbar} [\hat{a}, \hat{H}_{sys}] \underbrace{- i \sum_k g_k e^{-i\omega_k(t-t_0)} \cdot \hat{b}_k(t_0)}_{f_a(t)} - \sum_k g_k^2 \int_{t_0}^{t} dt' e^{-i\omega_k(t-t')} \cdot \hat{a}(t')$$

$$I \simeq - \sum_k g_k^2 \int_{t_0}^{t} dt' \cdot e^{-i\omega_k (t-t')} \cdot \hat{a}(t')$$

see Eq. (1.1.23) in Scully - Zubairy

$$\sum_k \to 2 \left(\frac{L}{2\pi}\right)^3 \cdot \int d^3k$$

$$d\mathcal{N} = 2\left(\frac{L}{2\pi}\right)^3 \cdot \frac{\omega_k^2 d\omega_k}{c^3} \cdot \int_0^{\pi} d\theta \cdot \sin\theta \cdot \int_0^{2\pi} d\phi = \boxed{\underbrace{\frac{L^3 \omega_k^2}{\pi^2 c^3}}_{D(\omega_k)} \cdot d\omega_k = d\mathcal{N}}$$

$$\Rightarrow I = - \frac{L^3}{\pi^2 c^3} \cdot \int_0^{\infty} d\omega_k \cdot \underbrace{\omega_k^2 g_k^2}_{\sim \text{Const. about } \omega_k \cong \omega} \cdot \int_{t_0}^{t} dt' \cdot e^{-i\omega_k(t-t')} a(t')$$

$$\Rightarrow I \cong - D(\omega) \cdot g^2(\omega) \cdot \int_{-\infty}^{\infty} d\omega_k \cdot \int_{t_0}^{t} dt' \cdot e^{-i\omega_k (t-t')} \hat{a}(t')$$

$$I \cong - D(\omega) \cdot g^2(\omega) \cdot 2\pi \int_{t_0}^{t} dt' \cdot \delta(t-t') \hat{a}(t')$$

$$\Rightarrow I \cong - D(\omega) \cdot g^2(\omega) \cdot \pi \cdot \hat{a}(t) = \frac{\eta}{2} \hat{a}(t)$$

$$\boxed{\eta = 2\pi D(\omega) \cdot g^2(\omega)}$$

$$\Rightarrow \dot{\hat{a}}(t) = \frac{-i}{\hbar}[\hat{a}, \hat{H}_{sys}] - i\sum_k g_k e^{-i\omega_k(t-t_0)} \cdot \hat{b}_k(t_0) - \frac{\eta}{2}\cdot\hat{a}(t)$$

$$\boxed{\hat{a}_{in}(t) = -i\sum_k e^{-i\omega_k(t-t_0)} \cdot \hat{b}_k(t_0)}$$

$$\Rightarrow \boxed{\dot{\hat{a}}(t) = -\frac{i}{\hbar}[\hat{a}, \hat{H}_{sys}] - \frac{\eta}{2}\cdot\hat{a}(t) + g(\omega)\cdot\hat{a}_{in}(t)}$$

$\eta \equiv \gamma$ in ...

$g(\omega) \equiv \sqrt{\gamma}$ in ...

$$\dot{\hat{a}}(t) = -\frac{i}{\hbar}[\hat{a}, \hat{H}_{sys}] - i\sum_k g_k \hat{b}_k$$

4b)

$$\Rightarrow \dot{\hat{a}}(t) = -\frac{i}{\hbar}[\hat{a}, H_{sys}] - i\sum_k g_k e^{-i\omega_k(t-t_1)} \cdot \hat{b}_k(t_1) + \sum_k g_k^2 \int_t^{t_1} dt' e^{-i\omega_k(t-t')} \cdot \hat{a}(t')$$

$$\boxed{\hat{a}_{out}(t) = +i\sum_k e^{-i\omega_k(t-t_1)} \hat{b}_k(t_1)}$$

$$\boxed{\dot{\hat{a}}(t) = -\frac{i}{\hbar}[\hat{a}, \hat{H}_{sys}] + \frac{\eta}{2}\cdot\hat{a}(t) - g(\omega)\cdot\hat{a}_{out}(t)}$$

$$\Rightarrow \boxed{g(\omega)\cdot[\hat{a}_{out}(t) + \hat{a}_{in}(t)] = \eta\cdot\hat{a}(t)}$$

$$\boxed{D(\omega) = \frac{L^3}{\pi^2 c^3}\cdot\omega^2}$$

$$\Rightarrow \hat{a}_{out}(t) + \hat{a}_{in}(t) = 2\pi\cdot D(\omega)\cdot g(\omega)\cdot\hat{a}(t)$$

B) Mechanical Coupling to vibrations

$\hat{H} = \hat{H}_m + \hat{H}_b$

$\hat{H}_m = \frac{p^2}{2m} + \frac{1}{2} m \omega_m^2 x^2$

$\hat{H}_{bm} = \sum_n \left[\frac{p_n^2}{2m_n} + \frac{1}{2} m_n \omega_n^2 (x - q_n)^2 \right]$

Gardner $\rightarrow$ Eq. (3.1.3)

$q_n = \tilde{p}_n / \sqrt{m_n \omega_n^2} \longrightarrow [\tilde{p}_n] = \sqrt{M/T^2} \cdot L = \frac{L}{T}\sqrt{M}$ $\boxed{\mathcal{K}_n = \sqrt{m_n \omega_n^2}}$

$p_n = -\tilde{q}_n \cdot \sqrt{m_n \cdot \omega_n^2} \longrightarrow [\tilde{q}_n] = M \frac{L}{T} \cdot \frac{1}{\sqrt{M/T^2}} = L\sqrt{M}$

$\Rightarrow \hat{H}_{bm} = \sum_n \frac{\tilde{q}_n^2 \cdot m_n \cdot \omega_n^2}{2m_n} + \frac{1}{2} m_n \omega_n^2 \left(x - \tilde{p}_n / \sqrt{m_n \omega_n^2} \right)^2$

$\Rightarrow \hat{H}_{bm} = \frac{1}{2} \sum_n \left[\omega_n^2 \cdot \tilde{q}_n^2 + (\tilde{p}_n - K_n x)^2 \right]$ $\boxed{K_n = \sqrt{m_n \omega_n^2}}$

$\boxed{[\tilde{q}_n, \tilde{p}_n] = i\hbar}$

Time evolution:

$i\hbar \dot{\tilde{q}}_n = [\tilde{q}_n, \hat{H}] = \frac{1}{2} \cdot \left([\tilde{q}_n, \tilde{p}_n^2] - 2K_n [\tilde{q}_n, \tilde{p}_n] x \right) = i\hbar \tilde{p}_n - K_n \cdot i\hbar \cdot x$

$\Rightarrow \boxed{\dot{\tilde{q}}_n = \tilde{p}_n - K_n x}$

$i\hbar \dot{\tilde{p}}_n = [\tilde{p}_n, \hat{H}] = \frac{1}{2} [p_n, \omega_n^2 \tilde{q}_n^2] = \frac{1}{2} \cdot \omega_n^2 \cdot 2(-i\hbar) \tilde{q}_n = -\omega_n^2 \, i\hbar \cdot \tilde{q}_n$

$\Rightarrow \boxed{\dot{\tilde{p}}_n = -\omega_n^2 \cdot \tilde{q}_n}$

If $H_{int} = -\hbar g_m \cdot \bar{\tilde{p}}_n \cdot \bar{x}$ $\Rightarrow g_n^{(m)} = \sqrt{\frac{m_n \omega_n}{m \omega_m}} \cdot \omega_n$

$H_{int} = -K_n \cdot \tilde{p}_n \cdot x = -\sqrt{m_n \omega_n^2} \cdot \sqrt{\hbar \omega_n} \cdot \sqrt{\frac{\hbar}{m \omega_m}} \cdot \bar{\tilde{p}}_n \bar{x}$

$= -\hbar \sqrt{\frac{m_n}{m} \frac{\omega_n}{\omega_m}} \cdot \omega_n$

$$\dot{\tilde{q}}_n = \tilde{p}_n - K_n \cdot \hat{x}$$

$$\dot{\tilde{p}}_n = -\omega_n^2 \cdot \tilde{q}_n$$

Define $\hat{a}_n = \dfrac{\omega_n \tilde{q}_n + i\tilde{p}_n}{\sqrt{2\hbar\omega_n}}$

$$[a_n] = \frac{\frac{1}{T} \cdot L\sqrt{M}}{\sqrt{M\frac{L^2}{T} \cdot \frac{1}{T}}} = 1 \rightarrow \text{dimensionless} \checkmark$$

$$\hat{a}_n^{\dagger} = \frac{\omega_n \tilde{q}_n - i\tilde{p}_n}{\sqrt{2\hbar\omega_n}}$$

$$\Rightarrow \tilde{q}_n = \frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{\omega_n}}\left(\hat{a}_n^{\dagger} + \hat{a}_n\right)$$

$$\tilde{p}_n = \frac{i}{\sqrt{2}}\sqrt{\hbar\omega_n} \cdot \left(\hat{a}_n^{\dagger} - \hat{a}_n\right)$$

$$\Rightarrow \boxed{\left[K_n\sqrt{\frac{\omega_n}{\hbar}}\right]_{dim} = \sqrt{\frac{M}{T^2} \cdot \frac{1}{T} \cdot \frac{1}{M \cdot L \frac{}{}/_{T} \cdot L}} = \frac{1}{L \cdot T} \checkmark}$$

$$\Rightarrow \dot{a}_n = -i\omega_n a_n - K_n\sqrt{\frac{\omega_n}{2\hbar}} \cdot \hat{x}$$

$$\boxed{a_n(t) = e^{-i\omega_n(t-t_0)} \cdot a_n(t_0) - K_n\sqrt{\frac{\omega_n}{2\hbar}}\int_{t_0}^{t} e^{-i\omega_n(t-t')}\,\hat{x}(t') \cdot dt'}$$

<u>Time evolution of $\hat{x}, \hat{p}$:</u>

$$i\hbar\dot{x} = [x, \hat{H}] = \frac{1}{2m}[x, p^2] = \frac{i\hbar}{m} \cdot p \quad \Rightarrow \quad \boxed{\dot{x} = p/m}$$

$$i\hbar\dot{p} = [\hat{p}, \hat{H}] = \frac{1}{2}m\omega_m^2[p, x^2] + \frac{1}{2}\left(-2K_n\tilde{p}_n[\hat{p}, \hat{x}] + K_n^2[p, x^2]\right)$$

$$= \frac{1}{2}m\omega_m^2 \cdot 2(-i\hbar) \cdot \hat{x} - K_n \cdot \tilde{p}_n \cdot (-i\hbar) + K_n^2(-i\hbar) \cdot \hat{x}$$

$$\Rightarrow \boxed{\dot{p} = -m\omega_m^2\hat{x} - \sum_n K_n^2\hat{x} + \sum_n K_n\tilde{p}_n}$$

Scale x, p:

$$\bar{x} = \frac{x}{\sqrt{\frac{\hbar}{m\omega_m}}} \qquad \bar{p} = \frac{p}{\sqrt{\hbar m \omega_m}} \qquad \bar{x}(t) = \omega_m \int_{t_0}^{t} \bar{p}(t') \cdot dt'$$

$$\Rightarrow \dot{\bar{x}} \cdot \sqrt{\frac{\hbar}{m\omega_m}} = \bar{p} \cdot \sqrt{\hbar m \omega_m} \cdot \frac{1}{m} \quad \Rightarrow \boxed{\dot{\bar{x}} = \omega_m \bar{p}}$$

$$\sqrt{\hbar m \omega_m} \cdot \dot{\bar{p}} = -m\omega_m^2 \cdot \sqrt{\frac{\hbar}{m\omega_m}} \cdot \bar{x} - \sum_n m_n \omega_n^2 \cdot \sqrt{\frac{\hbar}{m\omega_m}} \cdot \bar{x} + \sum_n k_n \cdot \tilde{p}_n$$

$$\dot{\bar{p}} = -\omega_m \cdot \bar{x} - \sum_n \underbrace{\frac{m_n \omega_n^2}{m \cdot \omega_m^2}}_{\bar{K}_n^2} \cdot \omega_m \bar{x} + \sum_n \sqrt{\frac{m_n \omega_n^2}{m \hbar \omega_m}} \cdot \tilde{p}_n$$

$$\Rightarrow \dot{\bar{p}} = -\omega_m \cdot \left(1 - \sum_n \bar{K}_n^2\right) \cdot \bar{x} + \sum_n \frac{K_n}{\sqrt{m\hbar\omega_m}} \cdot \frac{i}{\sqrt{2}} \sqrt{\hbar \omega_n} \, \times$$

$$\times \Big[\left(e^{i\omega_n (t-t_0)} \cdot a_n^\dagger(t_0) - e^{-i\omega_n (t-t_0)} \cdot \hat{a}_n(t_0) \right)$$

$$- K_n \sqrt{\frac{\omega_n}{2\hbar}} \int_{t_0}^{t} dt' \cdot \Big(\underbrace{e^{i\omega_n (t-t')} - e^{-i\omega_n (t-t')}}_{2i \sin[\omega_n (t-t')]} \Big) \cdot \sqrt{\frac{\hbar}{m\omega_m}} \cdot \bar{x}(t') \Big]$$

$$\Rightarrow \dot{\bar{p}} = -\omega_m \left(1 - \sum_n \bar{K}_n^2\right) \cdot \bar{x} + \sum_n \sqrt{\frac{m_n \omega_n^2}{m \hbar \omega_m} \cdot \frac{\hbar \omega_n}{2}} \cdot i \cdot \left(a_n^\dagger(t_0) e^{i\omega_n (t-t_0)} - a_n(t_0) e^{-i\omega_n (t-t_0)} \right)$$

$$+ \sum_n \left(\frac{m_n \omega_n^2}{m \hbar \omega_m} \cdot \frac{\hbar \omega_n}{2} \right)^{1/2} \cdot i \cdot \left(\frac{m_n \omega_n^2 \cdot \omega_n}{2\hbar} \right)^{1/2} \cdot (-1) \cdot \left(\frac{\hbar}{m\omega_m} \right)^{1/2} \cdot 2 \cdot i \cdot \int_{t_0}^{t} dt' \cdot \underbrace{\sin[\omega_n (t-t')]}_{du} \cdot \underbrace{\bar{x}(t')}_{v(t')}$$

$$\Rightarrow \dot{\bar{p}} = -\omega_m \cdot \left(1 - \sum_n \bar{\kappa}_n^2\right) \cdot \bar{x} + \sum_n g_n^{(m)} \cdot \frac{i}{\sqrt{2}} \cdot \left(\hat{a}_n^+(t_0)\, e^{i\omega_n(t-t_0)} - \hat{a}_n(t_0)\, e^{-i\omega_n(t-t_0)} \right)$$

$$I_3 \longrightarrow + \sum_n g_n^{(m)2} \cdot \left\{ \sin[\omega_n(t-t')]\, \bar{x}(t') \Big|_{t'=t_0}^{t} - \int_{t_0}^{t} dt' \cdot \frac{(-1)\cdot \cos[\omega_n(t-t')]}{\omega_n \cdot (-1)} \cdot \underbrace{\dot{\bar{x}}(t')}_{\omega_m \cdot p(t')} \right\}$$

$$\Rightarrow I_3 = \sum_n g_n^{(m)2} \cdot \left\{ \underbrace{-\sin[\omega_n(t-t_0)]\, \bar{x}(t_0)}_{=0 \text{ after } \sum_n \text{ integration}} - \frac{\omega_m}{\omega_n} \int_{t_0}^{t} dt' \cdot \cos[\omega_n(t-t')] \cdot \bar{p}(t') \right\}$$

include also degeneracy in each n. $\Rightarrow g_{n,r}^{(m)}$

$$\sum_{r=0}^{d_r} g_{n,r}^{(m)2} \cdot \frac{\omega_m}{\omega_n} = G_n(\omega_n)$$

$$\Rightarrow I_3 = -\int_{t_0}^{t} dt' \cdot \bar{p}(t') \cdot \sum_n G_n(\omega_n) \cdot \cos[\omega_n(t-t')]$$

$$I_3 = -\int_{t_0}^{t} dt' \cdot \bar{p}(t') \cdot \int_0^{\infty} d\omega \cdot \underbrace{\rho(\omega) \cdot G(\omega)}_{\simeq \text{Const (nonzero only some freq.)}} \cdot \cos[\omega(t-t')]$$

- $$\int_0^{\infty} d\omega \cdot \cos[\omega(t-t')] = \int_0^{\infty} d\omega \left(\frac{e^{i\omega(t-t')} + e^{-i\omega(t-t')}}{2} \right) = \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(t-t')}}{2} = \pi \cdot \delta(t-t')$$

$$\Rightarrow I_3 = -\rho(\omega) G(\omega) \cdot \int_{t_0}^{t} dt' \cdot \bar{p}(t') \cdot \pi \cdot \delta(t-t') = -\underbrace{\frac{\pi}{2}\rho(\omega)\, G(\omega)}_{\gamma_m} \cdot \bar{p}(t)$$

$$\boxed{\dot{\bar{p}} = -\omega_m\left(1-\sum_n \bar{k}_n^2\right)\cdot\bar{x} + g^{(m)}\cdot\hat{\mathcal{E}}_{in} - \gamma_m\cdot\bar{p}}$$

$$\hat{\mathcal{E}}_{in} = \sum_n \frac{i}{\sqrt{2}}\left(\hat{a}_n^{\dagger}(t_0)\, e^{i\omega_n(t-t_0)} - \hat{a}_n(t_0)\, e^{-i\omega_n(t-t_0)}\right)$$

↳ Behaves like momentum.

freq.

- $g^{(m)} = \sqrt{\dfrac{m_n\cdot\omega_n}{m\cdot\omega_m}}\cdot\omega_n$

freq^{-1} → freq.^{2}

- $\gamma_m = \dfrac{\pi}{2}\cdot\rho(\omega)\cdot G(\omega)$

- $G(\omega) = \displaystyle\sum_{r=0}^{d_r} g_{n,r}^{(m)2}\cdot\frac{\omega_m}{\omega_n}$ → since $G(\omega)\sim G(\omega_n) \Rightarrow G(\omega) \simeq \displaystyle\sum_{r=0}^{d_r} g_{n,r}^{(m)\,2}$

freq.^{2}

move dependency into $\rho(\omega)$ → $G_n(\omega_n) = \left(g_n^{(m)}\right)^2\cdot\dfrac{\omega_m}{\omega_n} \Rightarrow \boxed{G(\omega) \simeq \left(g^{(m)}\right)^2}$