Chain Conditions on Non-summands

Pınar AYDOĞDU, A. Çiğdem ÖZCAN
Hacettepe University Department of Mathematics
06800 Beytepe Ankara, Turkey
paydogdu@hacettepe.edu.tr,
ozcan@hacettepe.edu.tr
and
Patrick F. SMITH
University of Glasgow Department of Mathematics
G12 8QW Glasgow UK
pfs@maths.gla.ac.uk

Abstract

Let R be a ring. Modules satisfying ascending or descending chain conditions (resp. acc and dcc) on non-summand submodules belongs to some particular classes $\mathcal{X}$, such as the class of all R-modules, finitely generated, finite dimensional and cyclic modules, are considered. It is proved that a module M satisfies acc (resp. dcc) on non-summands if and only if M is semisimple or Noetherian (resp. Artinian). Over a right Noetherian ring R, a right R-module M satisfies acc on finitely generated non-summands if and only if M satisfies acc on non-summands; a right R-module M satisfies dcc on finitely generated non-summands if and only if M is locally Artinian. Moreover, if a ring R satisfies dcc on cyclic non-summand right ideals, then R is a semiregular ring such that the Jacobson radical J is left t-nilpotent.

Keywords: Noetherian, (locally) Artinian, regular, semisimple modules, semiregular rings.

1 Introduction

In this paper all rings have identity and all modules are unital right modules. Let R be a ring. By a *non–summand* of an R-module M we mean a submodule K which is not a direct summand of M. Among the non-summands of M we could mention proper essential submodules and non-zero superfluous submodules. This paper is concerned with the study of ascending and descending chain conditions (respectively, acc and dcc) on certain non-summands.

Recall that a submodule S of a module M is *superfluous* or *small* provided $M \neq S + L$ for every proper submodule L of M. The sum of all superfluous submodules of the module M is called the *radical* of M and will be denoted by

Rad(M). For a ring R, it will be denoted by J. The *socle* of the module M will be denoted by Soc(M). The module M will be called *finite dimensional* provided M does not contain an infinite direct sum of non-zero submodules. Recall that a non-zero module M is finite dimensional if and only if there exist a positive integer n and independent uniform submodules $U_i\,(1 \leq i \leq n)$ such that $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of M.

Let R be a ring. Goodearl [5, Proposition 3.6] proved that an R-module M satisfies acc on essential submodules if and only if $M/\mathrm{Soc}(M)$ is Noetherian. Goodearl's result has a dual due to Armendariz [2, Proposition 1.1] who proved that a module M satisfies dcc on essential submodules if and only if $M/\mathrm{Soc}(M)$ is Artinian. The results of Goodearl and Armendariz can also be found at [3, 5.15]. Varadarajan [9, Lemma 2.1] proved that a module M satisfies acc on superfluous submodules if and only if Rad(M) is Noetherian and Al-Khazzi and Smith [1, Theorem 5] proved that a module M satisfies dcc on superfluous submodules if and only if Rad(M) is Artinian. We shall give an example of a commutative von Neumann regular ring R such that R satisfies acc and dcc on essential ideals and on superfluous ideals but R satisfies neither acc nor dcc on non-summands (Example 3.5).

Let R be any ring and M an R-module which satisfies acc on finite dimensional non-summands. Let N be a non-zero finite dimensional submodule of M. Then $N = N_1 \oplus \cdots \oplus N_k$ is a direct sum of indecomposable submodules $N_i\,(1 \leq i \leq k)$, for some positive integer k. Let $1 \leq i \leq k$. If L is a proper non-zero submodule of N_i then L is not a direct summand of N_i and hence is not a direct summand of M. By hypothesis, N_i is Noetherian. Thus N is Noetherian. Thus every finite dimensional submodule of M is Noetherian. Such modules are studied in [6] and [7]. It is proved in [6, Theorem 2.15] that if R is a (commutative) Dedekind domain and M an R-module with torsion submodule T then every finite dimensional submodule of M is Noetherian if and only if T does not contain any non-zero injective submodule and every countably generated torsion-free submodule of M is projective.

2 Module classes

Let R be a ring and let N be any submodule of an R-module M. By Zorn's Lemma there exists a submodule K of M which is maximal in the collection of all submodules H of M such that $N \cap H = 0$. Such a submodule K is called a *complement* of N (in M) and it is well known (and easy to prove) that $N \oplus K$ is an essential submodule of M (see [3, 1.10] for more details).

Let R be a ring. By a *class* $\mathcal{X}$ of R-modules we mean a collection of R-modules which contains a zero module and which is closed under isomorphisms. If a module belongs to $\mathcal{X}$, then we say that it is an *$\mathcal{X}$–module*. By an *$\mathcal{X}$–submodule* (respectively, *$\mathcal{X}$–summand*, *$\mathcal{X}$–non-summand*) we mean an $\mathcal{X}$–module which is also a submodule (respectively, summand, non-summand) of M. This section is concerned with chain conditions on $\mathcal{X}$–non-summands of a module. In what follows R is an arbitrary ring and $\mathcal{X}$ any class of R-modules, unless stated otherwise. It is clear that every semisimple R-module satisfies both acc and dcc on $\mathcal{X}$–non-summands and that every Noetherian (respectively, Artinian) R-module satisfies acc (respectively, dcc) on $\mathcal{X}$–non-summands. Note the following elementary fact.

Proposition 2.1 *An R-module M satisfies acc (respectively, dcc) on $\mathcal{X}$–submodules if and only if M satisfies acc (respectively, dcc) both on $\mathcal{X}$–summands and on $\mathcal{X}$–non–summands.*

Proof Clear. $\square$

Recall that a module is Noetherian if and only if it satisfies acc on finitely generated submodules. Thus Proposition 2.1 shows that a module M is Noetherian if and only if M satisfies acc both on finitely generated summands and on finitely generated non-summands. Note too that every finite dimensional module satisfies acc and dcc on summands so that we have the following immediate corollary to Proposition 2.1.

Corollary 2.2 *A finite dimensional module satisfies acc (respectively, dcc) on $\mathcal{X}$–non-summands if and only if M satisfies acc (respectively, dcc) on $\mathcal{X}$–submodules.*

There is an analogue of Corollary 2.2 for modules with finite hollow dimension. For the definition of hollow dimension see [8].

Lemma 2.3 *Let M be a module which satisfies acc (respectively, dcc) on $\mathcal{X}$–non-summands. Then every submodule of M satisfies acc (respectively, dcc) on $\mathcal{X}$–non-summands.*

Proof Let N be any submodule of M. If K is an $\mathcal{X}$–non-summand of N then K is an $\mathcal{X}$–non-summand of M. The result follows. $\square$

Lemma 2.4 *Let $\mathcal{X}$ be a class of R-modules which is closed under extensions and let N be an $\mathcal{X}$–submodule of an R-module M. Suppose that M satisfies acc (respectively, dcc) on $\mathcal{X}$–non-summands. Then M/N satisfies acc (respectively, dcc) on $\mathcal{X}$–non-summands.*

Proof Let K be a submodule of M containing N such that K/N is an $\mathcal{X}$–non-summand of M/N. Then K is an $\mathcal{X}$–submodule of M because $\mathcal{X}$ is closed under extensions. Moreover, K is a non-summand of M. Thus K is an $\mathcal{X}$–non-summand of M. The result follows. $\square$

We shall see in the next section that if N is a submodule of an R-module M such that the modules N and M/N both satisfy acc (respectively, dcc) on non-summands then M need not satisfy acc (respectively, dcc) on non-summands. Indeed, more is true. We shall give an example of R-modules A_1 and A_2 which both satisfy acc on non-summands such that the module $A_1 \oplus A_2$ does not satisfy acc on non-summands and also an example of R-modules B_1 and B_2 which both satisfy dcc on non-summands but $B_1 \oplus B_2$ does not satisfy dcc on non-summands. However, in some situations the direct sum of modules with acc (respectively, dcc) on $\mathcal{X}$–non-summands also has the same property. For example, note the following result.

Lemma 2.5 *Let $\mathcal{X}$ be a class of R-modules such that, for each non-zero $\mathcal{X}$–module X, every non-zero submodule of X contains a non-zero $\mathcal{X}$-submodule. Let an R-module $M = M_1 \oplus M_2$ be a direct sum of submodules M_i $(i = 1,2)$ such that M_1 contains no non-zero $\mathcal{X}$–submodule and M_2 satisfies acc (respectively, dcc) on $\mathcal{X}$–non-summands. Then M satisfies acc (respectively, dcc) on $\mathcal{X}$–non-summands.*

Proof Let L be an $\mathcal{X}$–non-summand of M. By hypothesis, $L \cap M_1 = 0$. Let $\pi : M \to M_2$ denote the canonical projection. Then $\pi(L) \cong L$ so that $\pi(L)$ is an $\mathcal{X}$–submodule of M_2. Next note that $M_1 \oplus L = M_1 \oplus \pi(L)$ so that $\pi(L)$ is a non-summand of M and hence also of M_2, because L is a non-summand of M. Let $L_1 \subseteq L_2 \subseteq \dots$ be any ascending chain of $\mathcal{X}$–non-summands of M. Then $\pi(L_1) \subseteq \pi(L_2) \subseteq \dots$ is an ascending chain of $\mathcal{X}$–non-summands of M_2. Suppose that there exists a positive integer n such that $\pi(L_n) = \pi(L_{n+1}) = \dots$. Then $M_1 \oplus L_n = M_1 \oplus L_{n+1} = \dots$ and hence $L_n = L_{n+1} = \dots$. Thus if M_2 satisfies acc on $\mathcal{X}$–non-summands then so too does M. A similar result give the proof for descending chains. □

In particular, Lemma 2.5 applies to classes $\mathcal{X}$ which are closed under taking submodules. However, it applies more widely. For example, the class of finitely generated R-modules is not closed under taking submodules (if R is not right Noetherian) but satisfies the property of Lemma 2.5.

Lemma 2.6 *Let $\mathcal{X}$ be a class of modules closed under finite direct sums. Let M be a module which satisfies acc (respectively, dcc) on $\mathcal{X}$–non-summands. Let L and N be submodules of M such that $L \cap N = 0$. Then L satisfies acc (respectively, dcc) on $\mathcal{X}$–submodules or every $\mathcal{X}$-submodule of N is a direct summand of M and hence also of N.*

Proof Suppose that M satisfies acc on $\mathcal{X}$–non-summands. By Lemma 2.3 the module $L \oplus N$ also satisfies acc on $\mathcal{X}$–non-summands. Suppose there exists an $\mathcal{X}$–submodule K of N which is not a direct summand of M. Let $H_1 \subseteq H_2 \subseteq \dots$ be any ascending chain of $\mathcal{X}$-submodules of L. For each $i \geq 1$, $H_i \cap K = 0$ and $H_i \oplus K$ is an $\mathcal{X}$–non-summand of M (otherwise, K is a direct summand of M). Thus $H_1 \oplus K \subseteq H_2 \oplus K \subseteq \dots$ is an ascending chain of $\mathcal{X}$–non-summands of M and, by hypothesis, $H_n \oplus K = H_{n+1} \oplus K = \dots$ for some positive integer n. It follows that $H_n = H_{n+1} = \dots$. Thus L satisfies acc on $\mathcal{X}$–submodules. The proof for descending chains is similar. □

Theorem 2.7 *Let $\mathcal{X}$ be a class of R-modules which is closed under finite direct sums and under taking direct summands. Then an R-module M satisfies acc on $\mathcal{X}$–non-summands if and only if, for every $\mathcal{X}$–non-summand N of M, M satisfies acc on $\mathcal{X}$-submodules which contain N.*

Proof The sufficiency is clear. Conversely, suppose that M satisfies acc on $\mathcal{X}$–non-summands. Let L be any $\mathcal{X}$-submodule of M such that there exists a properly ascending chain $L = L_1 \subset L_2 \subset \ldots$ of $\mathcal{X}$-submodules of M. By hypothesis, there exists a positive integer n such that L_n is a direct summand of M. Let N be a submodule of M such that $M = L_n \oplus N$. For each $i \geq n$, $L_i = L_n \oplus (L_i \cap N)$. By hypothesis, $L_n \cap N \subset L_{n+1} \cap N \subset \ldots$ is a properly ascending chain of $\mathcal{X}$-submodules of N. By Lemma 2.6, L is a direct summand of M. The result follows. $\square$

Corollary 2.8 *Let $\mathcal{X}$ be a class of R-modules which is closed under extensions and also under taking homomorphic images. Then an R-module M satisfies acc on $\mathcal{X}$–non-summands if and only if M/N satisfies acc on $\mathcal{X}$–submodules for every $\mathcal{X}$–non-summand N of M.*

Proof Suppose first that M satisfies acc on $\mathcal{X}$–non-summands. Let N be any $\mathcal{X}$–non-summand of M. Let $\bar{L}_1 \subseteq \bar{L}_2 \subseteq \ldots$ be any ascending chain of $\mathcal{X}$–submodules of M/N. For each $i \geq 1$, $\bar{L}_i = L_i/N$ for some submodule L_i of M containing N. By hypothesis, L_i is an $\mathcal{X}$–submodule of M for all $i \geq 1$. By Theorem 2.7, $L_n = L_{n+1} = \ldots$ and hence $\bar{L}_n = \bar{L}_{n+1} = \ldots$ for some positive integer n. Thus M/N satisfies acc on $\mathcal{X}$–submodules.

Conversely, suppose that M/N satisfies acc on $\mathcal{X}$–submodules for each $\mathcal{X}$–non-summand N of M. Let L be any $\mathcal{X}$–non-summand of M and let $H_1 \subseteq H_2 \subseteq \ldots$ be any ascending chain of $\mathcal{X}$–submodules of M such that $L \subseteq H_1$. Then $H_1/L \subseteq H_2/L \subseteq \ldots$ is an ascending chain of $\mathcal{X}$–submodules of M/L. There exists a positive integer k such that $H_k/L = H_{k+1}/L = \ldots$ and hence $H_k = H_{k+1} = \ldots$. By Theorem 2.7, M satisfies acc on $\mathcal{X}$–non-summands. $\square$

The next result is a companion theorem to Theorem 2.7.

Theorem 2.9 *Let $\mathcal{X}$ be a class of R-modules which is closed under finite direct sums and under taking direct summands. Then an R-module M satisfies dcc on $\mathcal{X}$–non-summands if and only if every $\mathcal{X}$–non-summand of M satisfies dcc on $\mathcal{X}$–submodules.*

Proof The sufficiency is clear. Conversely, suppose that M satisfies dcc on $\mathcal{X}$–non-summands. Let N be any $\mathcal{X}$–non-summand of M. Suppose that N does not satisfy dcc on $\mathcal{X}$-submodules and let $N_1 \supset N_2 \supset \ldots$ be a properly descending chain of $\mathcal{X}$–submodules of N. By hypothesis, there exists a positive integer k such that N_k is a direct summand of M. Let L be a submodule of M such that $M = N_k \oplus L$. Now $N_k \supset N_{k+1} \supset \ldots$ is a properly descending chain of $\mathcal{X}$–submodules of N_k so that, by Lemma 2.6, every $\mathcal{X}$–submodule of L is a direct summand of L. However, $N = N_k \oplus (N \cap L)$ gives that $N \cap L$ is a direct summand of L and hence N is a direct summand of M, a contradiction. The result follows. $\square$

3 Special module classes

Let R be any ring. In this section we consider modules with ascending or descending chain conditions on $\mathcal{X}$–non-summands for some particular classes $\mathcal{X}$. We begin with the case $\mathcal{X} = Mod - R$. Lemma 2.6 has the following immediate consequence.

Lemma 3.1 *Let M be a module which satisfies acc (respectively, dcc) on non-summands and let L and N be submodules of M such that $L \cap N = 0$. Then L is Noetherian (respectively, Artinian) or N is semisimple.*

Now let R be a right Noetherian ring which is not semiprime Artinian and let U be any non-finitely generated semisimple R-module. Then the R-modules R and U both satisfy acc on non-summands but Lemma 3.1 shows that the module $R \oplus U$ does not satisfy acc on non-summands. In the same way if R is right Artinian (but not semiprime) then the R-modules R and U both satisfy dcc on non-summands but the module $R \oplus U$ does not satisfy dcc on non-summands by Lemma 3.1.

Theorem 3.2 *A module M satisfies acc on non-summands if and only if M is semisimple or Noetherian.*

Proof The necessity is clear. For the sufficiency assume that M satisfies acc on non-summands. Since M satisfies acc on essential submodules $M/\mathrm{Soc}(M)$ is Noetherian by the proof of Proposition 3.6 in [5] (see also [4, Lemma 2]). If $\mathrm{Soc}(M)$ is finitely generated, then M is Noetherian. Suppose that $\mathrm{Soc}(M)$ is not finitely generated. Then $\mathrm{Soc}(M) = S_1 \oplus S_2$ for some non-finitely generated submodules S_1, S_2. Because S_2 is not Noetherian, $M = S_1 \oplus L$ for some submodule L of M by Lemma 2.6. But S_1 not being Noetherian gives that L is semisimple by Lemma 3.1. Hence if $\mathrm{Soc}(M)$ is not finitely generated then M is semisimple. $\square$

Theorem 3.2 has the following analogue. The proof is rather similar but we give it for completeness.

Theorem 3.3 *A module M satisfies dcc on non-summands if and only if M is semisimple or Artinian.*

Proof The necessity is clear. For the sufficiency assume that M satisfies dcc on non-summands. Since M satisfies dcc on essential submodules, $M/Soc(M)$ is Artinian by [2, Proposition 1.1] (see also [3, 5.15]). If $\mathrm{Soc}(M)$ is finitely generated then M is Artinian. On the other hand, if $\mathrm{Soc}(M)$ is not finitely generated then M is semisimple by the proof of Theorem 3.2. $\square$

Theorems 3.2 and 3.3 have the following immediate consequence.

Corollary 3.4 *For any ring R, a finitely generated R-module M satisfies acc (respectively, dcc) on non-summands if and only if M is Noetherian (respectively, Artinian).*

In particular, for a ring R, if R_R satisfies dcc on non-summands, then R_R satisfies acc on non-summands. Now we give an example to show that there exist modules with acc (resp. dcc) on essential and on superfluous submodules but which do not have acc (resp. dcc) on non–summands.

Example 3.5 *Let K be any field and let S be the commutative ring which is the direct product of a countably infinite number of copies of K, that is $S = \prod_{i=1}^{\infty} K_i$, where $K_i = K$ for all $i \geq 1$. Let R denote the subring of S consisting of all elements $\{k_i\}$ such that $k_i \in K\,(i \in I)$ and $k_n = k_{n+1} = \ldots$ for some positive integer n. Then R is a commutative von Neumann regular ring which satisfies acc and dcc on essential ideals and on superfluous ideals but satisfies neither acc nor dcc on non-summand ideals.*

Proof It is clear that R is a von Neumann regular ring. Thus the Jacobson radical $J(R)$ of R is zero and trivially R satisfies acc and dcc on superfluous ideals. Moreover $\mathrm{Soc}(R_R)$ is the set of elements $\{k_i\}$ of R such that, for some positive integer n, $k_i = 0$ for all $i \geq n$. Thus $R/\mathrm{Soc}(R_R)$ is isomorphic to K and R satisfies acc and dcc on essential ideals. By Corollary 3.4 R does not satisfy acc on non-summand ideals and also does not satisfy dcc on non-summand ideals. $\square$

We now let $\mathcal{X}$ denote the class of finitely generated R-modules. Recall that a module M is called *regular* if every finitely generated submodule is a direct summand of M. Clearly regular modules satisfy both acc and dcc on finitely generated non-summands. Note that semisimple modules are clearly regular. We next prove a partial converse.

Lemma 3.6 *Let M be a regular module such that every cyclic submodule is finite dimensional. Then M is semisimple.*

Proof Let $M \neq 0$ and let $0 \neq m \in M$. Because mR is finite dimensional, there exist a positive integer n and non-zero indecomposable submodules $L_i\,(1 \leq i \leq n)$ of mR such that $mR = L_1 \oplus \cdots \oplus L_n$. Let $1 \leq i \leq n$ and let $0 \neq x \in L_i$. By hypothesis, xR is a direct summand of M and hence also of L_i so that $L_i = xR$. It follows that L_i is simple for all $1 \leq i \leq n$. Therefore mR is semisimple for all $m \in M$. It follows that M is semisimple. $\square$

Theorem 3.7 *The following statements are equivalent for a module M.*

(i) M satisfies acc on finitely generated non-summands.

(ii) M/L is Noetherian for every finitely generated non-summand L of M.

(iii) For every non-finitely generated submodule N of M, every finitely generated submodule of N is a direct summand of M.

Proof (i) $\Leftrightarrow$ (ii) By Corollary 2.8.

(ii) $\Rightarrow$ (iii) Let N be any non-finitely generated submodule of M. Let L be any finitely generated submodule of N. If L is not a direct summand of M then M/L is Noetherian by (ii) and hence N is finitely generated, a contradiction. Thus every finitely generated submodule L of N is a direct summand of M.

(iii) $\Rightarrow$ (ii) Let H be any finitely generated non-summand of M. By (iii), every submodule of M containing H is finitely generated and hence M/H is Noetherian. $\square$

Corollary 3.8 *Let R be a right Noetherian ring. Then the following statements are equivalent for a right R-module M.*

(i) M satisfies acc on non-summands.

(ii) M satisfies acc on finitely generated non-summands.

(iii) M is semisimple or Noetherian.

Proof By Theorems 3.2 and 3.7 and Lemma 3.6. $\square$

Corollary 3.9 *Let M be a module which satisfies acc on finitely generated non-summands. Then M is Noetherian or M contains an essential submodule N such that every finitely generated submodule of N is a direct summand of M.*

Proof If M is finite dimensional then M is Noetherian by Corollary 2.2. Suppose that M is not finite dimensional. Let a submodule $L = L_1 \oplus L_2 \oplus \dots$ be a direct sum of non-zero submodules L_i $(i \geq 1)$ of M. Let K be a complement of L in M and let $N = L \oplus K$. Then N is an essential submodule of M. Suppose that H is any finitely generated submodule of N. Note that $H \subseteq L_1 \oplus \cdots \oplus L_n \oplus K$ for some positive integer n and hence $L_{n+1} \oplus L_{n+2} \oplus \dots$ embeds in M/H. Thus M/H is not Noetherian so that H is a direct summand of M by Theorem 3.7. $\square$

Note that in Corollary 3.9, the essential submodule N of M is regular. Next we aim to give an example of a module which satisfies acc on finitely generated non-summands but which is neither Noetherian nor regular. First we state a well known lemma whose proof we shall include for completeness.

Lemma 3.10 *Let N be a finitely generated submodule of a module M such that every cyclic submodule of N is a direct summand of M. Then N is a direct summand of M.*

Proof There exist a positive integer k and elements $x_i \in N\,(1 \leq i \leq k)$ such that $N = x_1R + \cdots + x_kR$. If $k = 1$ then there is nothing to prove. Suppose that $k \geq 1$. There exists a submodule L of M such that $M = x_1R \oplus L$. Then $N = x_1R \oplus (N \cap L)$. If $\pi : N \to N \cap L$ is the canonical projection then $N \cap L$ is generated by the (k-1) elements $\pi(x_2), \ldots, \pi(x_k)$. By induction, $N \cap L$ is a direct summand of M and hence also of L and it follows that N is a direct summand of M. □

Example 3.11 *Let D be a commutative Noetherian domain with field of fractions $K \neq D$. Let T be the subring of the ring R of Example 3.5 consisting of all elements $\{k_i\}$ of R such that, for some positive integer n, $k_i \in D$ for all $i \geq n$. Then T is a commutative ring such that the T-module T satisfies acc on finitely generated non-summands but T is not Noetherian nor regular.*

Proof Note that $\mathrm{Soc}(T) = \mathrm{Soc}(R)$. Let L be any finitely generated non-summand of T_T. By Lemma 3.10 there exists $x \in L$ such that xT is not a direct summand of T. Then $x = \{k_i\}$ where $k_n = k_{n+1} = \ldots$ and k_n is a non-zero element of D, for some positive integer n. Then xT contains all elements of T of the form $\{h_i\}$ where $h_i = 0$ for all $1 \leq i \leq n-1$ and for all $i \geq m$ for some integer $m \geq n+1$. It follows that $\mathrm{Soc}(T)/(xT\cap$ $\mathrm{Soc}(R))$ is Noetherian. But $T/\mathrm{Soc}(T) \cong D$ so that $T/\mathrm{Soc}(T)$ is Noetherian. This implies that T/xT, and hence also T/L, is Noetherian. By Theorem 3.7 T_T satisfies acc on finitely generated non-summands. Clearly T is not Noetherian because $\mathrm{Soc}(T)$ is not finitely generated. Also if a is any non-zero non-unit element of D and s is the element $\{k_i\}$ of T with $k_i = a$ for all $i \geq 1$ then sT is not a direct summand of T so that T_T is not regular. □

We now consider modules which satisfy dcc on finitely generated non- summands. Recall that a module M satisfies dcc on finitely generated submodules if and only if M satisfies dcc on cyclic submodules, and in this case M is semiartinian. (see, for example, [10, 31.8]). A module M is *semiartinian* provided every non-zero homomorphic image of M has non-zero socle. As we have already remarked, regular modules satisfy dcc on finitely generated non-summands. Note the following result.

Theorem 3.12 *A module M is regular if and only if M satisfies dcc on finitely generated non-summands and every simple submodule is a direct summand of M.*

Proof The necessity is clear. Conversely, suppose that M satisfies the stated conditions. Suppose further that M is not regular. Then M contains a finitely generated non-summand. Let L be a minimal finitely generated non-summand of M. Note that $L \neq 0$. Let x be any non-zero element of L. Suppose that $L \neq xR$. It follows that xR is a direct summand of M so that $M = xR \oplus N$ for some submodule N of M. Now $L = xR \oplus (L \cap N)$ and hence $L \cap N$ is a finitely generated submodule of M. Clearly $L \neq L \cap N$ and this implies that $L \cap N$ is a direct summand of M and hence also of N, giving the contradiction that L is a direct summand of M. Thus $L = xR$ for every non-zero element x of L. It follows that L is a simple module, a contradiction. □

Theorem 2.9 gives the following result without further proof.

Theorem 3.13 *A module M satisfies dcc on finitely generated non-summands if and only if every finitely generated non-summand of M satisfies dcc on finitely generated submodules.*

Compare the next result with Corollary 3.9.

Proposition 3.14 *Let M be a module which satisfies dcc on finitely generated non-summands. Then M is regular or M has essential socle.*

Proof Suppose that M is not regular. By Theorem 3.12 M contains a simple submodule L which is not a direct summand. Let H be a complement of L in M so that $L \oplus H$ is an essential submodule of M. By Lemma 2.6, H satisfies dcc on finitely generated submodules and hence H has essential socle. It follows that M has essential socle. □

A module is called *locally Artinian* provided every finitely generated submodule is Artinian. Clearly locally Artinian modules satisfy dcc on finitely generated submodules. Compare the next result with Corollary 3.8.

Theorem 3.15 *Let R be a right Noetherian ring. Then a right R-module M satisfies dcc on finitely generated non-summands if and only if M is locally Artinian.*

Proof The sufficiency is clear. Conversely, suppose that M satisfies dcc on finitely generated non-summands. Let N be any finitely generated submodule of M. If $N \subseteq \mathrm{Soc}M$ then N is Artinian. Suppose that $N \nsubseteq \mathrm{Soc}M$. Then there exists a maximal submodule L of N such that $N \cap \mathrm{Soc}M \subseteq L$. Suppose that L is not Artinian and let $L_1 \supset L_2 \supset \ldots$ be any properly descending chain of submodules of L. Because N is finitely generated, so too is L_i for each $i \geq 1$ and hence L_k is a direct summand of M for some positive integer k. There exists a submodule H of M such that $M = L_k \oplus H$. By Lemma 2.6, every finitely generated submodule of H is a direct summand of H. Let $0 \neq h \in H$. Every submodule of hR is finitely generated, because R is right Noetherian, and hence is a direct summand of hR. Thus hR is semisimple for every non-zero $h \in H$. It follows that H is semisimple and thus $H \subseteq \mathrm{Soc}M$. It follows that $M = L_k + \mathrm{Soc}M$ and hence $N = L_k + (N \cap \mathrm{Soc}M) \subseteq L$, a contradiction. Thus L, and hence N, is Artinian. It follows that M is locally Artinian. □

The condition that R is right Noetherian cannot be removed in Theorem 3.15.

Example 3.16 *Let R be a commutative ring with unique maximal ideal J such that $J^2 = 0$ and J is not finitely generated. Then J is a non-finitely generated semisimple R-module. The R-module R is not Artinian and hence is not locally Artinian. Let A be a finitely generated non-summand of R. Then $A \neq R$ so that $A \subseteq J$. Thus A is Artinian because A is semisimple. Thus the R-module R satisfies dcc on finitely generated non-summands but is not locally Artinian and is not regular.*

Recall that a submodule N of a module M is *fully invariant* in M if $\varphi(N) \subseteq N$ for every endomorphism φ of M. If N is a fully invariant submodule of M then $N = (N \cap M_1) \oplus (N \cap M_2)$ and hence $M/N = ((M_1+N)/N) \oplus ((M_2+N)/N)$ for all submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$. Compare the following with Lemma 2.4.

Lemma 3.17 *If an R-module M satisfies dcc on finitely generated non-summands then so too does every factor module M/N, where N is a fully invariant submodule of M.*

Proof Let N be a nonzero fully invariant submodule of M. Let $\overline{M} = M/N$ and let $\overline{K} \subseteq \overline{L}$ be finitely generated submodules of $\overline{M}$. There exist positive integers s, t and elements x_i, y_j in $M\,(1 \leq i \leq s, 1 \leq j \leq t)$ such that $\overline{L} = (x_1 + N)R + \cdots + (x_s + N)R$ and $\overline{K} = (y_1 + N)R + \cdots + (y_t + N)R$. For each $1 \leq j \leq t$ there exist elements $r_{ij} \in R\,(1 \leq i \leq s)$ and $u_j \in N$ such that $y_j = \sum_{i=1}^{s} x_i r_{ij} + u_j$. Let $z_j = \sum_{i=1}^{s} x_i r_{ij}\,(1 \leq j \leq t)$. Then $\overline{K} = (z_1 + N)R + \cdots + (z_t + N)R$ and $z_1 R + \cdots + z_t R \subseteq x_1 R + \cdots + x_s R$.

Now let $\overline{L}_1 \supseteq \overline{L}_2 \supseteq \ldots$ be any descending chain of finitely generated non-summands of $\overline{M}$. By the above remarks we can suppose without loss of generality that $\overline{L}_i = (L_i + N)/N\,(i \geq 1)$ for some descending chain $L_1 \supseteq L_2 \supseteq \ldots$ of finitely generated submodules of M. Next note that the remarks preceding this lemma show that L_i is a non-summand of M for each $i \geq 1$. By hypothesis, there exists a positive integer k such that $L_k = L_{k+1} = \ldots$ and hence $\overline{L}_k = \overline{L}_{k+1} = \ldots$. □

Proposition 3.18 *If M satisfies dcc on finitely generated non-summands, then there exists a semiartinian submodule S of M such that M/S is regular.*

Proof Let $0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_\alpha \subseteq S_{\alpha+1} \subseteq \ldots$ be the socle series of M where for each ordinal $\alpha \geq 0$, $S_{\alpha+1}/S_\alpha = Soc(M/S_\alpha)$ and $S_\alpha = \bigcup_{0 \leq \beta < \alpha} S_\beta$ when α is a limit ordinal. Note that S_α is a fully invariant submodule of M for each ordinal $\alpha \geq 0$. Because M is a set there must exists an ordinal $\rho \geq 0$ such that $S_\rho = S_{\rho+1}$ and hence M/S_ρ has zero socle. Note that S_ρ is semiartinian.

Now suppose that M satisfies dcc on finitely generated non-summands. By Lemma 3.17, M/S_ρ satisfies dcc on finitely generated non-summands. Finally, by Proposition 3.14, M/S_ρ is regular. □

4 More special module classes

Theorem 4.1 *Let M be a module such that every non-zero submodule contains a uniform submodule. Then M_R satisfies acc on finite dimensional non-summands if and only if M is Noetherian or every uniform submodule of M is a direct summand.*

Proof ($\Leftarrow$) Suppose M is not Noetherian. Let L be a finite dimensional submodule of M. Suppose $L \neq 0$. Let $U \leq L$ and U be uniform. Then $M = U \oplus U'$ for some $U' \leq M$. Then $L = U \oplus (L \cap U')$ where $udim(L \cap U') < udim(L)$. By induction, $L \cap U'$ is a direct summand of U' so that L is a direct summand of M.

($\Rightarrow$) Suppose M satisfies acc on finite dimensional non-summands. Suppose M contains a (non-zero) finite dimensional non-summand. We shall show that M is Noetherian. Let H be a maximal finite dimensional non-summand of M. Then $M \neq H$ because H is a non-summand of M. Suppose that H is not essential in M. Then $H \cap L = 0$ for some non-zero submodule L. By hypothesis L contains a uniform submodule U. Then $H \oplus U$ is finite dimensional and hence, by the choice of H, a direct summand of M. This implies that H is a direct summand of M, a contradiction. Thus H is essential in M and M is finite dimensional. Every submodule of M is also finite dimensional. This gives that M satisfies acc on non-summands. By Theorem 3.2, M is Noetherian or semisimple and finite dimensional. Hence M is Noetherian. $\square$

A non-empty subset I of a ring R *acts t-nilpotently* on an R-module M if, for every sequence $a_1, a_2 \dots$ of elements in I and every $m \in M$, we have $ma_1a_2 \cdots a_{i-1}a_i = 0$ for some $i \in \mathbb{N}$ (depending on m) (see, for example, [10]). The set I is called *left t-nilpotent* if it acts t-nilpotently on R_R. Note that the ring R is left perfect if and only if its Jacobson radical J is left t-nilpotent and R/J is semisimple and this occurs if and only if R satisfies dcc on cyclic right ideals (see [10, 43.9]). By Proposition 2.1, we have the following.

Proposition 4.2 *A ring R is left perfect if and only if R satisfies dcc both on cyclic non-summand right ideals and cyclic summand right ideals.*

Example 3.5 shows that there exists a commutative ring satisfying dcc on finitely generated (cyclic) non-summands but which is not perfect.

Proposition 4.3 *Let M be an R-module satisfying dcc on cyclic non-summands. Then the Jacobson radical J of R acts t-nilpotently on M.*

Proof Let $a_1, a_2, \dots$ be a sequence of elements in J and $m \in M$. Consider the descending chain

$$ma_1R \supseteq ma_1a_2R \supseteq ma_1a_2a_3R \supseteq \cdots .$$

Assume that there exists a k such that $ma_1a_2 \dots a_kR$ is a direct summand of M. Since $ma_1a_2 \dots a_kR \subseteq mJ \subseteq RadM$, $ma_1a_2 \dots a_kR$ is superfluous and a direct summand of M. This implies that $ma_1a_2\dots a_k = 0$. Now we assume that for every k, $ma_1a_2\dots a_kR$ is not a direct summand of M. By hypothesis there exists i such that $ma_1a_2\dots a_iR = ma_1a_2\dots a_{i+1}R \subseteq ma_1a_2\dots a_iJ$. By Nakayama's Lemma, we have $ma_1a_2\dots a_i = 0$. □

Proposition 4.4 *Let M be an R-module satisfying dcc on cyclic non-summands and let $S = End_R(M)$. Suppose that the module M is a faithful right R-module and a finitely generated left S-module. Then J is left t-nilpotent.*

Proof Since M_R is faithful and ${}_SM$ is finitely generated it follows that R_R embeds in M_R^k for some positive integer k (see [10, 15.3 and 15.4]). By Proposition 4.3, J acts t-nilpotently on M_R^k. Hence J is left t-nilpotent. □

Corollary 4.5 *If R is a ring satisfying dcc on cyclic non-summand right ideals, then J is left t-nilpotent.*

An element a of a ring R is called *regular* if $a = aba$ for some $b \in R$. Note that if $a \in R$ and $a - aba$ is regular for some b in R then a is regular. For, there exists c in R such that $a - aba = (a - aba)c(a - aba)$ and thus $a = ada$ where $d = b + (1 - ba)c(1 - ab)$. The ring R is called *semiregular* if R/J is a von Neumann regular ring and idempotents can be lifted modulo J (see [10, 42.11]).

Theorem 4.6 *Let R be a ring with Jacobson radical J such that R satisfies dcc on cyclic non-summand right ideals. Then R is a semiregular ring such that J is left t-nilpotent.*

Proof By Corollary 4.5 J is left t-nilpotent. To prove that R is semiregular we can suppose without loss of generality that $J = 0$ (adapt the proof of Lemma 3.17). Let $0 \neq a \in R$. There exists a maximal right ideal M_1 of R such that $a \notin M_1$. Then $1 = ar + b$ for some $r \in R, b \in M_1$. It follows that $a_1 = a - ara = ba \in M_1$. Now suppose that $a_1 \neq 0$. By the same argument there exist a maximal right ideal M_2 and elements $r_1 \in R, b_1 \in M_2$ such that $a_2 = a_1 - a_1r_1a_1 = b_1a_1 \in M_1 \cap M_2$. If $a_2 \neq 0$ then repeat the argument. This gives a sequence of elements $a = a_0, a_1, a_2, \dots$ of R and a sequence of maximal right ideals $M_1, M_2, \dots$ of R such that, for each $i \geq 0$, $a_{i+1} = a_i - a_ir_ia_i$ for some $r_i \in R$ and $a_i \in M_1 \cap \dots \cap M_i$, $a_i \notin M_{i+1}$. Thus $a_0R \supset a_1R \supset \dots$. By hypothesis, there exists a positive integer n such that a_nR is a direct summand of R_R. There exists an idempotent e in R such that $a_nR = eR$. It can easily be checked that $a_n = a_nea_n$. Thus a_n is regular and by the remarks preceding this result so too is a. It follows that every element of R is regular and so R is von Neumann regular. □

The converse of Theorem 4.6 is false. Note the following fact.

Lemma 4.7 *Let R be a ring with Jacobson radical J. Let e be an idempotent in R such that $eR + J$ is a direct summand of R_R. Then $J \subseteq eR$.*

Proof Note that

$$eR+J=(eR+J)\cap[eR\oplus(1-e)R]=eR\oplus[(1-e)R\cap(eR+J)]=eR\oplus(1-e)J.$$

It follows that $(1-e)J=fR$ for some idempotent f in R. But $f\in J$ so that $f=0$. Hence $(1-e)J=0$ and $J\subseteq eR$. □

Example 4.8 *Let K be any field, let $K_i=K\,(i\geq 1)$ and let the ring $S=\prod_{i\geq 1}K_i$. Let U denote the simple ideal of S consisting of all elements in S of the form (k,0,0,...) with $k\in K$. Let R denote the trivial extension of S by U. Then R is a commutative semiregular ring with simple Jacobson radical J such that $J^2=0$ but R does not satisfy dcc on cyclic non-summand ideals.*

Proof The ring R consists of all elements of the form (s,u), with $s\in S$ and $u\in U$, with addition and multiplication defined by

$$(s,u)+(s',u')=(s+s',u+u')\;\text{ and }\;(s,u)(s',u')=(ss',su'+s'u)$$

for all $s,s'\in S,u,u'\in U$. It is well known that R is a commutative ring. Moreover, the set J of all elements of R of the form $(0,u)$ with $u\in U$ is an ideal of R such that $R/J\cong S$ so that R/J is von Neumann regular and $J^2=0$. Thus J is the Jacobson radical of R. Because U is a simple S-module, J is a simple R-module. Let $f_0=(1,0,0,...)$ and for each $i\geq 1$ let f_i denote the element $(0,0,...,0,1,1,1,...)$ of S with nth component 1 for all $n\geq i+1$. Let $e_i=(f_i,0)\in R$. Note that $U=Sf_0$ and for each $i\geq 1$, f_i is an idempotent of S such that $f_if_0=0$. Further note that $Sf_1\supset Sf_2\supset\ldots$. Let $i\geq 1$. Then $R(f_i,f_0)$ is a cyclic ideal of R such that $R(f_i,f_0)=Re_i+J$. Because e_i is an idempotent in R such that $Re_i=\{(sf_i,0)\,:\,s\in S\}$ so that $J\nsubseteq Re_i$ Lemma 4.7 gives that $R(f_i,f_0)$ is not a direct summand of R_R for each $i\geq 1$. Moreover $R(f_1,f_0)\supset R(f_2,f_0)\supset\ldots$. Thus R does not satisfy dcc on cyclic non-summand ideals. □

Acknowledgement: This paper arose out of discussions during a visit of the third author to Hacettepe University in November 2007. The third author would like to thank TUBITAK for their financial support and Hacettepe University for its hospitality. The first author is also grateful to TUBITAK for their grant.

References

[1] I. Al-Khazzi, P.F. Smith, Modules with chain conditions on superfluous submodule, Comm. Algebra 19 (1991) 2331-2351.

[2] E.P. Armendariz, Rings with dcc on essential left ideals, Comm. Algebra 8 (1980) 299-308.

[3] N.V. Dung, D.V. Huynh, P.F. Smith, R., Wisbauer, *Extending Modules*, Pitman Research Notes in Mathematics 313, Longman, Harlow, 1994.

[4] N. V. Dung, D.V. Huynh and R.Wisbauer, Quasi-injective modules with ACC or DCC on essential submodules, Arch. Math. 53 (1989), 252-255.

[5] K.R. Goodearl, Singular torsion and the splitting properties, Amer. Math. Soc. Memoirs 124 (1972).

[6] E. Sanchez Campos, P.F. Smith, Certain chain conditions in modules over Dedekind domains and related rings, Modules and Comodules, Trends in Mathematics, 125-141, Birkhäuser Verlag Basel/Switzerland, 2008.

[7] E. Sanchez Campos, P.F. Smith, Modules satisfying the ascending chain condition on submodules with bounded uniform dimension, Rings, modules and representations, 57–71, Contemp. Math., 480, Amer. Math. Soc., Providence, RI, 2009.

[8] T. Takeuchi, On cofinite-dimensional modules, Hokkaido Math. J. 5 (1976) 1-43.

[9] K. Varadarajan, Modules with supplements, Pacific J. Math. 82 (1979) 559-564.

[10] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, Reading, 1991.