DIRECT SUMS OF MODULES HAVING (S*)

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Abstract

A module M is said to satisfy the property (S*) if every submodule N of M is cosingular of a direct summand of M. In this study we investigate when a finite direct sum of modules with (S*) satisfies (S*). We prove that a module M is a direct sum of modules satisfying (S*) and $Z^*(M)$ has finite uniform dimension if and only if $M = M_1 \oplus M_2 \oplus M_3$ where M_1 is semisimple with $Z^*(M_1) = 0$, M_2 has finite uniform dimension with $Z^*(M_2) = M_2$ and M_3 has finite uniform dimension and is a finite direct sum of local submodules of M.

1. PRELIMINARIES

Direct sums of lifting modules have been studied by several authors for example [5, 7]. In [8], the property (S*) was introduced as a generalization of lifting modules. A module M is said to satisfy the property (S*) if every submodule N of M is cosingular of a direct summand of M. In this note we are interested in direct sums of modules with (S*). We prove that a direct sum of a semisimple module and a module with (S*) also satisfies (S*). Similarly, we show that a finite direct sum of projective modules with (S*) also satisfies the property (S*). On the other hand for a module M descending (ascending) chain conditions on small modules which are submodules of M are investigated. We prove that a module M is a direct sum of modules satisfying (S*) and $Z^*(M)$ has finite uniform dimension if and only if $M = M_1 \oplus M_2 \oplus M_3$ where M_1 is semisimple with $Z^*(M_1) = 0$, M_2 has finite uniform dimension with $Z^*(M_2) = M_2$ and M_3 has finite uniform dimension and is a finite direct sum of local submodules of M.

Throughout this note all rings have identity and all modules are unital right modules. Let R be a ring and M be a right R-module. For a small (essential) submodule N of M, we write $N \ll M$ ($N \leq_e M$). M is called a *small module* if it is a small submodule of some R-module. M is small if and only if M is small in its injective hull $E(M)$ [4]. We put

$$Z^*(M) = \{m \in M : mR \ll E(mR)\}$$

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The Jacobson radical Rad$M$ is a submodule of $Z^*(M)$. For further properties of $Z^*(.)$ see [8]. We call a module M *cosingular* if $Z^*(M) = M$. A ring R is called *right cosingular* if the right R-module R is cosingular. Clearly small modules are cosingular.

A module M is called *lifting* (or *a (D1)-module*) if for every submodule N of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll M$ (for example [5]). We shall say that M satisfies

(S*) if for every submodule N of M there exists a direct summand K of M such that $K \leq N$ and N/K is cosingular. [8]

A ring R satisfies (S*) if the (right) R-module R satisfies (S*). Lifting modules satisfy (S*). But for the converse, let R be the ring of integers Z. Since $Z^*(R) = R$ as an R-module, R satisfies (S*), but R is not lifting [5, p.56]. Note that every Z-module is cosingular and hence satisfies (S*).

Lemma 1 [8, Lemma 3.1] *Let M be an R-module. The following are equivalent.*
(i) M satisfies (S),*
(ii) For every submodule N of M, M has a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2$ is cosingular,
(iii) For every submodule N of M, N has a decomposition $N = N_1 \oplus N_2$ such that N_1 is a direct summand of M and N_2 is cosingular.

The class of modules satisfying (S*) is closed under submodules. If M satisfies (S*) and $Z^*(M) \ll M$ then M is lifting [8, Lemma 3.3]. If M satisfies (S*) then $M = M_1 \oplus M_2$ such that M_1 is semisimple with $Z^*(M_1) = 0$ and $Z^*(M_2) \leq_e M_2$ [8, Corollary 3.6].

Oshiro [7] calls a ring R a *right H-ring* if every injective right R-module is lifting. If every proper submodule of M is a small submodule then M is called *hollow*.

2. FINITE DIRECT SUMS

Example 2 *A finite direct sum of modules with (S*) does not necessarily satisfy (S*).*

Proof Let R be a right Artinian ring such that every indecomposable injective R-module is hollow but R is not a right H-ring (for the existence see [3, Example 5]). Then every indecomposable injective module is cyclic by [10, 41.4]. This implies that there exists a finitely generated injective R-module E which is not lifting by [7, Remark p.318]. Since Rad$E = Z^*(E) \ll E$, E does not satisfy (S*). Since R is right Artinian, E is a finite direct sum of indecomposable injective modules E_i [2, Theorem 25.6]. By hypothesis and since hollow modules are lifting, each E_i satisfies (S*). Hence E is the finite direct sum of modules with (S*) but E does not satisfy (S*). □

Theorem 3 *Let $M = M_1 \oplus M_2$ where M_1 is semisimple and M_2 satisfies (S^*). Then M satisfies (S^*).*

Proof Let $M = M_1 \oplus M_2$ where M_1 is semisimple and M_2 satisfies (S^*). Let $N \leq M$. Then $M_1 = (N \cap M_1) \oplus M'$ for some $M' \leq M_1$. Thus $M = (N \cap M_1) \oplus M' \oplus M_2$ and $N = (N \cap M_1) \oplus A$ where $A = N \cap (M' \oplus M_2)$. Since $(M_2 \oplus M')/M'$ satisfies (S^*), it follows that $(A + M')/M' = K/M' \oplus L/M'$ for some submodules K and L containing M' such that K/M' is a direct summand of $(M_2 \oplus M')/M'$ and, L/M' is cosingular. Thus K is a direct summand of M. But $K = M' \oplus (K \cap A)$, so that $K \cap A$ is also a direct summand of M. It is now clear that $(N \cap M_1) \oplus (K \cap A)$ is a direct summand of M. Moreover

$$N/((N \cap M_1) \oplus (K \cap A)) \cong A/(K \cap A) \cong (A + K)/K = (A + M')/K \cong L/M'$$

is cosingular. It follows that M satisfies (S^*). □

Corollary 4 *Let $M = M_1 \oplus M_2$ where M_1 is semisimple and M_2 is cosingular. Then M satisfies (S^*).*

Let P and M be modules. P is said to be *M-projective* if for any module N with an epimorphism $\pi : M \to N$ and homomorphism $\theta : P \to N$, there exists a homomorphism $\theta' : P \to M$ such that $\pi\theta' = \theta$. P is called *projective* if it is M-projective for every module M. If P is P-projective, P is called *quasi-projective*. A class of modules $\mathcal{C} = \{P_i : i \in I\}$ is called *relatively projective* if P_i is P_j-projective for all distinct $i, j \in I$.

Lemma 5 [10, 41.14] *Let M_1 and M_2 be modules and $M = M_1 \oplus M_2$. The following are equivalent.*
(i) M_1 is M_2-projective,
(ii) For every submodule N of M such that $M = N + M_2$, there exists a submodule N' of N such that $M = N' \oplus M_2$.

The following theorem is a generalization of Lemma 5.1 in [8].

Theorem 6 *Let $M = M_1 \oplus M_2$ be a direct sum of quasi-projective, relatively projective modules M_1, M_2 such that M_1 and M_2 satisfy (S^*). Then M satisfies (S^*).*

Proof Let $L \leq M$.
Case 1. If $M_1 \cap (L + M_2) = 0$, then $L \leq M_2$. Since M_2 satisfies (S^*), there exists $B_1 \leq L$ such that $M_2 = B_1 \oplus B_2$ and $L \cap B_2$ is cosingular for some submodule B_2 of M_2. Hence $M = M_1 \oplus B_1 \oplus B_2$ and $L \cap (M_1 \oplus B_2) = L \cap B_2$ is cosingular. It follows that M satisfies (S^*).
Case 2. $M_1 \cap (L + M_2) \neq 0$, there exists $A_1 \leq M_1 \cap (L + M_2)$ such that $M_1 = A_1 \oplus A_2$ and $M_1 \cap (L + M_2) \cap A_2 = A_2 \cap (L + M_2)$ is cosingular because M_1 satisfies (S^*). Then $M = A_1 \oplus A_2 \oplus M_2 = L + (M_2 \oplus A_2)$.

If $M_2 \cap (L + A_2) = 0$, then $L \cap A_2 \leq A_2$ and since A_2 satisfies (S*) there exists $C_1 \leq L \cap A_2$ such that $A_2 = C_1 \oplus C_2$, $L \cap A_2 \cap C_2 = L \cap C_2$ is cosingular. Then $M = (A_1 \oplus C_1) \oplus (C_2 \oplus M_2) = L + (C_2 \oplus M_2)$. Since M_1 is $M_1 \oplus M_2$-projective, A_1 is $C_2 \oplus M_2$-projective and C_1 is $C_2 \oplus M_2$-projective by [5]. Then $A_1 \oplus C_1$ is $C_2 \oplus M_2$-projective. This implies that there exists $L' \leq L$ such that $M = L' \oplus C_2 \oplus M_2$, $L \cap (C_2 \oplus M_2) \leq C_2 \cap (L + M_2) = L \cap C_2$ is cosingular. Hence M satisfies (S*).

If $M_2 \cap (L + A_2) \neq 0$, there exists $B_1 \leq M_2 \cap (L + A_2)$, $M_2 = B_1 \oplus B_2$, $B_2 \cap (L + A_2)$ is cosingular. Then $M = L + (A_2 \oplus M_2) = (A_1 \oplus B_1) \oplus (A_2 \oplus B_2)$ and $L \cap (A_2 \oplus B_2)$ is cosingular because $A_2 \cap (L + M_2)$ and $B_2 \cap (L + A_2)$ are cosingular. Since $A_1 \oplus B_1$ is $A_2 \oplus B_2$-projective there exists $L' \leq L$ such that $M = L' \oplus A_2 \oplus B_2$. Hence M satisfies (S*) □

Corollary 7 *Let $M = M_1 \oplus M_2$ be a projective module such that M_1 and M_2 satisfy (S*). Then M satisfies (S*).*

R is semiperfect if and only if the right (left) R-module R is lifting [5, Corollary 4.42]. Hence semiperfect rings satisfy (S*). It is well known that if R is semiperfect then every finitely generated projective R-module is lifting. As a corollary of Theorem 6 we have the following result for a ring satisfying (S*).

Corollary 8 *Let R be a ring satisfying (S*). Then every finitely generated projective R-module satisfies (S*).*

Proof Let P be a finitely generated projective R-module. Then P is isomorphic to a direct summand of a free R-module. Since Theorem 6 holds for a finite direct sum of modules, P satisfies (S*). □

3. SOME CHAIN CONDITIONS FOR Z*(.)

Al-Khazzi and Smith [1] investigated some chain conditions that Rad$M$ satisfies for a module M. From now on we shall consider the similar results for $Z^*(M)$ for a module M.

Clearly if $Z^*(M)$ is Artinian (Noetherian) then Rad$M$ is Artinian (Noetherian). But if Rad$M$ is Artinian (Noetherian) $Z^*(M)$ need not. For example, let M denote $\sum_p Z(1/p)/Z$ where p ranges over all prime integers. Since each $Z(1/p)/Z$ is simple, M is a semisimple Z-module and hence Rad$M = 0$. But $Z^*(M) = M$ is not Noetherian, and then not Artinian.

The following three propositions can be seen by the proof of Proposition 2, Proposition 3 and Theorem 5 in [1]. But we give the proofs for convenience.

Proposition 9 *The following are equivalent for a module M.*
(i) $Z^*(M)$ is Noetherian.
(ii) Every small module in M is Noetherian.
(iii) The ascending chain condition holds on small modules in M.
(iv) M satisfies the ascending chain condition on cosingular submodules.

Proof (i) $\Leftrightarrow$ (ii); (i) $\Rightarrow$ (iii), (iv) Clear.
(iii) $\Rightarrow$ (i) By (iii), M has a maximal small module K. Then $K \leq \mathrm{Z}^*(M)$. Let $x \in \mathrm{Z}^*(M)$. Since a finite sum of small modules is small, $K + xR \ll \mathrm{E}(M)$. Then $K = K + xR$ and $x \in K$. It follows that $\mathrm{Z}^*(M) = K$ and hence $\mathrm{Z}^*(M)$ is Noetherian.
(iv) $\Rightarrow$ (i) Since the class of cosingular modules is closed under submodules the proof is completed as in (iii) $\Rightarrow$ (i). □

A module M is called *locally Artinian* if every finitely generated submodule of M is Artinian.

Proposition 10 *The following are equivalent for a module M.*
(i) $Z^*(M)$ is Artinian,
(ii) Every small module in M is Artinian,
(iii) The descending chain condition holds on small modules in M.
(iv) M satisfies the descending chain condition on cosingular submodules.

Proof (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (i) $\Rightarrow$ (iv) Clear.
(iv) $\Rightarrow$ (i) $\mathrm{Z}^*(M)$ is cosingular. Hence every submodule of $\mathrm{Z}^*(M)$ is cosingular. By (iv), $\mathrm{Z}^*(M)$ is Artinian.
(iii) $\Rightarrow$ (i) Let N be a finitely generated submodule of $\mathrm{Z}^*(M)$. Then N is a small module and hence N is Artinian. It follows that $\mathrm{Z}^*(M)$ is locally Artinian. Let K be any proper submodule of $\mathrm{Z}^*(M)$. Let $x \in \mathrm{Z}^*(M) \setminus K$. Then xR is Artinian and $(xR + K)/K$ is a non-zero Artinian module. It follows that $\mathrm{Z}^*(M)/K$ has essential socle.

Suppose that $\mathrm{Z}^*(M)$ is not Artinian. Then there exists a submodule L of $\mathrm{Z}^*(M)$ such that $\mathrm{Z}^*(M)/L$ is not finitely cogenerated [2, Proposition 10.10]. Let P be a minimal submodule of $\mathrm{Z}^*(M)$ with respect to $\mathrm{Z}^*(M)/P$ not finitely cogenerated (by Zorn's Lemma). Let $\mathrm{Soc}(\mathrm{Z}^*(M)/P) = S/P$ where $S \leq \mathrm{Z}^*(M)$. We have seen that S/P is an essential submodule of $\mathrm{Z}^*(M)/P$. Therefore S/P is not finitely generated by [2, Proposition 10.7].

We claim that $P \ll M$. Let $M = P + Q$ for some $Q \leq M$. Then $S = P + (S \cap Q)$. Suppose that $P \cap Q \neq P$. Then $\mathrm{Z}^*(M)/(P \cap Q)$ is finitely cogenerated by the choice of P. But $S/P = (P + (S \cap Q))/P \cong (S \cap Q)/(P \cap Q) \leq \mathrm{Soc}(\mathrm{Z}^*(M)/(P \cap Q))$ and hence S/P is finitely generated, a contradiction. Thus $P \ll M$.

Now we claim that $S \ll \mathrm{E}(M)$. Let $\mathrm{E}(M) = S + V$ for some submodule V of $\mathrm{E}(M)$. Then $\mathrm{E}(M)/(P + V) = (S + V)/(P + V) \cong S/(P + (S \cap V))$. Thus $\mathrm{E}(M)/(P + V)$ is semisimple. If $\mathrm{E}(M) \neq P + V$ then there exists a maximal submodule W of $\mathrm{E}(M)$ such that $P + V \leq W$. But $S \leq \mathrm{Z}^*(M) \leq \mathrm{Rad}\mathrm{E}(M) \leq W$ and this gives that $\mathrm{E}(M) = W$, a contradiction. Thus $\mathrm{E}(M) = P + V$. Since $P \ll M$, $P \ll \mathrm{E}(M)$. This implies that $\mathrm{E}(M) = V$. Thus $S \ll \mathrm{E}(M)$ and, by hypothesis S is Artinian. It follows that S/P is finitely generated, a contradiction. Thus $\mathrm{Z}^*(M)$ is Artinian. □

A module M has *finite uniform dimension* k, for some non-negative integer k if M does not contain any infinite direct sum of non-zero submodules and k is the maximal number of summands in a direct sum of non-zero submodules of M.

Proposition 11 *The following are equivalent for a module M.*
(i) $Z^*(M)$ has finite uniform dimension,
(ii) Every small module in M has finite uniform dimension and there exists a positive integer k such that uniform dimension of $N \leq k$ for every $N \leq M, N \ll E(M)$,
(iii) M does not contain an infinite direct sum of non-zero small modules.

Proof (i) $\Rightarrow$ (ii) It is clear because if $N \leq M$, $N \ll \mathrm{E}(N)$, then $N \leq \mathrm{Z}^*(M)$ and dimension of $N \leq k$ where k is the uniform dimension of $\mathrm{Z}^*(M)$.
(ii) $\Rightarrow$ (iii) Let $N_1 \oplus N_2 \oplus \cdots$ be an infinite direct sum of non-zero small modules in M. Then $N_1 \oplus \cdots \oplus N_{k+1}$ is a small module. This implies that the uniform dimension of $N_1 \oplus \cdots \oplus N_{k+1} \geq k+1$, a contradiction.
(iii) $\Rightarrow$ (i) Let $N_1 \oplus N_2 \oplus \cdots$ be an infinite direct sum of non-zero submodules of $\mathrm{Z}^*(M)$. Let $x_i \in N_i$ for each $i \geq 1$. Then $x_iR \ll \mathrm{E}(x_iR)$ $(i \geq 1)$. This implies that $x_1R + x_2R + \cdots$ is an infinite direct sum of non-zero small modules in M. Hence $\mathrm{Z}^*(M)$ has finite uniform dimension. □

A module M is called *local* if M is hollow and $\mathrm{Rad}M \neq M$. Clearly if M is a local module then $M = mR$ for all $m \in M, m \notin \mathrm{Rad}M$ [10].

Proposition 12 *Let M be a module and $Z^*(M) \neq M$. If $M = mR$ for all $m \in M, m \notin Z^*(M)$, then M is hollow.*

Proof If N is a proper submodule of M, then by hypothesis, $N \leq \mathrm{Z}^*(M)$ so that $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$ is small. Then N is also small. □

Corollary 13 *Suppose $Z^*(M) \neq M$ for a module M. Then the following are equivalent.*
(i) $M = mR$ for all $m \in M, m \notin Z^*(M)$,
(ii) M is local.

Theorem 14 *The following are equivalent for a module M.*
(i) M is a direct sum of modules satisfying (S^*) and $Z^*(M)$ has finite uniform dimension.
(ii) M is a direct sum $M = M_1 \oplus M_2 \oplus M_3$ where M_1 is semisimple with $Z^*(M_1) = 0$, M_2 is cosingular and has finite uniform dimension and M_3 has finite uniform dimension and is a finite direct sum of local submodules of M.

Proof (ii) $\Rightarrow$ (i) Cosingular modules and local modules satisfy (S^*). Then (i) holds since $\mathrm{Z}^*(\oplus M_i) = \oplus \mathrm{Z}^*(M_i) (i \in I)$ for any family of modules M_i [8, Lemma 2.3].
(i) $\Rightarrow$ (ii) Suppose that $\mathrm{Z}^*(M)$ has finite uniform dimension and $M = \oplus_{i \in I} M_i$ where, for each $i \in I$, M_i satisfies (S^*). Since $\mathrm{Z}^*(\oplus_{i \in I} M_i) = \oplus_{i \in I} \mathrm{Z}^*(M_i)$, $\mathrm{Z}^*(M_i) = 0$ for all but a finite number of elements $i \in I$. It follows that M_i is semisimple for all but a finite number of elements $i \in I$. Then $M = M_1 \oplus ... \oplus M_k \oplus S$, S is semisimple with $\mathrm{Z}^*(S) = 0$. Let N be a module such that

$Z^*(N) \leq_e N$ and N satisfies (S*). Since $Z^*(M)$ has finite uniform dimension then N has finite uniform dimension. Suppose that N is uniform and $Z^*(N) \neq N$. Let $m \in N \backslash Z^*(N)$. Since N satisfies (S*), then there exist submodules K and L of N such that $N = K \oplus L$, $K \leq mR$ and $mR/K = Z^*(mR/K)$. If $K = 0$ then $m \in Z^*(N)$, a contradiction. Thus $K \neq 0$ and hence $L = 0$. In this case $N = K = mR$. By Corollary 13 , N is a local module.

Now suppose that each submodule having dimension less than or equal to $n-1$ in M is a direct sum of cosingular submodule and local submodule. Suppose that n is the uniform dimension of N and $Z^*(N) \neq N$. Let $x \in N \backslash Z^*(N)$. There exist submodules K and L of N such that $N = K \oplus L$, $K \leq xR$ and $xR/K = Z^*(xR/K)$. Because $x \notin Z^*(N)$, it follows that $K \neq 0$. If $L \neq 0$ then K and L are both a direct sum of local submodule and cosingular submodule. And hence N is a direct sum of two local submodules and a cosingular submodule. Now suppose that $L \neq 0$ for all $x \in N$, $x \notin Z^*(N)$. Then $N = K = xR$. Thus N is a local module. □

Corollary 15 *Let M be a module which is a direct sum of modules, each of which satisfies (S*). Suppose that $Z^*(M)$ is Noetherian. Then $M = M_1 \oplus M_2$ for some semisimple module M_1 with $Z^*(M_1) = 0$ and Noetherian module M_2.*

Finally we give a decomposition of a module M satisfying (S*) under which condition $Z^*(M)$ has ascending chain condition (acc) (descending chain condition (dcc)) on direct summands.

Lemma 16 *Let M be a module such that $Z^*(M) \leq_e M$. Let M_1 and M_2 be direct summands of M with $M_1 \leq M_2$. $Z^*(M_1) = Z^*(M_2)$ if and only if $M_1 = M_2$.*

Proof Let $M = M_1 \oplus M_1'$. Then $M_2 = M_1 \oplus (M_2 \cap M_1')$ and $Z^*(M_2) = Z^*(M_1) \oplus Z^*(M_2 \cap M_1')$. If $Z^*(M_1) = Z^*(M_2)$, $Z^*(M_2 \cap M_1') = (M_2 \cap M_1') \cap Z^*(M) = 0$. This implies that $M_2 \cap M_1' = 0$, by hypothesis. Hence $M_1 = M_2$. □

Proposition 17 *Let M be a module such that $Z^*(M) \leq_e M$. If $Z^*(M)$ has acc (dcc) on direct summands, then M has acc (dcc) on direct summands.*

Proof Clear by Lemma 16. □

Theorem 18 *Let M be a module satisfying (S*). Assume that $Z^*(M)$ has acc (dcc) on direct summands. Then $M = M_1 \oplus M_2$ where M_1 is semisimple and M_2 is a finite direct sum of indecomposable modules L_i ($i \in F, F$ is finite) such that every proper submodule of L_i is cosingular.*

Proof Let M be a module satisfying (S*). Then $M = M_1 \oplus M_2$ where M_1 is semisimple with $Z^*(M_1) = 0$ and $Z^*(M_2) \leq_e M_2$. By Proposition 17, M_2 has acc (dcc) on direct summands. By [2, Proposition 10.14], M_2 is a finite direct sum of indecomposable modules L_i ($i \in F, F$ is finite). Let $i \in F$ and K be a proper submodule of L_i. Since L_i satisfies (S*) and it is indecomposable, K is cosingular.
□

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