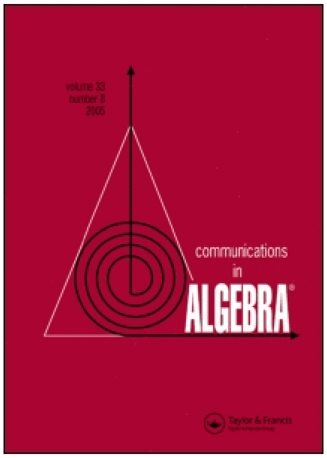

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

The Torsion Theory Cogenerated by δ-M-Small Modules and GCO-Modules

To cite this Article: , 'The Torsion Theory Cogenerated by δ-M-Small Modules and GCO-Modules', Communications in Algebra, 35:2, 623 - 633
xxxx:journal To link to this article: DOI: 10.1080/00927870601074871
URL: http://dx.doi.org/10.1080/00927870601074871

Communications in Algebra®, 35: 623–633, 2007

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870601074871

Taylor & Francis
Taylor & Francis Group

THE TORSION THEORY COGENERATED BY δ–M-SMALL MODULES AND GCO-MODULES

A. Çiğdem Özcan
Department of Mathematics, Hacettepe University, Beytepe, Ankara, Turkey

Let M be a module and K a submodule of a module N in $\sigma[M]$. We call K a δ–M-small submodule of N if whenever $N = K + L$, N/L is M-singular for any submodule L of N, we have $N = L$. Also we call N a δ–M-small module if N is a δ–M-small submodule of its M-injective hull. In this article, we consider $\overline{Z}_{\delta_M}(N) = \mathbf{Rej}(N, \mathfrak{D}\mathscr{M})$, the reject of $\mathfrak{D}\mathscr{M}$ in N, where $\mathfrak{D}\mathscr{M}$ is the class of all δ–M-small modules. We investigate the properties of $\overline{Z}_{\delta_M}(N)$ and consider the torsion theory $\tau_{\delta V}$ in $\sigma[M]$ cogenerated by $\mathfrak{D}\mathscr{M}$. We compare the $\tau_{\delta V}$ and the torsion theory τ_V cogenerated by M-small modules and finally we give a characterization of GCO-modules.

Key Words: GCO-modules; Lifting modules; Singular modules; Small modules.

2000 Mathematics Subject Classification: 16S90; 16D50; 16D60.

1. PRELIMINARIES

All rings we consider are rings with identity and all modules are unitary right modules. Let R be a ring and M an R-module.

An R-module N is *subgenerated* by M if N is isomorphic to a submodule of an M-generated module. $\sigma[M]$ denotes the full subcategory of Mod-R whose objects are all R-modules subgenerated by M. Let $N \in \sigma[M]$. *M-injective hull* of N or *injective hull of* N in $\sigma[M]$ is denoted by $\widehat{N}$. (see Wisbauer, 1991 or Dung et al., 1994).

We use the notation $N \leq_e M$ for an essential submodule and $N \leq_d M$ for a direct summand N of M. A module N in $\sigma[M]$ is called *M-singular* (or *singular in* $\sigma[M]$) if $N \cong L/K$ for an $L \in \sigma[M]$ and $K \leq_e L$. Every module $N \in \sigma[M]$ contains a largest M-singular submodule which is denoted by $Z_M(N)$ (Dung et al., 1994).

Let K be a submodule of M. K is called *small* in M if $K + L \neq M$ holds for every proper submodule L of M and denoted by $K \ll M$. A module N in $\sigma[M]$ is called *M-small* (or *small in* $\sigma[M]$) if $N \cong K \ll L$ for $K, L \in \sigma[M]$. In case $M = R$, instead of R-small, we just say *small*. A module N is M-small if and only if $N \ll \widehat{N}$. We denote the class of all M-small modules by $\mathscr{M}$. $\mathscr{M}$ is closed under submodules, homomorphic images and finite direct sums.

Received September 22, 2005; Revised April 10, 2006. Communicated by R. Wisbauer.
Address correspondence to A. Çiğdem Özcan, Department of Mathematics, Hacettepe University, 06532 Beytepe, Ankara, Turkey; Fax: +90-312–299-20-17; E-mail: ozcan@hacettepe.edu.tr

Note that the family of simple modules in $\sigma[M]$ splits into four disjoint classes by combining the exclusive choices [M-projective or M-singular] and [M-injective or M-small].

Let N be a submodule of a module M. N is called *δ-small* in M if whenever $M = N + K$ and M/K is singular for any $K \leq M$ we have $M = K$, denoted by $N \ll_\delta M$ (Zhou, 2000).

Recently Zhou generalized small submodules to δ-small submodules in Mod-R considering the class of all singular modules in place of the class of all R-modules. In this article, we consider δ-small submodules in the category $\sigma[M]$ for a module M and define it as a δ–M-small submodule. M-small modules are dual of M-singular modules. Also we define δ–M-small modules as a generalization of M-small modules. Rej$(N, \mathscr{M})$, the reject of $\mathscr{M}$ in N, is considered by Talebi and Vanaja (2002) and they investigate the torsion theory τ_V cogenerated by $\mathscr{M}$ and characterize lifting modules as an analogue of extending modules. In this article we deal with these subjects considering the class of δ–M-small modules.

In Section 2, we investigate general properties of δ–M-small (sub)modules and the reject of the class of all δ–M-small modules in N in $\sigma[M]$ and characterize lifting modules.

Section 3 is related with the torsion theory $\tau_{\delta V}$ cogenerated by the class of δ–M-small modules. We study some conditions for $\tau_{\delta V}$ to be cohereditary or split.

In the last section, GCO-modules are characterized.

2. δ–M-SMALL MODULES

In this section first we define δ–M-small (sub)modules and study the basic properties of them.

Definition. Let M be a module and $K \leq N \in \sigma[M]$. K is called *a δ–M-small submodule of N in $\sigma[M]$* if whenever $N = K + X$ and N/X is M-singular for $X \leq N$ we have $N = X$, we denote it by $K \ll_{\delta_M} N$.

Clearly, if $K \ll_\delta N \in \sigma[M]$, then $K \ll_{\delta_M} N$. The properties of δ-small submodules that are listed in Lemma 1.3 in Zhou (2000) also hold in $\sigma[M]$. We write them for convenience. Note that the class of M-singular modules is closed under submodules, homomorphic images and direct sums (Dung et al., 1994).

Lemma 2.1. *Let $N \in \sigma[M]$.*

a) *For modules $K, L \in \sigma[M]$ with $K \leq L \leq N$ we have*

$$L \ll_{\delta_M} N \quad \text{if and only if} \quad K \ll_{\delta_M} N \quad \text{and} \quad L/K \ll_{\delta_M} N/K.$$

b) *For $K, L \in \sigma[M]$,*

$$K + L \ll_{\delta_M} N \quad \text{if and only if} \quad K \ll_{\delta_M} N \quad \text{and} \quad L \ll_{\delta_M} N.$$

c) *If $K \ll_{\delta_M} N$ and $f : N \to L$ is a homomorphism, then $f(K) \ll_{\delta_M} L$. In particular, if $K \ll_{\delta_M} N \leq L$ then $K \ll_{\delta_M} L$.*

d) *If $K \leq L \leq_d N \in \sigma[M]$ and $K \ll_{\delta_M} N$ then $K \ll_{\delta_M} L$.*

The following lemma can be seen by a proof similar to Zhou (2000, Lemma 1.2).

Lemma 2.2. *Let $N \in \sigma[M]$. The following are equivalent*:

(1) $K \ll_{\delta_M} N$;
(2) *if $N = X + K$, then $N = X \oplus Y$ for a projective semisimple submodule Y with $Y \leq K$.*

Now as a generalization of M-small modules we define δ–M-small modules for a module M.

Definition. Let $N \in \sigma[M]$. N is called a *δ–M-small module* in $\sigma[M]$ if $N \cong K \ll_{\delta_M} L \in \sigma[M]$.

Note that a module N in $\sigma[M]$ is a δ–M-small module if and only if $N \ll_{\delta_M} \widehat{N}$ (Özcan, 2002, Lemma 2.4). We denote the class of all δ–M-small modules in $\sigma[M]$ by $\mathcal{DM}$. The class of δ–M-small modules is closed under submodules, homomorphic images and finite direct sums. If N is δ–M-small and M-singular, then N is M-small.

Let $\mathcal{C}$ be a class of modules in $\sigma[M]$. For any N in $\sigma[M]$, the *reject* of $\mathcal{C}$ in N is denoted by $\mathrm{Rej}(N, \mathcal{C}) = \bigcap\{\ker g : g \in \mathrm{Hom}(N, C), C \in \mathcal{C}\}$. Talebi and Vanaja (2002) define $\overline{Z}_M(N)$ as a dual of M-singular submodule as follows:

$$\overline{Z}_M(N) = \mathrm{Rej}(N, \mathcal{M}).$$

For $M = R$, we write $\overline{Z}(N)$ instead of $\overline{Z}_M(N)$. They call N *an M-cosingular* (*non-M-cosingular*) *module* if $\overline{Z}_M(N) = 0$ ($\overline{Z}_M(N) = N$).

Now we consider the class of δ–M-small modules and define

$$\overline{Z}_{\delta_M}(N) = \mathrm{Rej}(N, \mathcal{DM}).$$

Then $\overline{Z}_{\delta_M}(N) \leq \overline{Z}_M(N)$. For $M = R$, we write $\overline{Z}_{\delta}(N)$ instead of $\overline{Z}_{\delta_M}(N)$. We call N *a δ–M-cosingular* (*non-δ–M-cosingular*) *module* if $\overline{Z}_{\delta_M}(N) = 0$ ($\overline{Z}_{\delta_M}(N) = N$). Then every M-cosingular module is δ–M-cosingular and every non-δ–M-cosingular module is non-M-cosingular. Clearly, $N \in \sigma[M]$ is non-δ–M-cosingular if and only if every nonzero factor module of N is non-δ–M-small.

Definitions. Let $A \leq B \leq N \in \sigma[M]$. A is called *a coessential submodule* of B in N if $B/A \ll N/A$, denoted by $A \overset{ce}{\hookrightarrow} B$ in N. A submodule A of N is said to be *coclosed* in N if it has no proper coessential submodule in N, denoted by $A \overset{cc}{\hookrightarrow} N$. If A and B are submodules of N in $\sigma[M]$, it is said that A is *a coclosure* of B in N if A is a coessential submodule of B in N and A is a coclosed submodule of N.

The following two results can be seen by proofs similar to Lemma 2.3 and Proposition 2.4 in Talebi and Vanaja (2002).

Lemma 2.3. *Let $N \in \sigma[M]$. Then*:

(1) *If N is non-δ–M-cosingular and $X \leq N$ is a δ–M-small module, then $X \ll N$*;
(2) *Any non-δ–M-cosingular submodules of N is coclosed in N*;

(3) *A submodule of a non-δ–M-cosingular module is coclosed if and only if it is a non-δ–M-cosingular module*;
(4) *If M is non-δ–M-cosingular, then $\mathcal{DM} = \mathcal{M}$.*

Let A and B in $\sigma[M]$. A is called *a small cover* of B if there exists an epimorphism $f : A \to B$ with $\ker f \ll A$.

Proposition 2.4. *The class of all non-δ–M-cosingular modules is closed under homomorphic images, direct sums, extensions, small covers, and coclosed submodules.*

It follows that if M is a generator in $\sigma[M]$ and non-δ–M-cosingular, then every module in $\sigma[M]$ is non-δ–M-cosingular.

Now we list some properties of $\overline{Z}_{\delta_M}(N)$ that can be seen easily.

Proposition 2.5. *Let A, B and A_i $(i \in I)$ be modules in $\sigma[M]$. Then*:

(1) *If $A \le B$, then $\overline{Z}_{\delta_M}(A) \le \overline{Z}_{\delta_M}(B)$ and $(\overline{Z}_{\delta_M}(B) + A)/A \le \overline{Z}_{\delta_M}(B/A)$*;
(2) *If $f : B \to A$ is a homomorphism, then $f(\overline{Z}_{\delta_M}(B)) \le \overline{Z}_{\delta_M}(A)$*;
(3) $\overline{Z}_{\delta_M}(A/\overline{Z}_{\delta_M}(A)) = 0$;
(4) $\overline{Z}_{\delta_M}(\bigoplus_{i\in I} A_i) = \bigoplus_{i\in I} \overline{Z}_{\delta_M}(A_i)$;
(5) $\overline{Z}_{\delta_M}(\prod_{i\in I} A_i) \le \prod_{i\in I} \overline{Z}_{\delta_M}(A_i)$;
(6) *If $A = B + S$ where S is a δ–M-small module, then $\overline{Z}_{\delta_M}(B) = \overline{Z}_{\delta_M}(A)$*;
(7) *$\overline{Z}_{\delta_M}(A)$ is the smallest submodule of A such that $A/\overline{Z}_{\delta_M}(A)$ is δ–M-cosingular.*

Corollary 2.6. *The class of all δ–M-cosingular modules is closed under submodules, direct sums and direct products.*

A module M is called *hereditary* in $\sigma[M]$ if every submodule of M is projective in $\sigma[M]$.

Proposition 2.7. *Let M be a (projective) hereditary module in $\sigma[M]$. Then every M-singular injective module in $\sigma[M]$ is non-δ–M-cosingular.*

Proof. The homomorphic image of any injective module in $\sigma[M]$ is injective in $\sigma[M]$ by Wisbauer (1991, 39.6). Hence any nonzero factor module of any nonzero M-singular injective module in $\sigma[M]$ is not δ–M-small. Therefore, every M-singular injective module in $\sigma[M]$ is non-δ–M-cosingular. □

Proposition 2.8. *Let M be a non-M-singular module (i.e., $Z_M(M) = 0$) and S be a simple M-singular module in $\sigma[M]$. Then $\overline{Z}_{\delta_M}(\widehat{S}) \neq 0$.*

Proof. Since the class of M-singular modules is closed under extensions, $\widehat{S}$ is M-singular. Then $\overline{Z}_{\delta_M}(\widehat{S}) = \overline{Z}_M(\widehat{S}) \neq 0$ by Talebi and Vanaja (2002, Proposition 2.8). □

Corollary 2.9. *Let M be a module and S be a simple module in $\sigma[M]$ such that $\widehat{S}$ is M-singular. Then $\overline{Z}_{\delta_M}(\widehat{S}) \neq 0$.*

Note that if N is a projective injective simple module in $\sigma[M]$, then $\overline{Z}_{\delta_M}(N) = 0$ and $\overline{Z}_M(N) = N$ by definitions and Talebi and Vanaja (2002, Proposition 2.8).

Proposition 2.10. *Let $N \in \sigma[M]$ be indecomposable. Then either $\overline{Z}_{\delta_M}(N) = \overline{Z}_M(N)$ or N is projective injective simple module in $\sigma[M]$.*

Proof. If $\overline{Z}_{\delta_M}(N) = N$, then $\overline{Z}_{\delta_M}(N) = \overline{Z}_M(N)$. Assume $\overline{Z}_{\delta_M}(N) \neq N$. Let X be a submodule of N such that N/X is a δ–M-small module. We claim that N/X is an M-small module. Let $\widehat{N/X} = N/X + A$ for some submodule A of $\widehat{N/X}$. By Lemma 2.2, $\widehat{N/X} = Y/X \oplus A$ for a projective semisimple submodule Y/X with $Y/X \leq N/X$. Since Y/X is injective in $\sigma[M]$, $Y/X \leq_d N/X$. Let S/X be a submodule of N/X such that $N/X = Y/X \oplus S/X$. Then N/S is projective in $\sigma[M]$ and hence $S \leq_d N$.

If $S = 0$, then $X = 0$. Hence $N = Y$ is semisimple projective injective in $\sigma[M]$. Since N is indecomposable, N is simple.

If $S \neq 0$, then $S = N$. Then $Y/X = 0$ and so $N/X \ll \widehat{N/X}$. It follows that $\overline{Z}_{\delta_M}(N) = \overline{Z}_M(N)$. □

Corollary 2.11. *Let $N \in \sigma[M]$ be a direct sum of indecomposable modules. Then N has a decomposition $N = A \oplus B$ such that $\overline{Z}_{\delta_M}(A) = \overline{Z}_M(A)$ and B is semisimple projective in $\sigma[M]$.*

Remark 2.12. Now we list some examples in the literature so that a module N is a direct sum of indecomposable modules.

(a) Every local summand of N is a summand (Mohamed and Müller, 1990, Theorem 2.17).
(b) N is a locally noetherian CS-module (extending) (Dung et al., 1994, Corollary 8.3).
(c) N is a self-hereditary CS-module (Dung et al., 1994, Theorem 10.5).
(d) N is a $\sum$-CS-module (Gomez Pardo et al., 2001).
(e) N is a countable CS-module (Gomez Pardo et al., 2000).

The class of δ–M-cosingular modules need not be closed under homomorphic images. In Corollary 4.4, some conditions are given for the class of δ–M-cosingular modules to be closed under homomorphic images.

Example 2.13. Let $R = M = \mathbb{Z}$ and $N = \mathbb{Q}$. Since $\mathbb{Z}$ is a small module, $\overline{Z}_\delta(\mathbb{Z}) = \overline{Z}(\mathbb{Z}) = 0$. Since $\mathbb{Q}_\mathbb{Z}$ is uniform and not a projective $\mathbb{Z}$-module, $\overline{Z}_\delta(\mathbb{Q}) = \overline{Z}(\mathbb{Q}) = \mathbb{Q}$ by Proposition 2.10 and Talebi and Vanaja (2002, Proposition 2.7). So let $\mathbb{Z}^{(\mathbb{N})} \to \mathbb{Q}$ be an epimorphism. Then $\mathbb{Z}^{(\mathbb{N})}$ is δ-cosingular but $\mathbb{Q}$ is not.

The class of δ–M-cosingular modules need not be closed under extensions.

Example 2.14. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/4\mathbb{Z}$. Since M is uniform, $\overline{Z}_{\delta_M}(M) = \overline{Z}_M(M) = 2M$ by Proposition 2.10 and Özcan and Harmanci (2003, Example 4.4). Since $\mathbb{Z}/2\mathbb{Z}$ is M-small, $\overline{Z}_{\delta_M}(\mathbb{Z}/2\mathbb{Z}) = \overline{Z}_M(\mathbb{Z}/2\mathbb{Z}) = 0$.

Now we define $\overline{Z}^{\alpha}_{\delta_M}(N)$ for $N \in \sigma[M]$. We set $\overline{Z}^{0}_{\delta_M}(N) = N$, $\overline{Z}^{1}_{\delta_M}(N) = \overline{Z}_{\delta_M}(N)$ and define inductively $\overline{Z}^{\alpha}_{\delta_M}(N)$ for any ordinal α. Thus, if α is not a limit ordinal we set $\overline{Z}^{\alpha}_{\delta_M}(N) = \overline{Z}_{\delta_M}(\overline{Z}^{\alpha-1}_{\delta_M}(N))$, while if α is a limit ordinal we set $\overline{Z}^{\alpha}_{\delta_M}(N) = \bigcap_{\beta<\alpha} \overline{Z}^{\beta}_{\delta_M}(N)$. Hence there is a descending sequence

$$N = \overline{Z}^{0}_{\delta_M}(N) \supseteq \overline{Z}_{\delta_M}(N) \supseteq \overline{Z}^{2}_{\delta_M}(N) \supseteq \ldots \qquad \text{of submodules of } N.$$

The following example characterizes the $\overline{Z}_{\delta}(R_R)$ where R is the ring of 2×2 upper triangular matrices over a field.

Example 2.15. Let $R = M = \left[\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right]$ where F is a field. Let $A = \left[\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & F \end{smallmatrix}\right]$. Then $R_R = A \oplus B$, A is injective and B is simple. Hence $\overline{Z}_{\delta}(A) = \overline{Z}(A) = A$ and $\overline{Z}_{\delta}(B) = \overline{Z}(B) = 0$. This implies that $\overline{Z}_{\delta}(R_R) = \overline{Z}(R_R) = A$. Since R is a hereditary ring, $A = \overline{Z}(R_R) = \overline{Z}^2(R_R) = \overline{Z}^3(R_R)$ by Talebi and Vanaja (2002, Proposition 2.7). Also we have that $A = \overline{Z}_{\delta}(R_R) = \overline{Z}^2_{\delta}(R_R) = \overline{Z}^3_{\delta}(R_R)$.

Theorem 2.16. *Let $N \in \sigma[M]$ and $\alpha \geq 1$ be any ordinal. Then*

$$\overline{Z}^{\alpha+1}_{\delta_M}(N) = \bigcap \{X \mid X \overset{ce}{\hookrightarrow} \overline{Z}^{\alpha}_{\delta_M}(N) \text{ in } N\}$$

and the family of all coessential submodules of $\overline{Z}^{\alpha}_{\delta_M}(N)$ in N is closed under finite intersections.

Proof. Let $N \in \sigma[M]$ and $X \leq \overline{Z}^{\alpha}_{\delta_M}(N)$ be such that $\overline{Z}^{\alpha}_{\delta_M}(N)/X$ is δ–M-small. We claim that $\overline{Z}^{\alpha}_{\delta_M}(N)/X \ll N/X$. Let L be a submodule of N such that $N/X = \overline{Z}^{\alpha}_{\delta_M}(N)/X + L/X$. Then $N = \overline{Z}^{\alpha}_{\delta_M}(N) + L$. Consider the natural epimorphism $f : N \to N/L$. Then $N/L \leq (\overline{Z}_{\delta_M}(N) + L)/L = f(\overline{Z}_{\delta_M}(N)) \leq \overline{Z}_{\delta_M}(N/L)$ and hence $\overline{Z}_{\delta_M}(N/L) = N/L$. On the other hand if we consider the epimorphism $g : N/X \to N/L$, we have that $g(N/X) = g(\overline{Z}^{\alpha}_{\delta_M}(N)/X) = N/L$ is δ–M-small. This implies that $N = L$. Hence $\overline{Z}^{\alpha+1}_{\delta_M}(N) = \bigcap\{X \mid X \overset{ce}{\hookrightarrow} \overline{Z}^{\alpha}_{\delta_M}(N) \text{ in } N\}$. The other inclusion is obvious.

For the last part, let X and Y be coessential submodules of $\overline{Z}^{\alpha}_{\delta_M}(N)$ in N. Then $\overline{Z}^{\alpha}_{\delta_M}(N)/X$ and $\overline{Z}^{\alpha}_{\delta_M}(N)/Y$ are M-small. Since $\overline{Z}^{\alpha}_{\delta_M}(N)/(X \cap Y)$ is isomorphic to a submodule of $(\overline{Z}^{\alpha}_{\delta_M}(N)/X) \oplus (\overline{Z}^{\alpha}_{\delta_M}(N)/Y)$, then $X \cap Y \overset{ce}{\hookrightarrow} \overline{Z}^{\alpha}_{\delta_M}(N)$ by above. □

Corollary 2.17. *If $\overline{Z}_{\delta_M}(N)$ has a coclosure in N, then $\overline{Z}^2_{\delta_M}(N)$ is the unique coclosure of $\overline{Z}_{\delta_M}(N)$ and hence is the largest non-δ–M-cosingular submodule of N, i.e., $\overline{Z}^2_{\delta_M}(N) = \overline{Z}^3_{\delta_M}(N)$.*

Proof. It can be seen by the proof of Talebi and Vanaja (2002, Corollary 3.4) by taking $\overline{Z}_{\delta_M}(N)$ instead of $\overline{Z}_M(N)$. □

Definitions. A module M is called *lifting* if for every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2$ is small in M. If N and L are submodules of M, then N is called a *supplement* of L (in M) if $N + L = M$ and $N \cap L \ll N$. M is called an *amply supplemented* if, for all submodules N and L of M with $N + L = M$, N contains a supplement of L in M.

Theorem 2.18. *Let N be an amply supplemented module in $\sigma[M]$, L a subfactor of N and $f : L \to T$ an epimorphism. Then $\overline{Z}^2_{\delta_M}(L)$ is the largest non-δ–M-cosingular submodule of L and $f(\overline{Z}^2_{\delta_M}(L)) = \overline{Z}^2_{\delta_M}(T)$.*

Proof. It can be seen by the proof of Talebi and Vanaja (2002, Theorem 3.5) by taking $\overline{Z}_{\delta_M}(N)$ instead of $\overline{Z}_M(N)$.

Let $N \in \sigma[M]$. Lifting modules are characterized in Talebi and Vanaja (2002, Theorem 4.1) by using $\overline{Z}^2_M(N)$. N is lifting if and only if $N = \overline{Z}^2_M(N) \oplus K$ for some $K \leq N$ such that K and $\overline{Z}^2_M(N)$ are lifting, K is $\overline{Z}^2_M(N)$-projective and N is amply supplemented. By the technique of Talebi and Vanaja (2002, Theorem 4.1), we characterize lifting modules by using $\overline{Z}^2_{\delta_M}(N)$. Note that if N is a lifting module in $\sigma[M]$, then $\overline{Z}^2_{\delta_M}(N)$ need not be equal to $\overline{Z}^2_M(N)$, and as an example we may consider simple injective projective modules.

Theorem 2.19. *Let N be a module in $\sigma[M]$. Then N is lifting if and only if $N = \overline{Z}^2_{\delta_M}(N) \oplus K$ for some $K \leq N$ such that K and $\overline{Z}^2_{\delta_M}(N)$ are lifting, K is $\overline{Z}^2_{\delta_M}(N)$-projective and N is amply supplemented.*

3. THE TORSION THEORY COGENERATED BY δ–M-SMALL MODULES

Let $\mathscr{C}$ be a class of modules in $\sigma[M]$. *The torsion theory cogenereated by a class* of $\mathscr{C}$ of modules which is closed under isomorphisms and submodules in $\sigma[M]$ is $\tau_c = (\mathscr{T}_c, \mathscr{F}_c)$ where

$$\begin{aligned}\mathscr{T}_c &= \{T \in \sigma[M] : \forall C \in \mathscr{C}, \operatorname{Hom}(T, C) = 0\}\\ &= \{T \in \sigma[M] : \operatorname{Rej}(T, \mathscr{C}) = T\} \quad \text{and}\\ \mathscr{F}_c &= \{F \in \sigma[M] : \forall T \in \mathscr{T}_c, \operatorname{Hom}(T, F) = 0\}\\ &= \{F \in \sigma[M] : \forall 0 \neq U \leq F, \operatorname{Rej}(U, \mathscr{C}) \neq U\}.\end{aligned}$$

Talebi and Vanaja consider the torsion theory cogenerated by $\mathscr{M}$, we denote it by $\tau_V = (\mathscr{T}_V, \mathscr{F}_V)$ where

$$\begin{aligned}\mathscr{T}_V &= \{T \in \sigma[M] : \overline{Z}_M(T) = T\} \quad \text{and}\\ \mathscr{F}_V &= \{F \in \sigma[M] : \forall 0 \neq K \leq F, \overline{Z}_M(K) \neq K\}.\end{aligned}$$

In this article we consider the torsion theory cogenerated by the class of δ–M-small modules. We denote it by $\tau_{\delta V} = (\mathscr{T}_{\delta V}, \mathscr{F}_{\delta V})$ where

$$\begin{aligned}\mathscr{T}_{\delta V} &= \{T \in \sigma[M] : \overline{Z}_{\delta_M}(T) = T\} \quad \text{and}\\ \mathscr{F}_{\delta V} &= \{F \in \sigma[M] : \forall 0 \neq K \leq F, \overline{Z}_{\delta_M}(K) \neq K\}.\end{aligned}$$

Clearly, $\tau_{\delta V} \leq \tau_V$ (i.e., $\mathscr{T}_{\delta V} \subseteq \mathscr{T}_V$ and $\mathscr{F}_V \subseteq \mathscr{F}_{\delta V}$). Let us denote the radical associated with $\tau_{\delta V}$ by $\rho_{\delta V}$. Then $\rho_{\delta V}$ is the largest non-δ–M-cosingular submodule of $N \in \sigma[M]$.

Next we consider the modules having a projective cover and define the dual Lambek torsion theory τ_P because we deal with the relationship between $\tau_{\delta V}$ and τ_P.

Let M be a module. Assume that M has a projective cover P in $\sigma[M]$ and consider the torsion theory generated by P, $\tau_P = (\mathcal{T}_P, \mathcal{F}_P)$ where

$$\mathcal{F}_P = \{F \in \sigma[M] : \mathrm{Hom}(P, F) = 0\} \quad \text{and}$$
$$\mathcal{T}_P = \{T \in \sigma[M] : \forall F \in \mathcal{F}_P, \mathrm{Hom}(T, F) = 0\}.$$

This is cohereditary (i.e., $\mathcal{F}_P$ is closed under homomorphic images). Since $P \in \mathcal{T}_P$, $\mathrm{Gen}_M(P) \subseteq \mathcal{T}_P$ ($\mathrm{Gen}_M(P)$ is the class of objects in $\sigma[M]$ which are generated by P). Also we have

$$\mathcal{F}_P \subseteq \mathcal{M} \subseteq \mathcal{F}_V \subseteq \mathcal{F}_{\delta V}, \qquad \mathcal{T}_{\delta V} \subseteq \mathcal{T}_V \subseteq \mathcal{T}_P.$$

Applying Proposition 4.5 of Özcan and Harmanci (2003) and Proposition 2.4, we get the following proposition.

Proposition 3.1. *Assume a module M has a projective cover P in $\sigma[M]$. Then the following are equivalent*:

(1) $\overline{Z}_{\delta_M}(M) = M$;
(2) $\overline{Z}_{\delta_M}(P) = P$;
(3) $\mathcal{F}_P = \mathcal{D}\mathcal{M}$;
(4) $\mathcal{T}_P = \mathcal{T}_{\delta V}$;
(5) $Gen_M(P) \subseteq \mathcal{T}_{\delta V}$.

In this case,

$$\mathcal{F}_P = \mathcal{M} = \mathcal{D}\mathcal{M} = \mathcal{F}_{\delta V} = \left\{N \in \sigma[M] : \overline{Z}_{\delta_M}(N) = 0\right\} = \left\{N \in \sigma[M] : \overline{Z}_M(N) = 0\right\}.$$

A torsion theory $(\mathcal{T}, \mathcal{F})$ *splits* if every module N in $\sigma[M]$ has a decomposition $N = N_1 \oplus N_2$ such that $N_1 \in \mathcal{T}$, $N_2 \in \mathcal{F}$. In the following theorem, we give some examples of a module M where $\tau_{\delta V}$ splits. Also, in Corollary 4.3 we give more examples to such a module.

Theorem 3.2. *$\tau_{\delta V}$ splits in the following cases*:

(1) *Every δ–M-cosingular module is projective in $\sigma[M]$*;
(2) *M is a projective perfect module in $\sigma[M]$ and $\sigma[M]$ has no injective simple module*;
(3) *Every injective module in $\sigma[M]$ is lifting.*

Proof. (1) Let $N \in \sigma[M]$. Then $N/\overline{Z}_{\delta_M}(N)$ is projective in $\sigma[M]$. Let A be a submodule of N such that $N = \overline{Z}_{\delta_M}(N) \oplus A$. We have $\overline{Z}_{\delta_M}(A) = 0$. Then $\overline{Z}_{\delta_M}(N) = \overline{Z}^2_{\delta_M}(N)$ and hence $\overline{Z}_{\delta_M}(N) \in \mathcal{T}_{\delta V}$ and $A \in \mathcal{F}_{\delta V}$.

(2) By Talebi and Vanaja (2002, Theorem 3.8), $\mathcal{F}_V = \sigma[M]$ and then $\mathcal{F}_{\delta V} = \sigma[M]$.

(3) If every injective module is lifting and $N \in \sigma[M]$, then $N = E \oplus S$ where E is injective and S is M-small by Jayaraman and Vanaja (2000). Also $E = \bigoplus_{i\in I} E_i$ where each E_i is a local module. Let $I_1 = \{i \in I : \overline{Z}_{\delta_M}(E_i) = E_i\}$ and $I_2 = I - I_1$. Let $A = \bigoplus_{i\in I_1} E_i$ and $B = \bigoplus_{i\in I_2} E_i$. Then $\overline{Z}_{\delta_M}(A) = A$ and $\overline{Z}^2_{\delta_M}(B) = 0$ since every proper submodule of E_i is a small submodule. So $N = A \oplus (B \oplus S)$ where $A \in \mathcal{T}_{\delta V}$ and $B \oplus C \in \mathcal{F}_{\delta V}$. $\square$

Theorem 3.3. *Let M be a module such that every injective module in $\sigma[M]$ is amply supplemented. Then*:

(1) $\mathcal{T}_{\delta V} = \{N \in \sigma[M] \mid \overline{Z}^2_{\delta_M}(N) = N\}$;
(2) $\mathcal{F}_{\delta V} = \{N \in \sigma[M] \mid \overline{Z}^2_{\delta_M}(N) = 0\}$;
(3) $\rho_{\delta V} = \overline{Z}^2_{\delta_M}(N)$ *for all* $N \in \sigma[M]$, *and*;
(4) $\tau_{\delta V}$ *is cohereditary.*

Proof. For any $N \in \sigma[M]$, $\overline{Z}_{\delta_M}(N) = N \Leftrightarrow \overline{Z}^2_{\delta_M}(N) = N$, by definitions. Also if $\overline{Z}^2_{\delta_M}(N) = 0$, then $N \in \mathcal{F}_{\delta V}$. The converse holds by Theorem 2.18. Hence (1) and (2) are proved. (3) and (4) also follow from Theorem 2.18. $\square$

4. GCO-MODULES

In this section we give some characterizations of GCO-modules and then reach some other conditions so that $\tau_{\delta V}$ splits in addition to Theorem 3.2. Now we recall the definition.

M is called a *GCO-module* if every singular simple module is M-projective or M-injective (Dung et al., 1994). M is a GCO-module if and only if every M-singular simple module is M-injective (Dung et al., 1994, 16.4). In Özcan (2002) it is proved that M is a GCO-module if and only if every M-small module in $\sigma[M]$ is M-projective.

Theorem 4.1. *Let M be a module. The following are equivalent*:

(1) *M is a GCO-module*;
(2) *Every M-singular module is non-δ–M-cosingular*;
(3) *Every M-singular module is non-M-cosingular.*

Proof. (1) $\Rightarrow$ (2) Let $0 \neq N$ be an M-singular module and X be a proper submodule of N such that N/X is δ–M-small. By Dung et al. (1994, 16.3 and 16.4), every module in $\sigma[M]$ is a GCO-module and every submodule of a module has a maximal submodule. Then there exists a maximal submodule L of N such that $X \leq L$. Then N/L is simple M-singular. By hypothesis N/L is M-injective. Since N/L is a homomorphic image of N/X, N/L is M-small. But simple M-small modules can not be M-injective. Hence N is non-δ–M-cosingular.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) Let N be an M-singular simple module. By (3) N is non-M-cosingular. If N is M-small, then $\overline{Z}_M(N) = 0$, a contradiction. Hence N is M-injective. $\square$

Theorem 4.2. *Let M be a module. Consider the following conditions*:

(1) *M is a GCO-module*;
(2) *Every M-cosingular module is projective in $\sigma[M]$*;
(3) *Every δ–M-cosingular module is projective in $\sigma[M]$*;
(4) *Every module in $\sigma[M]$ is a direct sum of non-M-cosingular and a semisimple module*; *and*
(5) *Every module in $\sigma[M]$ is a direct sum of non-δ–M-cosingular and a semisimple module.*

Then $(5) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$ *and* $(5) \Rightarrow (4) \Rightarrow (2)$. *If in addition, the class of δ–M-cosingular modules is closed under homomorphic images, then* (1)–(5) *are all equivalent.*

Proof. $(5) \Rightarrow (4)$ It is clear.

$(4) \Rightarrow (2)$ Let N be an M-cosingular module in $\sigma[M]$ and $f : K \to N$ be an epimorphism where $K \in \sigma[M]$. Then K has a decomposition $K = A \oplus B$ where A is non-M-cosingular and B is semisimple. Since $f(A) \leq \overline{Z}_M(N) = 0$, $A \leq \ker f$. Then $\ker f = A \oplus (B \cap \ker f)$. Since B is semisimple, $\ker f$ is a direct summand of K. It follows that N is projective.

$(5) \Rightarrow (3)$ It can be seen as in $(4) \Rightarrow (2)$.

$(3) \Rightarrow (2)$ It is clear.

$(2) \Rightarrow (1)$ Let N be an M-singular simple module in $\sigma[M]$. If N is M-small then N is M-cosingular. By (2), N is projective in $\sigma[M]$, a contradiction. So N is M-injective.

Now assume the class of δ–M-cosingular modules is closed under homomorphic images.

$(1) \Rightarrow (3)$ Let N be a δ–M-cosingular module. We claim that N is semisimple. Let $x \in N$ and K be a maximal submodule of xR. Then xR/K is simple. If xR/K is M-singular, then $\overline{Z}_{\delta_M}(xR/K) = xR/K$ by Theorem 4.1. But since N is δ–M-cosingular, by assumption, $\overline{Z}_{\delta_M}(xR/K) = 0$, a contradiction. Hence xR/K is projective in $\sigma[M]$. It follows that xR is semisimple, i.e., N is semisimple. By the above theorem, N is projective in $\sigma[M]$.

$(2) \Rightarrow (4)$ Since the class of δ–M-cosingular modules is closed under homomorphic images, every δ–M-cosingular module is semisimple. Then by Talebi and Vanaja (2002, Theorem 3.8), (4) holds.

$(3) \Rightarrow (5)$ It can be seen as in $(2) \Rightarrow (4)$. □

Corollary 4.3. *Let M be a GCO-module. If* a) *the class of δ–M-cosingular modules is closed under homomorphic images, or* b) *$\tau_{\delta V}$ is cohereditary, then $\tau_{\delta V}$ and τ_V split.*

Proof. The proof of (a) follows from the equivalence of (1), (4), and (5) of Theorem 4.2.

By the hypothesis of (b), $\mathscr{F}_{\delta V}$ is closed under homomorphic images. As in the proof of (1) $\Rightarrow$ (3) of Theorem 4.2, every module in $\mathscr{F}_{\delta V}$ and hence in $\mathscr{F}_V$ is semisimple and projective. $\square$

Corollary 4.4. *The class of δ–M-cosingular modules is closed under homomorphic images, if* (1) *every injective module in $\sigma[M]$ is amply supplemented and M is a GCO-module, or* (2) *M has a projective cover in $\sigma[M]$ and $\overline{Z}_{\delta_M}(M) = M$.*

Proof. (1) Let N be a δ–M-cosingular module. We claim that N is semisimple. Let $x \in N$ and K be a maximal submodule of xR. If xR/K is M-singular, then $\overline{Z}_M(xR/K) = xR/K$ by Theorem 4.1. Then $\overline{Z}^2_M(xR/K) = xR/K$. But since $\overline{Z}_M(xR) = 0$, $\overline{Z}^2_M(xR/K) = 0$ by Talebi and Vanaja (2002, Theorem 3.6), a contradiction. Hence xR/K is projective in $\sigma[M]$. It follows that xR is semisimple, i.e., N is semisimple. By Theorem 4.1, N is projective in $\sigma[M]$.

Let X be a submodule of N and consider the natural epimorphism $\Pi : N \to N/X$. Then $\Pi(\overline{Z}^2_M(N)) = \overline{Z}^2_M(N/X) = 0$ by Theorem 2.18. Since $\overline{Z}_M(N/X)$ is a direct summand of N/X by above, we have $\overline{Z}^2_M(N/X) = \overline{Z}_M(N/X) = 0$.

(2) By Proposition 3.1. $\square$

REFERENCES

Dung, N. V., Huynh, D. V., Smith, P. F., Wisbauer, R. (1994). *Extending Modules*. Pitman Research Notes in Mathematics Series 313. Harlow: Longman Scientific & Technical.

Jayaraman, M., Vanaja, N. (2000). Harada modules. *Comm. Alg.* 28(8):3703–3726.

Gomez Pardo, J. L., Asensio, G., Pedro, A. (2000). Indecomposable decompositions of $\aleph$-$\sum$-CS-modules. *Algebra and Its Applications* (Athens, OH, 1999). Contemp. Math. 259. Providence, RI: Amer. Math. Soc., pp. 467–473.

Gomez Pardo, J. L., Asensio, G., Pedro, A. (2001). Every Σ-CS-module has an indecomposable decomposition. *Proc. Amer. Math. Soc.* 129(4):947–954.

Mohamed, S. H., Müller, B. J. (1990). *Continuous and Discrete Modules*. London Math. Soc. LN. 147. New York, Sydney: Cambridge University Press.

Özcan, A. Ç. (2002). On GCO-modules and M-small modules. *Commun. Fac. Sci. Univ. Ankara Series A1* 51(2):25–36.

Özcan, A. Ç., Harmanci, A. (2003). The torsion theory generated by M-small modules. *Alg. Coll.* 10(1):41–52.

Talebi, Y., Vanaja, N. (2002). The torsion theory cogenerated by M-small modules. *Comm. Alg.* 30(3):1449–1460.

Wisbauer, R. (1991). *Foundations of Module and Ring Theory*. Reading: Gordon and Breach.

Zhou, Y. (2000). Generalizations of perfect, semiperfect and semiregular rings. *Alg. Coll.* 7(3):305–318.