Input-output Formalism for multi-freq. cavity

$$\hat{H} = \hat{H}_{sys} + \hat{H}_{bath} + \hat{H}_{int}$$

c_1
c_2
c_3

$$\hat{H}_{bath} = \sum_k \hbar\omega_k b_k^\dagger b_k$$

$$\hat{H}_{int} = \hbar \sum_k g_k^{(1)} (b_k^\dagger \hat{c}_1 + c_1^\dagger b_k) + \hbar \sum_k g_k^{(2)} (b_k^\dagger c_2 + c_2^\dagger b_k) + \ldots$$

$$i\hbar \dot{b}_k = [b_k, \hat{H}] = \hbar\omega_k \hat{b}_k + \hbar g_k^{(1)} \hat{c}_1 + \hbar g_k^{(2)} \hat{c}_2 + \ldots$$

$$\Rightarrow \dot{b}_k = -i\omega_k \hat{b}_k - i g_k^{(1)} c_1 - i g_k^{(2)} c_2 \ldots$$

$$\Rightarrow \hat{b}_k(t) = e^{-i\omega_k(t-t_0)} \hat{b}_k(t_0) - i g_k^{(1)} \int_{t_0}^{t} dt' \, e^{-i\omega_k(t-t')} \cdot c_1(t') - i g_k^{(2)} \int_{t_0}^{t} dt' \, e^{-i\omega_k(t-t')} c_2(t')$$

for $t > t_0$

$$\hat{b}_k(t) = e^{-i\omega_k(t-t_1)} \hat{b}_k(t_1) + i g_k^{(1)} \int_{t}^{t_1} dt' \, e^{-i\omega_k(t-t')} \cdot \hat{c}_1(t')$$

$$+ i g_k^{(2)} \int_{t}^{t_1} dt' \, e^{-i\omega_k(t-t')} c_2(t') \ldots$$

for $t < t_1$

$$i\hbar \dot{c}_1(t) = [c_1, H] = \hbar g_k^{(1)} \hat{b}_k + [\hat{c}_1, H_{sys}]$$

$$\Rightarrow \dot{c}_1(t) = -\frac{i}{\hbar} [c_1, H_{sys}] - i \sum_k g_k^{(1)} \hat{b}_k$$

$$\dot{c}_1(t) = -\frac{i}{\hbar} [c_1, H_{sys}] - i \sum_k g_k^{(1)} e^{-i\omega_k(t-t_0)} b_k(t_0) - \sum_k \left(g_k^{(1)}\right)^2 \int_{t_0}^{t} dt' \, e^{-i\omega_k(t-t')} c_1(t')$$

$$- \sum_k g_k^{(1)} g_k^{(2)} \int_{t_0}^{t} dt' \, e^{-i\omega_k(t-t')} c_2(t')$$

$$I_1 = -\sum_k \left(g_k^{(1)}\right)^2 \int_{t_0}^{t} dt' \, e^{-i\omega_k(t-t')} \, \hat{c}_1(t')$$

$$= -\int d\omega \, D(\omega) \, g_k^{2}(\omega) \int_{t_0}^{t} dt' \, e^{-i\omega_k(t-t')} \, \hat{c}_1(t')$$

do not change much within the coupling window of ω_1

$$\Rightarrow I_1 = -g^{2(1)}(\omega_1) \cdot D(\omega_1) \cdot \int_{t_0}^{t} dt' \, c_1(t') \underbrace{\int_{-\infty}^{\infty} d\omega_k \, e^{-i\omega(t-t')}}_{2\pi\,\delta(t-t')}$$

$$\Rightarrow I_1 = -\left[g^{(1)}(\omega_1)\right]^2 \cdot D(\omega_1) \cdot \pi \; c_1(t)$$

$\rightarrow \kappa_1 \rightarrow$ cavity decay rate for mode ω_1

$$I_2 = -\sum_k g_k^{(1)} g_k^{(2)} \int_{t_0}^{t} dt' \, e^{-i\omega_k(t-t')} \, c_2(t')$$

$$\cong -g_k^{(1)}(\omega_1)\, g_k^{(2)}(\omega_2) \int_{t_0}^{t} dt' \, c_2(t') \underbrace{\int d\omega \, e^{-i\omega(t-t')}}_{2\pi\,\delta(t-t')}$$

$$I_2 = -\pi\, g_k^{(1)}(\omega_1)\, g_k^{(2)}(\omega_2)\, D(\omega_2)\, c_2(t)$$

- Transient oscillation freqs of $c_1(t)$ and $c_2(t)$ are optical order different ($|\omega_1-\omega_2| \sim$ PHz) $\Rightarrow$ coupling of $I_2(t)$ term to $\dot{c}_1(t)$ can be safely neglected.

Note that $g_k^{(1)}(\omega_k)$ and $g_k^{(2)}(\omega_k)$ are in fact the same functions, but calculated at different optical frequencies, ω_1 and ω_2, which may deviate two values, $g_k(\omega_1)$ and $g_k(\omega_2)$, non-negligibly.

Hence input-output formalism can be rewritten as:

$$\dot{\hat{c}}_1(t) = -\frac{i}{\hbar}[c_1, H_{sys}] - K_1 \hat{c}_1(t) + g(\omega_1)\, \hat{a}_{in}(t) \rightarrow \omega_1 \mp K_1$$

MHz

$$\dot{\hat{c}}_2(t) = -\frac{i}{\hbar}[c_2, H_{sys}] - K_2 \hat{c}_2(t) + g(\omega_2) \cdot \hat{a}_{in}(t) \rightarrow \omega_2 \mp K_2$$

• Note the $\hat{a}_{in}(t) = \sum_k e^{-i\omega_k(t-t_0)} b_k(t_0)$ are the same in both lines. But, coupling (noise transfer and leakage) happens to different freq. b_k modes.

Since the three optical freq.s are seperated with optical frequencies ($\sim$ PHz), this separation is much larger than the bandwidth ($\sim$ MHz) of the leakage to outer vacuum modes. Therefore, $c_1(t)$, $c_2(t)$ will not mix due to coupling to the same vacuum $[a_{in}(t)]$.

⇒ The same is true for the output field.

Since the 3 modes are seperated with frequency of order PHz and room temperature fluctuations are only 10^{-2} PHz, each mode couples to the spectral width $\sim g$ of the vacuum centered around ω_i (or 0.01 PHz for room temp.), due to the adiabatic elimination considerations (see the supplementary "some additions to Input-Output formalism") since the vacuum modes out of this extent ($\Delta \sim g$) do not contribute to the integrals; one can safely seperate the input and output for each 3 modes as

$$\hat{b}^{(i)}_{in}(t) \quad \text{and} \quad \hat{b}^{(i)}_{out}(t) \, .$$