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A Generalization of Semiregular and Semiperfect Modules

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Abstract. Let U be a submodule of a module M. We call U a strongly lifting submodule of M if whenever $M/U = (A+U)/U \oplus (B+U)/U$, then $M = P \oplus Q$ such that $P \leq A$, $(A+U)/U = (P+U)/U$ and $(B+U)/U = (Q+U)/U$. This definition is a generalization of strongly lifting ideals defined by Nicholson and Zhou. In this paper, we investigate some properties of strongly lifting submodules and characterize U-semiregular and U-semiperfect modules by using strongly lifting submodules. Results are applied to characterize rings R satisfying that every (projective) left R-module M is $\tau(M)$-semiperfect for some preradicals τ such as Rad, Z_2 and δ.

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1 Introduction

Semiregular and semiperfect rings were generalized to I-semiregular and I-semiperfect rings for an ideal I of a ring R by Yousif and Zhou [15]. After that Nicholson and Zhou [11] defined the concept of strongly lifting left ideals. A left ideal I is called *strongly lifting* (or *idempotents lift strongly modulo* I) if whenever $a^2 - a \in I$, there exists $e^2 = e \in Ra$ (equivalently, $e^2 = e \in aR$) such that $e - a \in I$. Then they proved that a ring R is I-semiregular (I-semiperfect, respectively) if and only if R/I is regular (semisimple) and I is strongly lifting. Note that being I-semiregular or I-semiperfect for an ideal I of a ring R is left-right symmetric by Theorems 28 and 36 in [11].

In [1] and [12], U-semiregular and U-semiperfect modules are defined as module theoretic versions of I-semiregular and I-semiperfect rings by considering any fully invariant submodule U of a module, and so some properties of I-semiregular and I-semiperfect rings are generalized to modules.

In Section 2, we investigate strongly lifting submodules and U-semipotent modules for a submodule U of a module. We call a submodule U of a module M *strongly lifting* if whenever $M/U = (A+U)/U \oplus (B+U)/U$, then M has a decomposition $M = P \oplus Q$ such that $P \leq A$, $(A+U)/U = (P+U)/U$ and $(B+U)/U = (Q+U)/U$. We prove that an ideal I of a ring R is a strongly lifting ideal if and only if I is a strongly lifting submodule of ${}_RR$ (Proposition 2.2). M is called *U-semipotent* if for every submodule A of M such that $A \not\subseteq U$, there exists a summand B of M such that $B \leq A$ and $B \not\subseteq U$. We prove that if $U \leq M$ and M is U-semipotent, then for any submodule N of M with $N \not\subseteq U$, N is indecomposable if and only if N is local (Proposition 2.9).

In Section 3, we give a new characterization of U-semiregular and U-semiperfect modules by considering strongly lifting submodules for a projection-invariant submodule U. We prove that if M is finitely generated and projective, then M is U-semiregular if and only if every finitely generated submodule of M/U is a summand and U is strongly lifting (Corollary 3.3). If M is projective, then M is U-semiperfect if and only if M/U is semisimple and U is strongly lifting (Corollary 3.8).

In Section 4, rings R satisfying the property that every (projective) R-module M is $\tau(M)$-semiperfect are characterized for some preradicals τ such as Rad, Z_2 and δ. We prove that every left R-module M is $Z_2(M)$-semiperfect if and only if R is $Z_2({}_RR)$-semiperfect; every projective left R-module M is $\delta(M)$-semiperfect if and only if R is left δ-perfect; and a ring R is $Z({}_RR)$-semiperfect and $Z_2({}_RR)$ is injective if and only if R is semiperfect and left self-injective.

Throughout this paper, R denotes an associative ring with identity and modules M are unitary left R-modules. For a module M, $\mathrm{Rad}(M)$, $\mathrm{Soc}(M)$, $Z(M)$ and $Z_2(M)$ are the Jacobson radical, the socle, the singular submodule and the Goldie torsion submodule of M, respectively. We write $J(R)$ for the Jacobson radical of R. A submodule N of M is called *small* in M, denoted by $N \ll M$, whenever for any submodule L of M, $N + L = M$ implies $L = M$. For a (direct) summand K of M, we write $K \leq^{\oplus} M$. An element x in M is called *regular* if $(x\alpha)x = x$ for some $\alpha \in M^*$. Zelmanowitz [16] calls a module *regular* if each of its elements is regular, equivalently, if every finitely generated submodule is a projective summand. A submodule U of M is called *projection-invariant* if for every projection π of M, $(U)\pi \leq U$.

Lemma 1.1. [6, Exercise 4.d, p. 50] *Let $M = M_1 \oplus M_2$ and U be any projection-invariant submodule of M. Then $U = (U \cap M_1) \oplus (U \cap M_2)$.*

2 Strongly Lifting Submodules and U-Semipotent Modules

Definition 2.1. Let U be a submodule of a module M. U is called a *strongly lifting submodule* of M if whenever $M/U = (A+U)/U \oplus (B+U)/U$, then M has a decomposition $M = P \oplus Q$ such that $P \leq A$, $(A+U)/U = (P+U)/U$ and $(B+U)/U = (Q+U)/U$.

Proposition 2.2. *Let I be an ideal of R, $\overline{R} = R/I$ and $\overline{r} = r + I$ for any $r \in R$. The following are equivalent*:

(1) *I is strongly lifting.*

(2) *I is a strongly lifting submodule of ${}_RR$.*

Proof. (1)⇒(2) Let $\overline{R} = \overline{A} \oplus \overline{B}$. Let $\overline{1} = \overline{a} + \overline{b}$, where $a \in A$ and $b \in B$. Then $\overline{a}$ and $\overline{b}$ are orthogonal idempotents. By [11, Proposition 11], there exist orthogonal idempotents e_1 and e_2 in R such that $\overline{e_1} = \overline{a}$, $\overline{e_2} = \overline{b}$ and $e_1 \in Ra$, $e_2 \in Rb$. Then $R = Re_1 \oplus R(1-e_1)$ and $Re_1 \leq Ra$, $\overline{Re_1} = \overline{Ra} = \overline{A}$, $\overline{R}(\overline{1}-\overline{e_1}) = \overline{R}(\overline{1}-\overline{a}) = \overline{Rb} = \overline{B}$. Hence, (2) holds.

(2)⇒(1) Let $\overline{e}^2 = \overline{e} \in \overline{R}$. Then $\overline{R} = \overline{Re} \oplus \overline{R}(\overline{1-e})$. By hypothesis, $R = P \oplus Q$, where $P \leq Re$, $\overline{P} = \overline{Re}$ and $\overline{Q} = \overline{R}(\overline{1-e})$. Then there exists an idempotent f in R such that $P = Rf$ and $Q = R(1-f)$. Since $\overline{P} = \overline{Rf} = \overline{Re}$, we have $\overline{f} = \overline{ae}$ and $\overline{e} = \overline{bf}$ for some $\overline{a}, \overline{b}$ in $\overline{R}$. This implies that $\overline{e}\overline{f} = \overline{e}$. Since $\overline{Q} = \overline{R}(\overline{1-f})$ and $\overline{f} = \overline{e}\overline{f} + (\overline{1-e})\overline{f}$, we have $\overline{f} = \overline{e}$. Hence, I is strongly lifting. □

Proposition 2.3. *Let M be a self-projective module and $U \leq M$. If U is a summand of M, then U is strongly lifting.*

Proof. Let N be such that $M = U \oplus N$, and $M/U = (A+U)/U \oplus (B+U)/U$. Let $f : N \to M/U$ be the isomorphism. Then there exist submodules B_1 and B_2 of N such that $f(B_1) = (A+U)/U = (B_1+U)/U$, $f(B_2) = (B+U)/U = (B_2+U)/U$. Then $M/U = (B_1+U)/U \oplus (B_2+U)/U$. Since $B_1 \cap B_2 \leq (B_1+U) \cap (B_2+U) = U$, $B_1 \cap B_2 = 0$. Also, $N = B_1 + B_2$. Hence, $M = U \oplus N = U \oplus B_1 \oplus B_2$. Since $U \oplus B_1 = U + A$ is self-projective, there exists a submodule L of A such that $U \oplus B_1 = U \oplus L$ by [14, 41.14]. Thus, $M = U \oplus L \oplus B_2$, where $L \leq A$, $(L+U)/U = (A+U)/U$ and $(B_2+U)/U = (B+U)/U$, i.e., U is strongly lifting. □

A left R-module M is said to have the *exchange property* if for any module X and decompositions $X = M' \oplus Y = \oplus_{i \in I} N_i$, where $M' \simeq M$, there exist submodules $N_i' \leq N_i$ for each i such that $X = M' \oplus (\oplus N_i')$. If this condition holds for finite sets I (equivalently, for $|I| = 2$), the module M is said to have the *finite exchange property*. Note that a self-projective module M has the finite exchange property if and only if whenever $M = A + B$, there exists a decomposition $M = P \oplus Q$ such that $P \leq A$ and $Q \leq B$ [3, Theorem 3].

Theorem 2.4. *Let M be a self-projective module. Then the following are equivalent:*

(1) *M has the finite exchange property.*
(2) *Every submodule of M is strongly lifting.*

Proof. (1)⇒(2) Let $N \leq M$ and $M/N = (A+N)/N \oplus (B+N)/N$. Then $M = A + B + N$. By [3, Theorem 3], there is a decomposition $M = P_1 \oplus P_2$ with $P_1 \leq A$ and $P_2 \leq B + N$. Then $(P_1+N)/N = (A+N)/N$ and $(P_2+N)/N = (B+N)/N$. Hence, N is strongly lifting.

(2)⇒(1) Let $M = M_1 + M_2$ and $N = M_1 \cap M_2$. Since $M/N = M_1/N \oplus M_2/N$, there is a decomposition $M = P \oplus Q$ such that $P \leq M_1$, $(P+N)/N = M_1/N$ and $(Q+N)/N = M_2/N$. Then $Q \leq M_2$. By [3, Theorem 3], M has the finite exchange property. □

Definition 2.5. Let U be a submodule of M. M is called *U-semipotent* if for every submodule A of M such that $A \not\subseteq U$, there exists a summand B of M such that $B \leq A$ and $B \nsubseteq U$. A ring R is called *semipotent* if R is $J(R)$-semipotent. M is called *U-potent* if M is U-semipotent and U is a strongly lifting submodule of M.

There exists a U-semipotent module M, where U is not strongly lifting (see [11, Example 23]).

Hence, M is 0-potent if every nonzero submodule of M contains a nonzero summand of M. Every regular module is 0-potent. In fact, let M be regular and $0 \neq A \leq M$. Then there exist $0 \neq a \in A$ and $\alpha \in \mathrm{Hom}_R(M, R)$ such that $(a\alpha)a = a$. This implies that Ra is a nonzero summand of M in A.

On the other hand, modules M with zero radical and essential socle are 0-potent. In fact, let $0 \neq A \leq M$. Then A contains a simple submodule S. Since S is not small in M, S is a summand of M.

Proposition 2.6. *Let U be a projection-invariant submodule of a module M. If M is U-semipotent, then M/U is 0-potent. The converse holds if U is strongly lifting.*

Proof. Let $0 \neq A/U \leq M/U$. Then $A \nsubseteq U$, and by hypothesis there exists a summand B of M such that $B \leq A$ and $B \nsubseteq U$. Let B' be such that $M = B \oplus B'$. Since U is projection-invariant, $U = (B \cap U) \oplus (B' \cap U)$ by Lemma 1.1. This implies that $(B + U) \cap (B' + U) = [B + (B' \cap U)] \cap [B' + (B \cap U)] = U$. Hence, $(B + U)/U$ is a nonzero summand of M/U in A/U. The converse is clear. □

Proposition 2.7. *Let U be a submodule of M. If M is U-semipotent, then for every submodule N of M with $N \nsubseteq U$, N is $U \cap N$-semipotent.*

Proof. Assume that M is U-semipotent. Let $N \leq M$ and $X \leq N$ be such that $X \nsubseteq U \cap N$. Then $X \nsubseteq U$. By assumption, there exists a summand Y of M such that $Y \leq X$ and $Y \nsubseteq U$. Then Y is a summand of N such that $Y \leq X$ and $Y \nsubseteq U \cap N$. Hence, N is $U \cap N$-semipotent. □

Proposition 2.8. *If a module M is self-projective with the finite exchange property, then M is $\mathrm{Rad}(M)$-semipotent.*

Proof. Let $N \leq M$ be such that $N \nsubseteq \mathrm{Rad}(M)$. Let $n \in N \setminus \mathrm{Rad}(M)$. Then there exists a maximal submodule K of M such that $M = Rn + K$. By [3, Theorem 3], there is a decomposition $M = P \oplus Q$ such that $P \leq Rn$ and $Q \leq K$. If $P \leq \mathrm{Rad}(M)$, then $P \leq K$, and so $M = K$, a contradiction. Hence, $P \nsubseteq \mathrm{Rad}(M)$, and so the proof is completed. □

A module M is called *indecomposable* if $M \neq 0$ and it is not a direct sum of two nonzero submodules. If M has a largest proper submodule, i.e., a proper submodule which contains all other proper submodules, then M is called a *local* module. Any local module is indecomposable. By [14, Theorem 41.4], a nonzero module M is local if and only if M is hollow (i.e., every proper submodule of M is small) and cyclic.

Proposition 2.9. *Let U be a submodule of a module M and assume that M is*

U-semipotent. Then the following are equivalent for a submodule N of M with $N \nsubseteq U$:

(1) *N is indecomposable.*
(2) *For any submodule A of N with $A \nsubseteq U$, $A = N$.*
(3) *N is local.*

Proof. (3)⇒(1) It is obvious.

(1)⇒(2) Let $A \leq N$ with $A \nsubseteq U$. Then there exists a summand B of M such that $B \leq A$ and $B \nsubseteq U$. So B is a summand of N. If $B = 0$, then $B \leq U$, a contradiction. Then $B = N$. This implies that $A = N$.

(2)⇒(3) Since $N \nsubseteq U$, by (2), N is cyclic. Now let K be a proper submodule of N and $N = K + L$ for some L. We claim that $L = N$. Assume $L \leq U$. If $K \leq U$, then $N = U$, a contradiction. If $K \nsubseteq U$, then $K = N$, again a contradiction. Hence, $L \nsubseteq U$ and so $L = N$. By [14, Theorem 41.4], N is local. □

Proposition 2.10. *If M is $\mathrm{Rad}(M)$-semipotent, then every indecomposable summand N of M with $N \nsubseteq \mathrm{Rad}(M)$ is local.*

Proof. Let N be an indecomposable summand of M with $N \nsubseteq \mathrm{Rad}(M)$. We claim that for every proper submodule K of N, $K \leq \mathrm{Rad}(N)$. Let K be a proper submodule of N and assume $K \nsubseteq \mathrm{Rad}(N)$. Since $\mathrm{Rad}(N) = N \cap \mathrm{Rad}(M)$, $K \nsubseteq \mathrm{Rad}(M)$. Since M is $\mathrm{Rad}(M)$-semipotent, there exists a summand X of M such that $X \leq K$ and $X \nsubseteq \mathrm{Rad}(M)$. Then X is a summand of N. Since N is indecomposable, we have $X = N = K$, a contradiction. Hence, N is local. □

Proposition 2.11. *Let U be a projection-invariant submodule of a module M. If M is U-semipotent, then for any indecomposable summand $(A+U)/U$ of M/U, there exists a summand P of M such that $P \leq A$ and $(P+U)/U = (A+U)/U$.*

Proof. Let $(A+U)/U$ be an indecomposable summand of M/U. Then $A \nsubseteq U$. Since M is U-semipotent, there exists a summand P of M such that $P \leq A$ and $P \nsubseteq U$. Since U is projection-invariant, $(P+U)/U$ is a summand of M/U and then a summand of $(A+U)/U$. Since $(P+U)/U \neq 0$, $(P+U)/U = (A+U)/U$. □

3 *U*-Semiregular and *U*-Semiperfect Modules

Let U be a submodule of a module M. M is called *U-semiperfect* (*U-semiregular*, respectively) if for any (finitely generated) submodule N of M, there exists a decomposition $M = A \oplus B$ such that A is projective, $A \leq N$ and $N \cap B \leq U$. If U is a projection-invariant submodule of M, then this is equivalent to that for any (finitely generated) submodule N of M, there exists a decomposition $N = A \oplus B$ such that A is a projective summand of M and $B \leq U$ (see also [1] and [12]). Clearly, U-semiperfect modules are U-semiregular. Note that M is semiregular if and only if M is $\mathrm{Rad}(M)$-semiregular. If M is projective and $\mathrm{Rad}(M) \ll M$, then M is semiperfect if and only if M is $\mathrm{Rad}(M)$-semiperfect.

Let U and N be any submodules of a module M. Following [11], we say that U *respects* N if there exists a summand A of M contained in N such that $M = A \oplus B$ and $B \cap N \leq U$.

Lemma 3.1. *Let U be a projection-invariant submodule of M and N any submodule of a module M. Then the following are equivalent*:

(1) *U respects N.*

(2) *There exists a summand A of M contained in N such that $N = A \oplus B$ and $B \leq U$.*

(3) *There exists $\pi^2 = \pi$ in $\mathrm{End}_R(M)$ with $(M)\pi \leq N$ such that $(N)(1-\pi) \leq U$.*

Proof. By Lemma 1.1, it is obvious. □

Recall that a module M is called *lifting* (or (D1)) (see [7]) if for any submodule N of M, N has a decomposition $N = A \oplus B$, where $A \leq^{\oplus} M$ and $B \ll M$. Then $B \leq \mathrm{Rad}(M)$. Hence, if M is lifting, then $\mathrm{Rad}(M)$ respects every submodule of M.

First we want to characterize U-semiregular modules. Clearly, if M is U-semiregular, then U respects every finitely generated submodule of M. If M is projective, then the converse is true.

Theorem 3.2. *Let U be a projection-invariant submodule of a module M and $\overline{M} = M/U$. Consider the following conditions*:

(1) (i) *Every finitely generated submodule of $\overline{M}$ is a summand.*
 (ii) *If $\overline{M} = \overline{A} \oplus \overline{B}$, where $\overline{A}$ is finitely generated, then there exists a decomposition $M = P \oplus Q$ such that $P \leq A$, $\overline{P} = \overline{A}$ and $\overline{Q} = \overline{B}$.*

(2) *U respects every finitely generated submodule of M.*

Then (1)⇒(2); *and* (2)⇒(1) *if M is self-projective.*

Proof. (1)⇒(2) Let N be a finitely generated submodule of M. Then $\overline{M} = \overline{N} \oplus \overline{B}$ for some submodule $\overline{B}$. By hypothesis, $M = P \oplus Q$ such that $P \leq N$, $\overline{P} = \overline{N}$, $\overline{Q} = \overline{B}$. Since $N = P + (N \cap U)$ and $U = (U \cap P) \oplus (U \cap Q)$, we have $Q \cap N \leq U$. So (2) follows.

(2)⇒(1)(i) Let $X/U \leq M/U$ be finitely generated. Choose a finitely generated submodule N of M such that $X/U = (N+U)/U$. By (2), $M = A \oplus B$ such that $A \leq N$ and $B \cap N \leq U$. Then $X/U = (A+U)/U$. Since $U = (U \cap A) \oplus (U \cap B)$ and $(B+U) \cap (A+U) = \big(B + (U \cap A)\big) \cap \big(A + (U \cap B)\big) = U$, we get $\overline{A} \oplus \overline{B} = \overline{M}$. So $\overline{X}$ is a summand of $\overline{M}$.

For (ii), let $\overline{M} = \overline{A} \oplus \overline{B}$, where $\overline{A}$ is finitely generated. Let N be a finitely generated submodule of A such that $\overline{A} = \overline{N}$. Then $M = C \oplus D$ such that $C \leq N$ and $D \cap N \leq U$. Since $N = C \oplus (D \cap N)$, $M = (A+U) + B = (C+U) + B$. Since C is a summand of M and M is self-projective, there exists a summand Q of M such that $M = C \oplus Q$ and $Q \leq U + B$ [14, 41.14]. Now it can be seen that $C \leq A$, $\overline{C} = \overline{A}$ and $\overline{Q} = \overline{B}$. □

Corollary 3.3. *Let U be a projection-invariant submodule of a projective module M and $\overline{M} = M/U$. Then the following are equivalent*:

(1) *M is U-semiregular.*

(2) (i) *Every finitely generated submodule of $\overline{M}$ is a summand.*
 (ii) *If $\overline{M} = \overline{A} \oplus \overline{B}$, where $\overline{A}$ is finitely generated, then there exists a decomposition $M = P \oplus Q$ such that $P \leq A$, $\overline{P} = \overline{A}$ and $\overline{Q} = \overline{B}$.*

In addition, if M is finitely generated, then they are equivalent to

(3) (i) *Every finitely generated submodule of* $\overline{M}$ *is a summand.*
(ii) U *is strongly lifting.*

Corollary 3.4. *Let* U *be a submodule of a module* M*. If* M *is* U*-semiregular, then* M *is* U*-semipotent. If in addition,* M *is finitely generated and self-projective, then* M *is* U*-potent.*

Proof. Let A be a submodule of M with $A \nsubseteq U$. Let $a \in A \setminus U$. Then $M = X \oplus Y$, where $X \leq Ra$ and $Y \cap Ra \leq U$. This implies that $Ra = X \oplus (Y \cap Ra)$ and so $X \nsubseteq U$. Hence, M is U-semipotent. If M is finitely generated self-projective, by the proof of (2)⇒(1)(ii) in Theorem 3.2, U is strongly lifting. □

U-semipotent modules need not be U-semiregular even if M/U is regular (see [11, Example 52]).

Proposition 3.5. *Let* U *be a proper submodule of a module* M*. If* M *is indecomposable and* $\mathrm{Rad}(M) \ll M$*, then the following are equivalent*:
(1) U *respects every finitely generated submodule of* M*.*
(2) M *is* U*-semipotent.*
(3) M *is local and* $U = \mathrm{Rad}(M)$*.*

Proof. (1)⇒(2) By the proof of Corollary 3.4.

(2)⇒(3) By Proposition 2.9, M is local. Since $\mathrm{Rad}(M)$ is maximal, we have $U \leq \mathrm{Rad}(M)$. Now let $x \in \mathrm{Rad}(M) \setminus U$. Then there exists a summand B of M such that $B \leq Rx$ and $B \nsubseteq U$. Since $Rx \ll M$, we have $B \ll M$. Then $B = 0$, a contradiction. Hence, $\mathrm{Rad}(M) = U$.

(3)⇒(1) Let N be a finitely generated submodule of M. If $N = M$, there is nothing to prove. Assume $N \neq M$. Then $N \leq \mathrm{Rad}(M)$. Hence, the decomposition $M = 0 \oplus M$ completes the proof. □

In [1, Proposition 2.2], it is proved that for any fully invariant submodule U of M, M is U-semiregular if and only if for any $x \in M$, there exists a regular element $y \in Rx$ such that $x - y \in U$ and $Rx = Ry \oplus R(x - y)$. The same proof shows that the condition "$Rx = Ry \oplus R(x - y)$" is removable, even for a projection-invariant submodule U of M. We give below its proof for completeness. Also, it is proved in [1, Corollary 2.7] that with some conditions, M is U-semiregular if and only if for any $x \in M$, there exists a regular element $y \in M$ such that $x - y \in U$.

Theorem 3.6. *Let* U *be a projection-invariant submodule of a module* M*. Then the following are equivalent*:
(1) M *is* U*-semiregular.*
(2) *For any* $x \in M$*, there exists a regular element* $y \in Rx$ *such that* $x - y \in U$*.*

Proof. (1)⇒(2) See the proof of (2)⇒(4) in [1, Proposition 2.2].

(2)⇒(1) Let x and y be as in (2) and let $\alpha \in \mathrm{Hom}_R(M, R)$ be such that $(y\alpha)y = y$. Then by [8, Lemma 1.1], $M = Ry \oplus W$, where $W = \{w \in M \mid (w\alpha)y = 0\}$. Hence, $Rx = Ry \oplus (Rx \cap W)$. Let $\pi : M \to W$ be the projection map. Then $Rx \cap W = (Rx \cap W)\pi = (Rx)\pi = (R(x - y))\pi \leq U\pi \leq U$. □

Now we consider U-semiperfect modules. If M is U-semiperfect, then U respects every submodule of M. If M is projective, then the converse is true. The following theorem generalizes Theorem 36 in [11]. The proof of some of the implications is similar to that of [11, Theorem 36] but we give it for completeness.

Theorem 3.7. *Let U be a projection-invariant submodule of a module M, $\overline{M} = M/U$ and $S = \mathrm{End}_R(M)$. Consider the following conditions:*

(1) *$\overline{M}$ is semisimple and U is strongly lifting.*
(2) *U respects every submodule of M.*
(3) *U respects every countably generated submodule of M.*
(4) *M is U-semipotent and U respects $\oplus_{i=1}^{\infty}(M)\pi_i$ for any orthogonal idempotents $\pi_i \in S$.*
(5) *M is U-semipotent and there is no infinite orthogonal family of idempotents $\pi_i \in S$ such that $(M)\pi_i \not\subseteq U$.*
(6) *M is U-semipotent and $\overline{M}$ is semisimple.*

Then (1)⇒(2)⇒(3), (5)⇒(2)⇒(6). If M is self-projective, then (2)⇒(1). If M is finitely generated, then (3)⇒(4)⇒(5). If M is finitely generated and self-projective, then (6)⇒(1).

Proof. (1)⇒(2) Let N be a submodule of M. Since $\overline{M}$ is semisimple, there exists $B \leq M$ such that $U \leq B$ and $\overline{M} = \overline{N} \oplus \overline{B}$. By hypothesis, M has a decomposition $M = P \oplus Q$ such that $P \leq N$, $\overline{P} = \overline{N}$ and $\overline{Q} = \overline{B}$. Now we show $Q \cap N \leq U$. Since $N = N \cap (N+U) = N \cap (P+U) = P + (N \cap U)$, we have $Q \cap N = Q \cap (P + (N \cap U)) \leq Q \cap \big(P + (P \cap U) + (Q \cap U)\big) = Q \cap (P + (Q \cap U)) = (Q \cap U) + (Q \cap P) = Q \cap U \leq U$.

(2)⇒(1) By a proof similar to that of (2)⇒(1) in Theorem 3.2.

(2)⇒(3) It is clear.

(3)⇒(4) By the proof of Corollary 3.4.

(4)⇒(5) Assume that M is finitely generated. Let $\{\pi_i\}_{i=1}^{\infty}$ be a family of orthogonal idempotents in S such that $(M)\pi_i \not\subseteq U$. By (4), $\oplus_{i=1}^{\infty}(M)\pi_i = A \oplus B$, where A is a summand of M and $B \leq U$. Since A is finitely generated, A is contained in $\oplus_{i=1}^{n}(M)\pi_i$ for some n. Then $\oplus_{i=1}^{\infty}(M)\pi_i = \oplus_{i=1}^{n}(M)\pi_i + B$. Let $k > n$ and $(m)\pi_k = (m_1)\pi_1 + \cdots + (m_n)\pi_n + b$, where $m, m_i \in M$, $i = 1, \ldots, n$ and $b \in B$. Then $(m)\pi_k = (b)\pi_k$. Since U is projection-invariant, $(m)\pi_k \in U$. Hence, $(M)\pi_k \leq U$, a contradiction.

(5)⇒(2) Assume that (2) is not satisfied. By Lemma 3.1, there exists $N \leq M$ such that $N \cap (M)(1-\pi) \not\subseteq U$ for all $\pi^2 = \pi \in S$ with $(M)\pi \leq N$. Since $N \not\subseteq U$, there exists a summand A_1 of M such that $A_1 \leq N$ and $A_1 \not\subseteq U$. Let $M = A_1 \oplus B_1$ and let $\pi_1 : M \to A_1$ be the projection onto A_1 along B_1. Then $N = (M)\pi_1 \oplus (N \cap B_1)$ and $N_1 = N \cap B_1 \not\subseteq U$. Let A_2 be a summand of M such that $A_2 \leq N_1$ and $A_2 \not\subseteq U$. If $M = A_2 \oplus B_2$ and $\alpha : M \to A_2$ is the projection onto A_2 along B_2, then $\alpha\pi_1 = 0$. Let $\pi_2 = (1-\pi_1)\alpha$. Then $\{\pi_1, \pi_2\}$ is an orthogonal set such that $(M)\pi_i \leq N$ for $i = 1, 2$. Since $\alpha\pi_2 = \alpha$, $(M)\pi_2 \not\subseteq U$. Continuing the construction, suppose that $\pi_1, \ldots, \pi_n$ are orthogonal idempotents in S such that $(M)\pi_i \leq N$ and $(M)\pi_i \not\subseteq U$ for $i = 1, \ldots, n$. Let $\pi = \pi_1 + \cdots + \pi_n$. Then π is an idempotent, $(M)\pi \leq N$ and so $N \cap (M)(1-\pi) \not\subseteq U$. Let Y be a summand of M

such that $Y \leq N \cap (M)(1-\pi)$ and $Y \not\subseteq U$. If $M = Y \oplus Y'$ and $\beta : M \to Y$ is the projection onto Y along Y', then let $\pi_{n+1} = (1-\pi)\beta$. This implies that $\{\pi, \pi_{n+1}\}$ is an orthogonal set of idempotents in S such that $(M)\pi \not\subseteq U$ and $(M)\pi_{n+1} \not\subseteq U$ since $\beta\pi_{n+1} = \beta$. Hence, $\pi_1, \dots, \pi_n, \pi_{n+1}$ are orthogonal idempotents in S such that $(M)\pi_i \not\subseteq U$ for $i = 1, \dots, n+1$, and by induction, we have a contradiction.

(2)⇒(6) By the proof of Corollary 3.4, M is U-semipotent, and by the proof of (2)⇒(1)(i) in Theorem 3.2, $\overline{M}$ is semisimple.

(6)⇒(1) Assume that M is finitely generated and self-projective. Let $\overline{M} = \overline{A} \oplus \overline{B}$. We show that there exists a decomposition $M = P \oplus Q$ such that $P \leq A$, $\overline{P} = \overline{A}$ and $\overline{Q} = \overline{B}$.

If $A \subseteq U$, then $\overline{M} = \overline{B}$ and hence $M = 0 \oplus M$ is the desired decomposition.

If $A \not\subseteq U$, then there exists a summand Y_1 of M such that $Y_1 \leq A$ and $Y_1 \not\subseteq U$. Let W_1 be such that $M = Y_1 \oplus W_1$. Then $A = Y_1 \oplus (A \cap W_1)$.

If $A \cap W_1 \subseteq U$, then $(A+U)/U = (Y_1+U)/U$. Also, we have $M = A + B + U$ $= Y_1 + (A \cap W_1) + B + U = Y_1 + B + U$. Since M is self-projective, there exists a submodule $X \subseteq B + U$ such that $M = Y_1 \oplus X$ by [14, 41.14]. Since $\overline{M} = \overline{A} \oplus \overline{X} = \overline{A} \oplus \overline{B}$, we have $\overline{X} = \overline{B}$. Thus, we obtain $M = Y_1 \oplus X$, $Y_1 \leq A$, $\overline{Y_1} = \overline{A}$ and $\overline{X} = \overline{B}$.

If $A \cap W_1 \not\subseteq U$, then there exists a summand Y_2 of M such that $Y_2 \leq A \cap W_1$ and $Y_2 \not\subseteq U$. Let W_2 be such that $M = Y_2 \oplus W_2$. Then $W_1 = Y_2 \oplus (W_1 \cap W_2)$. So $M = Y_1 \oplus W_1 = Y_1 \oplus Y_2 \oplus (W_1 \cap W_2)$ implies that $A = Y_1 \oplus Y_2 \oplus (A \cap W_1 \cap W_2)$. This process produces a strictly ascending chain $\overline{Y_1} \subset \overline{Y_1} \oplus \overline{Y_2} \subset \cdots \subset \overline{M}$. Since $\overline{M}$ is Noetherian, this process must stop so that $A \cap W_1 \cap \ldots \cap W_n \subseteq U$ for some positive integer n. Hence, the proof is completed. □

Corollary 3.8. *Let M be projective and U a projection-invariant submodule of M. The following are equivalent*:

(1) *M is U-semiperfect.*

(2) *M/U is semisimple and U is strongly lifting.*

Now we characterize semiperfect modules. Recall that a projective module M with $\text{Rad}(M) \ll M$ is semiperfect if and only if $\text{Rad}(M)$ respects every submodule of M.

A ring R is called *clean* if every element of R is written as the sum of an idempotent and a unit in R. A module M is called *discrete* if M is lifting and if for any submodule A of M such that M/A is isomorphic to a summand of M, A is a summand of M (see [7]).

Theorem 3.9. *Let M be a projective module with* $\text{Rad}(M) \ll M$ *and let* $S = \text{End}_R(M)$. *Consider the following conditions*:

(1) *Every indecomposable summand of M is local and there is no infinite orthogonal family of idempotents $\pi_i \in S$ such that $(M)\pi_i \not\subseteq \text{Rad}(M)$.*

(2) $\text{End}_R(M)$ *is clean and there is no infinite orthogonal family of idempotents $\pi_i \in S$ such that $(M)\pi_i \not\subseteq \text{Rad}(M)$.*

(3) *M has the finite exchange property and there is no infinite orthogonal family of idempotents $\pi_i \in S$ such that $(M)\pi_i \not\subseteq \text{Rad}(M)$.*

(4) *M is semiperfect.*

Then (1)⇔(2)⇔(3)⇒(4). *In addition, if M is finitely generated, then* (4)⇒(1).

Proof. (1)⇒(2) Since there is no infinite orthogonal family of idempotents $\pi_i \in S$ such that $(M)\pi_i \not\subseteq \mathrm{Rad}(M)$, M is a finite direct sum of indecomposable submodules M_i such that $M_i \not\subseteq \mathrm{Rad}(M)$. Then each M_i is local. By [7, Corollary 4.54], M is discrete. By [4, Corollary 4.2], $\mathrm{End}_R(M)$ is clean.

(2)⇒(3) Since $\mathrm{End}_R(M)$ is clean, M has the finite exchange property by Proposition 1.8 and Theorem 2.1 in [9].

(3)⇒(1) By Propositions 2.8 and 2.10, every indecomposable summand of M is local.

(1)⇒(4) By Corollaries 4.54 and 4.43 in [7], M is semiperfect.

(4)⇒(1) Assume that M is finitely generated. By Theorem 3.7 and Proposition 2.10, (1) holds. □

A ring R is called *I-finite* if R has no infinite set of orthogonal idempotents. If $_RR$ has the finite exchange property, then R is called an *exchange ring*.

By Theorems 3.7 and 3.9, we have the following corollary. For the equivalences of (1)–(4), see [10], and the equivalences of (1), (5) and (6) are given in [5].

Corollary 3.10. *The following are equivalent for a ring R*:

(1) *R is semiperfect.*
(2) *R is semipotent and $R/J(R)$ is semisimple.*
(3) *R is semipotent and I-finite.*
(4) *Every primitive idempotent in R is local and R is I-finite.*
(5) *R is clean and I-finite.*
(6) *R is an exchange ring and I-finite.*

4 Every Projective Module is $\tau(\ )$-Semiperfect

A functor τ from R-Mod to itself is called a *preradical* on R-Mod if it satisfies the following properties:

(i) $\tau(M)$ is a submodule of M for every left R-module M.

(ii) If $f : M' \to M$ is a homomorphism in R-Mod, then $f(\tau(M')) \leq \tau(M)$ and $\tau(f)$ is the restriction of f to $\tau(M')$.

Note that any fully invariant submodule defines a preradical (see [13]).

In this section, we characterize rings R for which every projective R-module M is $\tau(M)$-semiperfect for some preradicals τ on R-Mod.

By definition, every projective module M is $\tau(M)$-semiperfect if and only if for every projective module M, $\tau(M)$ respects every submodule of M.

Now we consider the preradical Rad. It is well known that a ring R is left perfect if and only if every projective left R-module is semiperfect (see Theorem 4.41 and Corollary 4.43 in [7]). Also, if a projective module M is semiperfect, then M is $\mathrm{Rad}(M)$-semiperfect. The converse is true if $\mathrm{Rad}(M) \ll M$. The following theorem may be known but we do not have a reference.

Theorem 4.1. *Let R be a ring. Then the following are equivalent*:

(1) *Every projective left R-module M is* $\mathrm{Rad}(M)$*-semiperfect.*

(2) *R is left perfect.*

Proof. (2)$\Rightarrow$(1) It is clear.

(1)$\Rightarrow$(2) By the above remark, it is enough to prove that for any projective R-module P, $\mathrm{Rad}(P) \ll P$. Let Y be a submodule of P such that $P = \mathrm{Rad}(P)+Y$. By hypothesis, $P = A \oplus B$, where $A \leq Y$ and $B \cap Y \leq \mathrm{Rad}(P)$. Then $Y = A \oplus (B \cap Y)$ and so $P = \mathrm{Rad}(P) + A$. Since A is a summand of P, there exists a submodule X of $\mathrm{Rad}(P)$ such that $P = X \oplus A$ by [14, 41.14]. Then $\mathrm{Rad}(X) = X \cap \mathrm{Rad}(P) = X$. Since X is projective, $X = 0$. So $P = Y$. □

For the singular submodule $Z(M)$ of a module M, the following theorem is given in [15, Proposition 3.3].

Theorem 4.2. *Let R be a ring. Then the following are equivalent*:

(1) *Every projective left R-module M is $Z(M)$-semiperfect.*

(2) *R is left perfect and $Z(_RR) = J(R)$.*

There exists a left perfect ring R with $Z(_RR) \neq J(R)$, for example, the ring of 2×2 upper triangular matrices over a field. Hence, this ring does not satisfy (1) of Theorem 4.2.

Note also that in [12, Corollary 3.8], it is proved that R is a QF-ring (i.e., every projective R-module is injective) if and only if every left R-module M is $Z(M)$-semiperfect.

For the Goldie torsion submodule, we have the following result.

Theorem 4.3. *Let R be a ring. The following are equivalent*:

(1) *R is $Z_2(_RR)$-semiperfect.*

(2) *For any module $_RM$, $M = Z_2(M) \oplus X$, where $_RX$ is semisimple.*

(3) *Every nonsingular left R-module is injective.*

(4) *Every projective left R-module M is $Z_2(M)$-semiperfect.*

(5) *Every left R-module M is $Z_2(M)$-semiperfect.*

Proof. The equivalences of (1)–(4) are given by [11, Theorem 49].

(5)$\Rightarrow$(1) It is clear.

(1)$\Rightarrow$(5) Let M be an R-module and N a submodule of M. Then by (2), $N = Z_2(N) \oplus X$ for some semisimple submodule X. So X is nonsingular and projective. By (3), X is injective and hence a projective summand of M. It follows that N has a decomposition $N = A \oplus B$ such that $A \leq^{\oplus} M$, A is projective and $B \leq Z_2(M)$. Hence, M is $Z_2(M)$-semiperfect. □

Lemma 4.4. *If R is $Z_2(_RR)$-semiperfect and $Z_2(_RR)$ is injective, then every finitely generated projective left R-module is injective. In particular, R is left self-injective.*

Proof. Let P be a finitely generated projective left R-module. Then P is a summand of a finitely generated free R-module. Since $Z_2(_RR)$ is injective, we have that $Z_2(P)$ is injective. Hence, $P = Z_2(P) \oplus X$ for some submodule X. On the other hand, $P/Z_2(P)$ is injective by Theorem 4.3. Then X is injective and so P is injective. □

Theorem 4.5. *Let R be a ring. Then the following are equivalent*:

(1) *R is $Z({}_RR)$-semiperfect and $Z_2({}_RR)$ is injective.*
(2) *R is $Z_2({}_RR)$-semiperfect, $Z_2({}_RR)$ is injective and R is I-finite.*
(3) *R is semiperfect and left self-injective.*

Proof. (1)⇒(2) By [15, Theorem 2.5], R is $Z({}_RR)$-semiperfect if and only if R is semiperfect and $J(R) = Z({}_RR)$. Hence, (2) follows.

(2)⇒(3) By Lemma 4.4, R is left self-injective. By [4, Corollary 3.12], any left self-injective ring is clean. Hence, by Corollary 3.10, R is semiperfect.

(3)⇒(1) Since R is left self-injective, $J(R) = Z({}_RR)$. Then R is $Z({}_RR)$-semiperfect. Since $Z_2({}_RR)$ is closed in R, we have that $Z_2({}_RR)$ is injective. □

Theorem 4.6. *Let R be a ring. Then the following are equivalent*:
(1) *R is a QF-ring.*
(2) *R is $Z_2({}_RR)$-semiperfect, and for every projective left R-module P, $Z_2(P)$ is injective.*
(3) *R is $Z_2({}_RR)$-semiperfect, $Z_2({}_RR)$ is injective and R is left Noetherian.*

Proof. We first assume (1), and prove (2) and (3). Since R is QF, R is semiperfect and $J(R) = Z({}_RR) \leq Z_2({}_RR)$. Then R is $Z_2({}_RR)$-semiperfect. Let P be a projective left R-module. Then P is injective. Since $Z_2(P)$ is closed in P, we have $Z_2(P) \leq^{\oplus} P$. Hence, $Z_2(P)$ is injective.

(2)⇒(1) Let P be a projective left R-module. Then P is a summand of a free R-module $R^{(\Lambda)}$ for some index set Λ. Since $Z_2(R^{(\Lambda)})$ is injective by hypothesis, this implies that $Z_2(P)$ is injective. Hence, there exists a submodule X of P such that $P = Z_2(P) \oplus X$. Since $P/Z_2(P)$ is nonsingular, X is injective by Theorem 4.3. Hence, P is injective.

(3)⇒(1) Let P be a projective left R-module. Then P is a summand of a free R-module $R^{(\Lambda)}$ for some index set Λ. Since R is left Noetherian, $Z_2(R^{(\Lambda)}) = Z_2({}_RR)^{(\Lambda)}$ is injective. Hence, $Z_2(P)$ is injective. By the proof of (2)⇒(1), P is injective. □

Following [17], a submodule N of a module M is called *δ-small* in M, denoted by $N \ll_\delta M$, if $N + K \neq M$ for any submodule K of M with M/K singular. The sum of all δ-small submodules of M is a fully invariant submodule of M, and it is denoted by $\delta(M)$. Also, $\delta(M) = \cap\{N \leq M \mid M/N$ is singular simple$\}$. Clearly, $\mathrm{Rad}(M) \leq \delta(M)$. A pair (P, p) is called a *projective δ-cover* of the module M if P is projective and p is an epimorphism of P onto M with $\ker(p) \ll_\delta P$. A ring R is called *δ-semiperfect* if every simple R-module has a projective δ-cover. A ring R is called *left δ-perfect* if every left R-module has a projective δ-cover (see [17]). In the following theorem, we give a new characterization of a left δ-perfect ring.

Theorem 4.7. *Let R be a ring. Then the following are equivalent*:
(1) *Every projective left R-module M is $\delta(M)$-semiperfect.*
(2) *R is left δ-perfect.*

Proof. (2)⇒(1) Let R be a left δ-perfect ring. Then for any submodule N of a projective module P, P/N has a projective δ-cover. By [17, Lemma 2.4], P is $\delta(P)$-semiperfect.

(1)⇒(2) If every projective left R-module M is $\delta(M)$-semiperfect, then R is δ-semiperfect, and so idempotents lift modulo $\delta({}_RR)$ by [17, Theorem 3.6]. By [17, Theorem 3.8], it is enough to prove that $\overline{R} = R/\mathrm{Soc}({}_RR)$ is left perfect. Since $J(\overline{R}) = \delta({}_RR)/\mathrm{Soc}({}_RR)$, $\overline{R}/J(\overline{R})$ is semisimple.

We claim that for every projective left R-module P, $\delta(P) \ll_\delta P$. Let P be a projective R-module and $P = \delta(P) + Y$, where P/Y is singular. By hypothesis, $P = A \oplus B$ such that $A \leq Y$ and $B \cap Y \leq \delta(P)$. Then $Y = A \oplus (B \cap Y)$ and so $P = \delta(P) + Y = \delta(P) + A$. Since A is a summand of P, there exists a submodule $X \leq \delta(P)$ such that $P = X \oplus A$ by [14, 41.14]. Since $\delta(X) = X \cap \delta(P) = X$, X is semisimple projective by [12, Proposition 2.13]. Since P/Y is an epimorphic image of $P/A \cong X$, P/Y is projective. Since it is singular, we have $P = Y$. Hence, $\delta(P) \ll_\delta P$.

Now by the proof of [17, Theorem 3.7], it can be seen that $J(\overline{R})$ is left T-nilpotent. By [2, Theorem 28.4], $\overline{R}$ is left perfect. □

By [12, Corollary 3.10], R is semisimple if and only if every left R-module M is $\delta(M)$-semiperfect, if and only if every left R-module M is $\delta(M)$-semiregular.

For the socle, the following results are given in Corollaries 2.24 and 3.5 of [12]: Every projective left R-module M is $\mathrm{Soc}(M)$-semiperfect if and only if R is $\mathrm{Soc}({}_RR)$-semiperfect. R is a QF-ring with $J(R)^2 = 0$ if and only if $J(R) \leq Z({}_RR)$ and every left R-module M is $\mathrm{Soc}(M)$-semiperfect.

Finally, we note that for an ideal I of a ring R, R is I-semiperfect if and only if every finitely generated projective R-module M is IM-semiperfect by [12, Corollary 2.11].

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