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Strongly J-Clean Rings with Involutions

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Abstract

A ring with an involution $*$ is called strongly J-$*$-clean if every element is a sum of a projection and an element of the Jacobson radical that commute. In this article, we prove several results characterizing this class of rings. It is shown that a $*$-ring R is strongly J-$*$-clean, if and only if R is uniquely clean and strongly $*$-clean, if and only if R is uniquely strongly $*$-clean, that is, for any $a \in R$, there exists a unique projection $e \in R$ such that $a - e$ is invertible and $ae = ea$.

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1 Introduction

A ring R is *strongly clean* provided that every element is a sum of an idempotent and a unit that commute with each other ([9]). A ring R is *strongly J-clean* provided that every element is a sum of an idempotent and an element in its Jacobson radical that commute ([2]). A ring R is *uniquely strongly clean* provided that for any $a \in R$, there exists a unique idempotent $e \in R$ such that $a - e$ is invertible and $ae = ea$ ([6]). Note that $\{$uniquely strongly clean rings$\} \subsetneqq \{$ strongly J-clean rings$\} \subsetneqq \{$ strongly clean rings$\}$. The above mentioned classes of rings have been getting much attention. Their relations with other classes of rings, such as unit-regular rings, strongly π-regular rings and others, have been studied in the past.

An *involution* of a ring R is an anti-automorphism whose square is the identity map. A ring R with involution $*$ is called a *$*$-ring.* All C^*-algebras and Rickart $*$-rings are $*$-rings. Furthermore, every commutative ring can be regarded as a $*$-ring with the identity involution $*$. An element e in a $*$-ring R is called a *projection* if $e^2 = e = e^*$. A $*$-ring R is *strongly $*$-clean* if each of its elements is a sum of a unit and a projection that commute with each other (see [8, 11]).

The main purpose of this paper is to explore strong J-cleanness for $*$-rings. We call a $*$-ring R *strongly J-$*$-clean* provided that every element is a sum of a projection and an element in its Jacobson radical that commute with each other. We show several results characterizing this class of rings. It is proved that a $*$-ring R is strongly J-$*$-clean, if and only if R is uniquely clean and

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strongly $*$-clean (Theorem 3.2). We say that a $*$-ring R is *uniquely strongly $*$-clean* provided that for any $a \in R$, there exists a unique projection $e \in R$ such that $a - e$ is invertible and $ae = ea$. We show that strongly J-$*$-clean rings and uniquely strongly $*$-clean rings are equivalent notions unlike their non-involutive counterparts (Theorem 3.2). Also it is proved that R is strongly J-$*$-clean if and only if R is strongly $*$-clean and the Jacobson radical $J(R) = \{x \in R \mid 1 - x$ is invertible $\}$ (Theorem 3.4). As consequences, various properties of strongly J-$*$-clean rings are derived.

Throughout, all rings are associative with identity. We use $U(R)$ to denote the set of all invertible elements in the ring R and $J(R)$ always stands for the Jacobson radical of R.

2 Strongly J-$*$-Clean Rings

The aim of this section is to characterize strongly J-$*$-clean rings by means of strong $*$-cleanness. A $*$-ring R is J-$*$-clean in the case that every element is a sum of an idempotent and an element in this Jacobson radical. We begin with the following result.

Proposition 2.1 *Let R be a $*$-ring. Then the following are equivalent:*

(1) *R is strongly J-$*$-clean.*

(2) *R is strongly J-clean and strongly $*$-clean.*

(3) *R is abelian and R is J-$*$-clean.*

Proof (1) $\Rightarrow$ (2) Clearly, R is strongly J-clean. For any $a \in R$, there exists a projection $e \in R$ such that $u := a - e \in J(R)$ and $ae = ea$. Then $a = (1 - e) + (2e - 1 + u)$. Obviously, $1 - e \in R$ is a projection. As $(2e - 1)^2 = 1$, we see that $2e - 1 + u \in U(R)$. Thus, R is strongly $*$-clean.

(2) $\Rightarrow$ (3) Since R is strongly $*$-clean, it follows from [8, Theorem 2.2] that R is abelian and every idempotent is a projection. Therefore R is J-$*$-clean.

(3) $\Rightarrow$ (1) is obvious. $\square$

Example 2.2 (1) Let the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Define $\sigma : R \to R$ by $\sigma(x, y) = (y, x)$. Consider the ring $T_2(R, \sigma) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R \right\}$ with the following operations:

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ 0 & a+c \end{pmatrix}, \quad \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} ac & ad + b\sigma(c) \\ 0 & ac \end{pmatrix}.$$

Then $J(T_2(R, \sigma)) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in R \right\}$ is nilpotent, and $T_2(R, \sigma)/J(T_2(R, \sigma)) \cong R$ is Boolean. Define $* : R \to R$ by $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^* = \begin{pmatrix} a & \sigma(b) \\ 0 & a \end{pmatrix}$. As $\sigma^2 = 1_R$, we easily check that $*$ is an involution of $T_2(R, \sigma)$. For any $a, b \in R$, $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ is a projection. Furthermore, $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in J(T_2(R, \sigma))$. Therefore, $T_2(R, \sigma)$ is a J-$*$-clean ring. But it is not strongly J-$*$-clean, as $\begin{pmatrix} (0,1) & (0,0) \\ (0,0) & (0,1) \end{pmatrix}$ is not central.

(2) Let $R = \mathbb{Z}_{(3)}$ be the localization, and $* = 1_R$, the identical automorphism of R. Since R is local, it is strongly $*$-clean. Since $R/J(R)$ is not Boolean, for example $(\frac{2}{1})^2 - \frac{2}{1} \notin J(R) = 3R$, R is not strongly J-$*$-clean (see Proposition 2.6).

(3) Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. By [9, Example 2] R is strongly clean, and it is clear that $R/J(R)$ is Boolean. Then R is strongly J-clean by [2, Theorem 2.3]. But R cannot be strongly J-$*$-clean for any involution $*$ on R because it is not abelian.

Let R be a $*$-ring, and let $C(R) = \{a \in R \mid ax = xa \text{ for any } x \in R\}$. Then $C(R)$ is a subring of the ring R. It is easy to check that $* : C(R) \to C(R)$ is also an anti-automorphism. Thus, $C(R)$ is a $*$-ring.

Corollary 2.3 *Let R be a $*$-ring. Then R is strongly J-$*$-clean if and only if*

(1) *$C(R)$ is J-$*$-clean;*

(2) *$R = C(R) + J(R)$.*

Proof Suppose that R is strongly J-$*$-clean. Then R is strongly $*$-clean by Proposition 2.1. Thus, every projection in R is central from [8, Theorem 2.2]. For any $a \in C(R)$, there exist a projection $e \in R$ and an element $u \in J(R)$ such that $a = e + u$. Hence, $e \in C(R)$ is a projection. This implies that $u \in J(R) \cap C(R) \subseteq J\big(C(R)\big)$. As a result, $C(R)$ is J-$*$-clean. On the other hand, it follows from Proposition 2.1 that R is abelian strongly J-clean. In view of [4, Corollary 16.4.16], R uniquely clean. According to [10, Proposition 25], $R = C(R) + J(R)$.

Conversely, assume that (1) and (2) hold. For any idempotent $e \in R$, there exist some $b \in C(R)$ and $c \in J(R)$ such that $e = b + c$. As $C(R)$ is J-$*$-clean, we can find a projection $f \in C(R)$ and a $w \in J\big(C(R)\big)$ such that $b = f + w$. Hence, $e = f + (w + c)$, and so $e - f = w + c$. Obviously, $(e - f)\big(1 - (e - f)^2\big) = 0$. On the other hand, $(e - f)^2 \in J\big(C(R)\big) + J(R)$, and so $1 - (e - f)^2 \in U(R)$. Therefore $e = f$, i.e., every idempotent in R is a projection. For any $a \in R$, write $a = s + t, s \in C(R), t \in J(R)$. We have a projection $g \in R$ such that $s = g + v$ with $v \in J\big(C(R)\big)$. This shows that $a = g + (t + v)$, and so R is J-$*$-clean. According to [8, Lemma 2.1] and Proposition 2.1, we complete the proof. □

According to [3, Theorem 2.1], a ring R is uniquely clean if and only if R is an abelian exchange ring and $R/M \cong \mathbb{Z}_2$ for all maximal ideals M of R. For strongly J-$*$-clean rings, we derive the following result.

Proposition 2.4 *Let R be a $*$-ring. Then R is strongly J-$*$-clean if and only if*

(1) *R is strongly $*$-clean ring;*

(2) *$R/M \cong \mathbb{Z}_2$ for all maximal ideals M of R.*

Proof Suppose that R is strongly J-$*$-clean. Then R is an abelian strongly J-clean ring by Proposition 2.1. Thus, R is uniquely clean from [4, Corollary 16.4.16]. Therefore $R/M \cong \mathbb{Z}_2$ for all maximal ideals M of R by [3, Theorem 2.1].

Conversely, assume that (1) and (2) hold. Then every idempotent in R is a projection. Further, R is an abelian exchange ring. According to [3, Theorem 2.1], R is uniquely clean, and so R is strongly J-clean. Accordingly, R is strongly J-$*$-clean by Proposition 2.1. □

Corollary 2.5 *Let R be a $*$-ring. Then R is strongly J-$*$-clean if and only if*

(1) *R is strongly $*$-clean ring;*

(2) *For all maximal ideal M of R, 1 is not the sum of two units in R/M.*

Proof One direction is obvious by the previous result. Conversely, assume that (1) and (2) hold. Let M be a maximal ideal of R. Clearly, R/M is an abelian exchange ring; hence, R/M is an exchange ring with artinian primitive factors. As R/M is simple, $J(R/M) = 0$. Let $f \in R/M$ be an idempotent. Then $(R/M)f(R/M) = 0$ or R/M, and so $f = \overline{0}$ or $\overline{1}$. This means that R/M is indecomposable. Therefore R/M is simple artinian. Thus, $R/M \cong M_n(D)$, where D is a division ring. As every idempotent in R/M is central, we deduce that $n = 1$. This implies that $R/M \cong D$. If $|D| \geq 3$, then we can find a set $\{0, 1, x\} \subseteq D$, where $x \neq 0, 1$. This shows that $1 - x \in U(R/M)$, a contradiction. Hence, $R/M \cong \mathbb{Z}_2$. According to Proposition 2.4, we complete the proof. □

Proposition 2.6 *Let R be a $*$-ring. Then the following are equivalent:*

(1) *R is strongly J-$*$-clean.*

(2) *$R/J(R)$ is Boolean and R is strongly $*$-clean.*

Proof (1) ⇒ (2) In view of Proposition 2.1, R is strongly $*$-clean and R is strongly J-clean. By virtue of [4, Proposition 16.4.15], $R/J(R)$ is Boolean.

(2) ⇒ (1) As R is strongly $*$-clean, it is strongly clean. According to [4, Proposition 16.4.15], R is strongly J-clean. In light of Proposition 2.1, R is a strongly J-$*$-clean ring. □

Corollary 2.7 *Let R be a $*$-ring. Then R is strongly J-$*$-clean if and only if*

(1) *R is strongly $*$-clean;*

(2) *Every nonzero idempotent in R is not the sum of two units.*

Proof Suppose that R is a strongly J-$*$-clean ring. By virtue of Proposition 2.6, R is a strongly $*$-clean ring, and $R/J(R)$ is Boolean. Let $0 \neq e \in R$ be an idempotent. If $e = u + v$ for some $u, v \in U(R)$. Then $\overline{e} = \overline{u} + \overline{v}$ in $R/J(R)$. As $\overline{u} = \overline{v} = \overline{1}$, we see that $2 - e \in J(R)$. In light of [2, Proposition 3.1], $2 \in J(R)$; hence, $e \in J(R)$. This implies that $e = 0$, a contradiction. Therefore every nonzero idempotent in R is not the sum of two units.

Conversely, assume that (1) and (2) hold. Then R is an exchange ring. According to [7, Theorem 13], $R/J(R)$ is Boolean. Therefore we complete the proof by Proposition 2.6. □

Recall that a ring R is *local* if R has only one maximal right ideal. As is well known, a ring R is local if and only if $a + b = 1$ in R implies that either a or b is invertible.

Corollary 2.8 *Let R be a local $*$-ring. Then the following are equivalent:*

(1) *R is strongly J-$*$-clean.*

(2) *R is strongly J-clean.*

(3) *R is uniquely clean.*

(4) *$R/J(R) \cong \mathbb{Z}_2$.*

(5) *1 is not the sum of two units in R.*

Proof (1) $\Rightarrow$ (2) is trivial.

(2), (3) and (4) are equivalent by [2, Lemma 4.2].

(4) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (1) Since R is a local $*$-ring, R is strongly $*$-clean. Therefore the result follows from Corollary 2.7. □

A $*$-ring R is called *$*$-regular* if R is (von Neumann) regular and the involution is proper, equivalently for every x in R there exists a projection p such that $xR = pR$ (see [1]).

Corollary 2.9 *Let R be a $*$-regular ring. Then R is strongly J-$*$-clean if and only if R is Boolean.*

Proof Suppose that R is Boolean. Then $R/J(R)$ is Boolean. For any idempotent $e \in R$, there exists a projection p such that $eR = pR$. As R has stable range one, we have a unit $u \in R$ such that $e = pu$. Clearly, $u = 1$, and so $e = p$. This implies that R is strongly $*$-clean. According to Proposition 2.6, R is strongly J-$*$-clean.

Conversely, assume that R is strongly J-$*$-clean. Then $R/J(R)$ is Boolean by Proposition 2.6. But R is regular, and so $J(R) = 0$. Therefore R is Boolean. □

3 Uniqueness of Projections

We start this section by studying the relationship between strong J-$*$-cleanness and uniqueness which will be repeatedly used in the sequel.

Lemma 3.1 *Let R be a $*$-ring. Then the following are equivalent:*

(1) *R is strongly J-$*$-clean.*

(2) *R is uniquely clean and for any $a \in R$, $a - a^* \in J(R)$.*

(3) *R is uniquely clean and for any $a \in R$, $a + a^* \in J(R)$.*

Proof (1) $\Rightarrow$ (2) In view of Proposition 2.1, R is an abelian strongly J-clean ring, whence it is uniquely clean [3, Corollary 3.4]. For any $a \in R$, there exist a projection $e \in R$ and an element $u \in J(R)$ such that $a = e + u, ae = ea$. Thus, $a^* = e^* + u^*$. As $\big(J(R)\big)^* \subseteq J(R)$, we see that $a - a^* = (e - e^*) + (u - u^*) \in J(R)$.

(2) $\Rightarrow$ (3) Since R is uniquely clean, it follows from [10, Lemma 18], $2 \in J(R)$. Therefore $a + a^* = (a - a^*) + 2a^* \in J(R)$, as desired.

(3) $\Rightarrow$ (1) Since R is uniquely clean, R is abelian strongly J-clean (see [4, Corollary 16.4.16]). For any idempotent $e \in R$, $e + e^* \in J(R)$. Thus, $(e - e^*)(e + e^*)^2 = e - e^*$; hence, $(e - e^*)\big(1 - (e + e^*)^2\big) = 0$. This implies that $e = e^*$, i.e., every idempotent is a projection. Hence, R is strongly J-$*$-clean. $\square$

By the referee's suggestion, we say that a $*$-ring is *uniquely J-$*$-clean* provided that for any $a \in R$, there exists a unique projection $e \in R$ such that $a - e \in J(R)$. We come now to prove the following main result.

Theorem 3.2 *Let R be a $*$-ring. Then the following are equivalent:*

(1) *R is strongly J-$*$-clean.*

(2) *R is uniquely clean and R is strongly $*$-clean.*

(3) *R is uniquely strongly $*$-clean.*

(4) *R is uniquely J-$*$-clean.*

(5) *For any $a \in R$, there exists a unique idempotent $e \in R$ such that $a - e \in U(R), ae = ea, ae^* = e^*a$ and $e - e^* \in J(R)$.*

Proof (1) $\Rightarrow$ (2) is clear from Lemma 3.1 and Proposition 2.1.

(2) $\Rightarrow$ (3) For any $a \in R$, it follows from [10, Theorem 20] that there exists a unique idempotent $e \in R$ such that $a - e \in J(R)$. Since R is strongly $*$-clean, e is a central projection. Therefore there exists a unique projection $e \in R$ such that $a - e \in U(R)$ and $ae = ea$, as required.

(3) $\Rightarrow$ (1) Clearly, R is strongly $*$-clean. In light of [8, Theorem 2.2], R is abelian and every idempotent in R is a projection. Thus, R is uniquely clean. Let $a \in R$. According to [10, Theorem 20], there exists an idempotent $e \in R$ such that $a - e \in J(R)$. Thus, we have a projection $e \in R$ such that $a - e \in J(R)$ and $ae = ea$. Therefore R is strongly J-$*$-clean.

(1) $\Rightarrow$ (4) is trivial by Lemma 3.1.

(4) $\Rightarrow$ (1) For any idempotent e in R, there exists a projection $g \in R$ such that $e - g \in J(R)$. Thus, $e^* - g \in J(R)$. Therefore $e - e^* = (e - g) - (e^* - g) \in J(R)$. Clearly, there exists a projection $h \in R$ such that $2 = h + w$ where $w \in J(R)$. Thus, $1 - h = -1 + w \in U(R)$. Hence $1 - h = 1$, and so $h = 0$. This implies that $2 \in J(R)$. As a result, we deduce that $e + e^* = (e - e^*) + 2e^* \in J(R)$.

Set $z = 1 + (e - e^*)^*(e - e^*)$. Write $t = z^{-1}$. Since $z^* = z$, $t^* = t$. Also $e^*z = e^*ee^* = ze^*$, and so $e^*t = te^*$, and $et = te$. Set $f = e^*et = te^*e$. Then

$$f^* = f, f^2 = e^*ete^*et = e^*ee^*(tet) = e^*ztet = e^*et = f, fe = f \text{ and } ef = ee^*et = ezt = e.$$

Now $e = f + (e - f)$ and $e - f = e - e^*et = ee^*et - e^*et = (e - e^*)e^*et \in J(R)$. Here $f = f^* = f^2$. In addition, $f = e^*e[1 + (e^* - e)(e - e^*)]^{-1}$.

Set $z' = 1 + (e^* - e)^*(e^* - e)$. Write $t' = (z')^{-1}$. Since $(z')^* = z'$, $(t')^* = t'$. Also $ez' = ee^*e = z'e$. Set $f' = ee^*t' = t'ee^*$. As in the preceding proof, we see that $f' = (f')^2 = (f')^*$ and $ef' = f'$, $f'e = e$. In addition,

$$e - f' = f'e - f' = t'ee^*(e - e^*) \in J(R),$$

where $f' = [1 + (e - e^*)(e^* - e)]^{-1}ee^*$.

Thus we get $e = f + (e - f) = f' + (e - f')$ with $e - f, e - f' \in J(R)$, f and f' are projections. By the uniqueness, we get

$$e^*e\big[1 + (e^* - e)(e - e^*)\big]^{-1} = \big[1 + (e - e^*)(e^* - e)\big]^{-1}ee^*.$$

This shows that

$$[1 + (e - e^*)(e^* - e)]e^*e = ee^*[1 + (e^* - e)(e - e^*)].$$

Obviously, $(e - e^*)(e^* - e)e^*e = -e^*e + e^*ee^*e$ and $ee^*(e^* - e)(e - e^*) = -ee^* + ee^*ee^*$. Consequently, $e^*ee^*e = ee^*ee^*$. One easily checks that

$$(e - e^*)^3 - (e - e^*) = -ee^*e + e^*ee^*;$$
$$\big[(e - e^*)^3 - (e - e^*)\big](e + e^*) = (e - e^*)^3 - (e - e^*).$$

Thus $(e - e^*)((e - e^*)^2 - 1)((e + e^*) - 1) = 0$. As $e - e^*$, $e + e^* \in J(R)$, we see that $(e - e^*)^2 - 1, (e + e^*) - 1 \in U(R)$, and so $e = e^*$. Therefore every idempotent is a projection, and so R is abelian by [8, Lemma 2.1], hence the result follows.

(2) + (3) $\Rightarrow$ (5) By (3), there exists an idempotent $f \in R$ such that $a - f \in U(R), af = fa, af^* = f^*a$ and $f - f^* = 0 \in J(R)$. The uniqueness of such idempotent immediately follows from the uniquely cleanness of R.

(5) $\Rightarrow$ (3) Let $a \in R$. Then there exists an idempotent $e \in R$ such that $u := a - e \in U(R), ae = ea, ae^* = e^*a$ and $e - e^* \in J(R)$. Let $p = 1 + (e^* - e)^*(e^* - e)$. As $ae = ea, ae^* = e^*a$, we see that $ap = pa$. Clearly, $p \in U(R)$. Write $q = p^{-1}$. Then $p^* = p$, this implies $q^* = q$. Further,

$$ep = e(1 - e - e^* + ee^* + e^*e) = ee^*e = (1 - e - e^* + ee^* + e^*e)e = pe.$$

Thus, we see that $eq = qe$ and $e^*q = qe^*$. Set $g = ee^*q$. Then

$$g^2 = ee^*qee^*q = qee^*ee^*q = qpee^*q = ee^*q = g.$$

In addition, $g^* = q^*ee^* = ee^*q = g$, i.e., $g \in R$ is a projection. As $aq = qa$, we see that $ag = ga$. One easy checks that $eg = g$ and $ge = ee^*qe = ee^*eq = epq = e$. This implies that

$$e - g = e - ee^*p^{-1} = e\big(e(e^* - e)^* - 1\big)(e^* - e)p^{-1} \in J(R),$$

and so $e - g + u \in U(R)$. Thus we have a projection $g \in R$ such that $a - g \in U(R)$ and $ga = ag$. The uniqueness in (5) completes the proof. □

A ring element is said to be *clean* if it is a sum of a unit and an idempotent. Other related concepts can be defined for ring elements analogously. With such definitions in mind, let us note that the conditions of Theorem 3.2 are not equivalent when referring to a single element.

The following example displays a ring with an element that is uniquely strongly $*$-clean and not uniquely clean. Thus, conditions (2) and (3) are not equivalent when applied to a single element.

Let $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{Z})$, where $M_2(\mathbb{Z})$ is a $*$-ring with the involution $* : A \mapsto A^T$. One easily checks that there exists a unique projection $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{Z})$ such that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(\mathbb{Z})$ is invertible. But we have an idempotent $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{Z})$ such that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$.

We note that the unique projection in (3) or (4) of Theorem 3.2 can not be replaced by the unique idempotent even for a commutative $*$-ring as the following example shows:

Example 3.3 Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ where $0, 1 \in \mathbb{Z}_2$. Define $* : R \to R$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b & b \\ a+b+c+d & b+d \end{pmatrix}$. Then R is a commutative $*$-ring with the usual matrix addition and multiplication. In fact, R is Boolean, and so, for any $a \in R$, there exists a unique idempotent $e \in R$ such that $a - e \in U(R)$ (or, $a - e \in J(R)$) and $ae = ea$. But R is not strongly J-$*$-clean, even not a $*$-clean ring.

Theorem 3.4 *Let R be a $*$-ring. Then R is strongly J-$*$-clean if and only if*

(1) *R is strongly $*$-clean;*

(2) $J(R) = \{x \in R \mid 1 - x \in U(R)\}$.

Proof Suppose that R is strongly J-$*$-clean. By virtue of Lemma 3.1, R is uniquely clean. Obviously, $J(R) \subseteq \{x \in R \mid 1 - x \in U(R)\}$. Suppose that $1 - x \in U(R)$. If $x \notin J(R)$, then $0 \neq xR \nsubseteq J(R)$. In view of [10, Lemma 17], there exists an idempotent $0 \neq e \in xR$. Write $e = xr$ for an $r \in R$. Then $e = (exe)(ere)$ as every idempotent in R is central. It is easy to see that R is directly finite. Thus, $exe \in U(eRe)$. In view of [10, Corollary 5], eRe is uniquely clean. Clearly, $0 + exe = e + e(x-1)e$. The uniqueness implies that $0 = e$, a contradiction. Therefore $x \in J(R)$, and so $\{x \in R \mid x - 1 \in U(R)\} \subseteq J(R)$. Thus, $J(R) = \{x \in R \mid 1 - x \in U(R)\}$.

Conversely, assume that (1) and (2) hold. Let $a \in R$. Then we can find a projection $e \in R$ such that $(a-1) - e \in U(R)$ and $e(a-1) = (a-1)e$. That is, $(1-a) + e \in U(R)$. As $1 - (a - e) \in U(R)$, by hypothesis, $a - e \in J(R)$. In addition, $ea = ae$. Therefore R is strongly J-$*$-clean. □

Corollary 3.5 *Let R be a $*$-ring. Then R is strongly J-$*$-clean if and only if*

(1) *R is strongly $*$-clean;*

(2) *For any $a \in R$, $a + a^* \in J(R)$;*

(3) $J(R) = \{x \in R \mid 1 + xx^* \in U(R)\}$.

Proof Suppose that R is strongly J-$*$-clean. By virtue of Proposition 2.1 and Lemma 3.1, R is a strongly $*$-clean ring with $2 \in J(R)$. For any $x \in R$, it follows from Lemma 3.1 that $x + x^* \in J(R)$. If $1 + xx^* \in U(R)$, then $(1+x)(1+x^*) = 1 + x + x^* + xx^* \in U(R)$. This implies that $1 + x \in R$ is right invertible. Similarly, $1 + x \in R$ is left invertible. In view of Theorem 3.4, $-x \in J(R)$, and so $x \in J(R)$. This shows that $J(R) = \{x \in R \mid 1 + xx^* \in U(R)\}$.

Conversely, assume that (1), (2) and (3) hold. If $1 + x \in U(R)$, then $1 + x^* \in U(R)$; hence, $(1 + x)(1 + x^*) \in U(R)$. As $x + x^* \in J(R)$, we see that $1 + xx^* = (1 + x)(1 + x^*) - (x + x^*) = (1 + x)(1 + x^*)\big(1 - ((1 + x)(1 + x^*))^{-1}(x + x^*)\big) \in U(R)$. This implies that $x \in J(R)$, and so $-x \in J(R)$. As a result, $J(R) = \{x \in R \mid 1 - x \in U(R)\}$. According to Theorem 3.4, the result follows. $\square$

Recall that a group G with an identity e is *torsion* provided that for any $g \in G$ there exists some $n \in \mathbb{N}$ such that $g^n = e$.

Corollary 3.6 *Let R be a $*$-ring. Then R is strongly J-$*$-clean if and only if*

(1) *R is strongly $*$-clean;*

(2) *$2 \in J(R)$;*

(3) *$U\big(R/J(R)\big)$ is torsion.*

Proof If R is strongly J-$*$-clean, then R is strongly $*$-clean and $2 \in J(R)$. In addition, $R/J(R)$ is Boolean. Thus, $U\big(R/J(R)\big) = \{\overline{1}\}$ is torsion.

Conversely, assume that (1), (2) and (3) hold. Assume that $1 - x \in U(R)$. Then $\overline{1 - x} \in U\big(R/J(R)\big)$. By hypothesis, there exists some $n \in \mathbb{N}$ such that $(\overline{1 - x})^n = \overline{1}$, and so $(\overline{1 - x})^{2n} = \overline{1}$. As $2 \in J(R)$, we see that $x^{2n} \in J(R)$. Clearly, R is an abelian exchange ring, and then so is $R/J(R)$. This implies that $R/J(R)$ is reduced, i.e. it has no nonzero nilpotent elements, and so $x \in J(R)$. This implies that $J(R) = \{x \in R \mid 1 - x \in U(R)\}$. Accordingly, R is strongly J-$*$-clean by Theorem 3.4. $\square$

We say that an ideal I of a $*$-ring R is a *$*$-ideal* provided that $I^* \subseteq I$. If I is a $*$-ideal of a $*$-ring, it is easy to check that R/I is also a $*$-ring.

Theorem 3.7 *Let I be a $*$-ideal of a $*$-ring R. If $I \subseteq J(R)$, then R is strongly J-$*$-clean if and only if*

(1) *R/I is strongly J-$*$-clean;*

(2) *R is abelian;*

(3) *Every idempotent lifts modulo I.*

Proof Suppose R is strongly J-$*$-clean. Then R is an abelian exchange ring, and so every idempotent lifts modulo I. We attempt to prove that R/I is strongly J-$*$-clean, by showing that (3) of Theorem 3.2. For any $a \in R$, there exist a projection $e \in R$ and a unit $u \in R$ such that $a = e + u$; hence, $\overline{a} = \overline{e} + \overline{u}$ in R/I. Assume that there exist a projection $\overline{f} \in R/I$ and a unit $\overline{v} \in R/I$ such that $\overline{a} = \overline{f} + \overline{v}$. Then, we can find an idempotent $g \in R$ such that $f = g + r$ for some $r \in I$. Hence, $a = g + (v + r + t)$ for some $t \in J(R)$. Obviously, $g - g^* = f - r - f^* + r^* \in J(R)$.

As $ag=ga, ag^*=g^*a$ and $v+r+t\in U(R)$, it follows by Theorem 3.2(5) that $g=e$, and so $\overline{f}=\overline{e}$ in R/I. Therefore R/I is strongly J-$*$-clean.

Conversely, assume that (1), (2) and (3) hold. For any $a\in R$, it follows from Theorem 3.2 that there exist a projection $\overline{e}\in R/I$ and a unit $\overline{u}\in R/I$ such that $\overline{a}=\overline{e}+\overline{u}$. As $e-e^2\in I$, by hypothesis, there exists an idempotent $f\in R$ such that $e-f\in I$. Since every unit lifts modulo I, we may assume that $u\in U(R)$. Thus, $a=f+u+r$ for some $r\in I$. Set $v=u+r$. Then $a=f+v$ with $f=f^2\in R, v\in U(R)$. As R is abelian, $af=fa$ and $af^*=f^*a$. Further, $f-f^*\equiv e-e^*\equiv 0\,(\mathrm{mod}\ I)$ and so $f-f^*\in J(R)$. Suppose that $a=g+w$ with $g=g^2\in R, w\in U(R)$, $ag=ga, ag^*=g^*a$ and $g-g^*\in J(R)$. Then $\overline{a}=\overline{g}+\overline{w}$ in R/I. Clearly, R/I is uniquely clean, and so $f-g\in I\subseteq J(R)$. As $fg=gf$, we see that $(f-g)^3=(f-2fg+g)(f-g)=f-g$, and so $f=g$. In light of Theorem 3.2(5), R is strongly J-$*$-clean. □

Recall that a $*$-ring is $*$-*Boolean* in the case that every element is a projection.

Corollary 3.8 *Let R be a $*$-ring R. Then R is strongly J-$*$-clean if and only if*

(1) *$R/J(R)$ is $*$-Boolean;*

(2) *R is abelian;*

(3) *Every idempotent lifts modulo $J(R)$.*

Proof One easily checks that $J(R)$ is a $*$-ideal of R, and thus establishing the claim by Theorem 3.7. □

Let $P(R)$ be the *prime radical* of R, i.e., the intersection of all prime ideals of R. Recall that $a\in R$ is *strongly nilpotent* if for every sequence $a_0, a_1, \cdots, a_i, \cdots$ such that $a_0=a$ and $a_{i+1}\in a_iRa_i$, there exists an n with $a_n=0$. As is well known, the prime radical $P(R)$ is the set of all strongly nilpotent elements in R.

Corollary 3.9 *A $*$-ring R is strongly J-$*$-clean if and only if R is abelian and $R/P(R)$ is strongly J-$*$-clean.*

Proof Let $a\in P(R)$. For every sequence $a_0, a_1, \cdots, a_i, \cdots$ such that $a_0=a^*$ and $a_{i+1}\in a_iRa_i$, we get a sequence $a_0^*, a_1^*, \cdots, a_i^*, \cdots$ such that $a_0^*=a$ and $a_{i+1}^*\in a_i^*Ra_i^*$. As $a\in R$ is strongly nilpotent, we can find some n such that $a_n^*=0$, and so $a_n=0$. This implies that a^* is strongly nilpotent; hence, $a^*\in P(R)$. We infer that $P(R)$ is a $*$-ideal. As every idempotent lifts modulo $P(R)$, we complete the proof by Theorem 3.7. □

In [8], Li and Zhou proved that a $*$-ring R is strongly $*$-clean if and only if $R/J(R)$ is strongly $*$-clean, every projection is central and every projection lifts to a projection modulo $J(R)$. Analogous to the previous discussion, we easily prove that a $*$-ring R is strongly $*$-clean if and only if $R/J(R)$ is strongly $*$-clean; R is abelian and every idempotent lifts modulo $J(R)$.

4 Certain Extensions

By applying the preceding results, we will construct various examples of strongly J-$*$-clean rings. Let R be a $*$-ring, and let $R[i] = \{a + bi \mid a, b \in R, i^2 = -1\}$. Then $R[i]$ is also a $*$-ring by defining $* : a + bi \mapsto a^* + b^* i$.

Lemma 4.1 *Let R be a ring with $2 \in J(R)$. Then $U\big(R[i]\big) = \{a + bi \mid a + b \in U(R)\}$.*

Proof Assume that $(a+bi)(c+di) = 1$. Then $ac-bd = 1$ and $ad+bc = 0$. Thus, $(a+b)(c+d) = ac + bd + ad + bc = (ac - bd) + 2bd = 1 + 2bd \in U(R)$. This implies that $a + b \in R$ is right invertible. As a result, we show that $U\big(R[i]\big) \subseteq \{a + bi \mid a + b \in U(R)\}$.

Assume that $a + b \in U(R)$. Then $a - b = (a + b) - 2b \in U(R)$. Clearly, $a(a - b)^{-1}a = (a - b + b)(a - b)^{-1}(a - b + b) = \big(1 + b(a - b)^{-1}\big)(a - b + b) = a + b + b(a - b)^{-1}b$. Therefore $a(a - b)^{-1}a(a + b)^{-1} - b(a - b)^{-1}b(a + b)^{-1} = 1$. Likewise, $b(a - b)^{-1}a - a(a - b)^{-1}b = 0$. It is easy to check that

$$\begin{aligned} & (a + bi)(a - b)^{-1}(a - bi)(a + b)^{-1} \\ = \ & \big(a(a - b)^{-1}a(a + b)^{-1} + b(a - b)^{-1}b(a + b)^{-1}\big) + \big(b(a - b)^{-1}a - a(a - b)^{-1}b\big)i \\ = \ & 1 + 2b(a - b)^{-1}b(a + b)^{-1} \\ \in \ & U(R). \end{aligned}$$

Thus, $a + bi \in R[i]$ is right invertible. Analogously, $(a - b)^{-1}(a - bi)(a + b)^{-1}(a + bi) \in U(R)$. Therefore $a + bi \in U\big(R[i]\big)$, as required. □

Proposition 4.2 *Let R be a $*$-ring. Then $R[i]$ is strongly J-$*$-clean if and only if so is R.*

Proof Suppose that $R[i]$ is strongly J-$*$-clean. Then $2 \in J(R)$. Further, every idempotent in $R[i]$ is a projection, and it is central. Let $a \in R$. Then we can find a projection $e + fi \in R[i]$ and an element $u + vi \in J\big(R[i]\big)$ such that $a = (e + fi) + (u + vi)$ and $a(e + fi) = (e + fi)a$. Thus, $a = e + u$ and $ae = ea$. As $e + fi \in R[i]$ is central, $e \in R$ is central. Since $(e + fi)^* = e + fi$, we see that $e^* = e$. From $e + fi = (e + fi)^2$, we get $e^2 - f^2 = e$ and $2ef = f$. This implies that $(2e - 1)f = 0$, and then $f = 0$. Hence, $e \in R$ is a projection. It is easy to verify that $u \in J(R)$, and therefore R is strongly J-$*$-clean.

Conversely, assume that R is strongly J-$*$-clean. Then R is an abelian exchange ring. In addition, every idempotent in R is a projection and $2 \in J(R)$. Let $a + bi \in R[i]$. By hypothesis, there exist projections $e, f \in R$ and $u, v \in J(R)$ such that $a = e + u, b = f + v, ae = ea, bf = fb$. Thus, $a + bi = (e + f) + \big(u - f + (f + v)i\big)$. Clearly, $(e + f)^2 - (e + f) = 2ef \in J(R)$. As every idempotent lifts modulo $J(R)$, we can find an idempotent $g \in R$ such that $e + f = g + r$ where $r \in J(R)$. Thus, $a + bi = g + \big(r + u - f + (f + v)i\big)$ where $g = g^2 = g^*$ and $(a + bi)g = g(a + bi)$.

Write $x = r + u - f$ and $y = f + v$. Then $x + y = r + u + v \in J(R)$. For any $c + di \in R[i]$, we see that $1 - (x + yi)(c + di) = (1 - xc + yd) - (xd + yc)i$. As $(1 - xc + yd) - (xd + yc) = (1 - xc - yd) - (xd + yc) + 2yd = 1 - (x + y)(c + d) + 2yd \in U(R)$. In light of Lemma 4.1, $1 - (x + yi)(c + di) \in U\big(R[i]\big)$, and so $x + yi \in J\big(R[i]\big)$. Therefore $R[i]$ is strongly J-$*$-clean, as asserted. □

Let R be a $*$-ring. Then $*$ induces an involution of the power series ring $R[[x]]$, denoted by $*$, where $\big(\sum\limits_{i=0}^{\infty} a_i x^i\big)^* = \sum\limits_{i=0}^{\infty} a_i^* x^i$. This induces the involution on $R[[x]]/(x^n)$ $(n \geq 1)$.

Proposition 4.3 *Let R be a $*$-ring. Then the following are equivalent:*

(1) *R is strongly J-$*$-clean.*

(2) *$R[[x]]$ is strongly J-$*$-clean.*

(3) *$R[[x]]/(x^n)$ is strongly J-$*$-clean for all $n \geq 2$.*

(4) *$R[[x]]/(x^2)$ is strongly J-$*$-clean.*

Proof (1) $\Rightarrow$ (2) Since R is strongly J-$*$-clean, $R[[x]]$ is strongly $*$-clean by [8, Corollary 2.10]. $R[[x]]$ is also strongly J-clean by [4, Example 16.4.17]. Hence $R[[x]]$ is strongly J-$*$-clean by Proposition 2.1.

(2) $\Rightarrow$ (3) In view of Lemma 3.1, $R[[x]]$ is uniquely clean, and then so is $S := R[[x]]/(x^n)$ by [10, Theorem 22]. For any $f \in S$, it follows from Lemma 3.1 that $f(0) - \big(f(0)\big)^* \in J(R)$, and so $f - f^* = f(0) - (f(0))^* + \sum\limits_{i=1}^{n-1} b_i x^i \in J(S)$. By using Lemma 3.1 again, $R[[x]]/(x^n)$ is a strongly J-$*$-clean ring.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1) Let $S = R[[x]]/(x^2)$. For any $a = a + 0x \in S$, there exists a projection $e + fx \in S$ such that $a(e+fx) = (e+fx)a$ and $a-(e+fx) \in J(S)$. This implies that $e \in R$ is a projection, $ae = ea$ and $a - e \in J(R)$. Therefore R is a strongly J-$*$-clean ring. □

Since $R[x]/(x^2)$ and $R[[x]]/(x^2)$ are isomorphic, Proposition 4.3 also implies that a $*$-ring is strongly J-$*$-clean if and only if $R[x]/(x^2)$ is.

Let R be a $*$-ring and G be a group. Then $*$ induces an involution of the group ring RG, denoted by $*$, where $(\sum_g a_g g)^* = \sum_g a_g^* g^{-1}$ (see [8, Lemma 2.12]). A group G is called *locally finite* if every finitely generated subgroup of G is finite. A group G is a 2-*group* if the order of each element of G is a power of 2.

Proposition 4.4 *Let R be a $*$-ring, and let G be a locally finite group. Then RG is strongly J-$*$-clean if and only if R is strongly J-$*$-clean and G is a 2-group.*

Proof Suppose that RG is strongly J-$*$-clean. Then it is uniquely clean. By virtue of [5, Theorem 12], R is uniquely clean and G is a 2-group. In view of Lemma 3.1, $2 \in J(RG)$, and so $2 \in J(R)$. In addition, every idempotent in RG is central, and so every idempotent in R is central in R. According to [5, Lemma 11], every idempotent in RG is in R. Therefore every idempotent in R is a projection. According to Theorem 3.2, R is strongly J-$*$-clean.

Conversely, assume that R is a strongly J-$*$-clean ring and G is a 2-group. By virtue of [5, Theorem 13], RG is uniquely clean. As R is strongly J-$*$-clean, we see that $2 \in J(R)$. Further, every idempotent in R is central. According to [5, Lemma 11], every idempotent in RG is in R. Therefore every idempotent in RG is a projection. By using Proposition 2.1, RG is strongly J-$*$-clean. □

Corollary 4.5 *Let R be a $*$-ring, and let G be a solvable group. Then RG is strongly J-$*$-clean if and only if R is strongly J-$*$-clean and G is a 2-group.*

Proof The proof of necessity is the same as in Proposition 4.4. Conversely, assume that R is a strongly J-$*$-clean ring and G is a 2-group. Analogously to the consideration in [5, Theorem 13], G is locally finite, and then the result follows from Proposition 4.4. □

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