# ON GCO-MODULES AND M-SMALL MODULES

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**Abstract**

Let $M$ be a right $R$-module. Define $\mathrm{Z}^*_M(N)$ ($\delta^*_M(N)$) to be the set of elements $n \in N$ for any $R$-module $N$ in $\sigma[M]$ such that $nR$ is an $M$-small (respectively $\delta$-$M$-small) module. In this note it is proved that $M$ is a GCO-module if and only if every $M$-small module in $\sigma[M]$ is $M$-projective if and only if every $\delta$-$M$-small module in $\sigma[M]$ is $M$-projective. Also, if $M/\delta^*_M(M)$ is semisimple then $M$ is a GCO-module if and only if $M$ is an SI-module.

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For a right $R$-module $M$, the submodule $\mathrm{Z}^*(M)$ is defined to be the set of elements $m \in M$ such that $mR$ is a small module (see [4]). Some further properties of $\mathrm{Z}^*(.)$ were studied in [4, 8, 9, 10]. In this paper we think this submodule in the category $\sigma[M]$, and therefore the corresponding definition of $\mathrm{Z}^*(.)$ in $\sigma[M]$ is defined by $\mathrm{Z}^*_M(N)$ to be the set of elements $n \in N$ for a module $N \in \sigma[M]$ such that $nR$ is $M$-small. In Section 1 we prove that $M$ is a GCO-module if and only if every $M$-small module in $\sigma[M]$ is $M$-projective (Theorem 1.5). Also if $M/\mathrm{Z}^*_M(M)$ is semisimple, then $M$ is a GCO-module if and only if $M$ is an SI-module if and only if $\mathrm{Z}^*_M(M)$ is semisimple $M$-projective (Theorem 1.12). In Section 2, we define $\delta$-$M$-small modules and $\delta^*_M(N)$ as a generalization of $M$-small modules and $\mathrm{Z}^*_M(N)$ in $\sigma[M]$ being inspired from [14]. Most of the results in Section 1 hold for $\delta$-$M$-small modules and $\delta^*_M(N)$ but the characterization of V-modules (Example 2.6).

Throughout this paper, $R$ will be an associative ring with unit and all modules be unitary right $R$-modules.

Let $M$ be an $R$-module. For a direct summand $N$ of $M$ we write $N \leq_d M$ and for essential submodule $N$ of $M$, $N \leq_e M$.

An $R$-module $N$ is *subgenerated* by $M$ if $N$ is isomorphic to a submodule of an $M$-generated module. $\sigma[M]$ is denoted by the full subcategory of Mod-$R$ whose objects are all $R$-modules subgenerated by $M$ [12].

Let $\widehat{N}$ be the $M$-injective hull of $N$ in $\sigma[M]$ and let $\mathrm{E}(M)$ be an $R$-injective hull of $M$.

A module $N$ in $\sigma[M]$ is called *$M$-singular* (or *singular in $\sigma[M]$*) if $N \cong L/K$ for an $L \in \sigma[M]$ and $K \leq_e L$ (see [3]). In case $M = R$, instead of $R$-singular, we just say *singular*. Every module $N \in \sigma[M]$ contains a largest $M$-singular submodule which is denoted by $Z_M(N)$.

Let $\mathcal{G}(M)$ be the singular torsion theory in $\sigma[M]$, that is, $\mathcal{G}(M)$ is the smallest torsion class in $\sigma[M]$ which contains all $M$-singular modules (see [11]). $\mathcal{G}(M)$ is closed under $M$-injective hulls by [11, 2.4(3)], and hence $\mathcal{G}(M) = \{N \in \sigma[M] : Z_M(N) \leq_e N\}$.

Following Hirano a module $M$ is called a *V-module* (or *co-semisimple*) if every simple module (in $\sigma[M]$) is $M$-injective. A module $M$ is called a *GV-module* if every singular simple module is $M$-injective. $M$ is a GV-module if and only if every simple module is projective or $M$-injective [5]. As a generalization of GV-modules a module $M$ is called a *GCO-module* if every singular simple module is $M$-projective or $M$-injective [3]. $M$ is a GCO-module if and only if every $M$-singular simple module is $M$-injective [3, 16.4]. Obviously any V-module is a GV-module and any GV-module is a GCO-module. $M$ is called an *SI-module* if every $M$-singular module is $M$-injective [3]. Clearly SI-modules are GCO-modules. Note that a right GCO-ring coincides with a right GV-ring.

# 1 $M$-small Modules

Let $K$ be a submodule of a module $M$. $K$ is called *small* in $M$ if $K + L \neq M$ holds for every proper submodule $L$ of $M$ and denoted by $K \ll M$. We write Rad$M$, which is the sum of all small submodules in $M$, for the radical of $M$ (see [1]).

An $R$-module $N$ is called *$M$-small* (or *small in $\sigma[M]$*) if $N \cong K \ll L$ for $K, L \in \sigma[M]$. Note that $M$-small modules are dual notion to that of $M$-singular modules. In case $M = R$, instead of $R$-small, we just say *small*. $M$-small modules are small, since the class of small modules is closed under isomorphism. An $R$-

module $N$ is $M$-small if and only if $N \ll \widehat{N}$. Every simple $R$-module is $M$-injective or $M$-small. The class of $M$-small modules is closed under submodules, homomorphic images and finite direct sums. (see [6])

Let $M$ be an $R$-module. Denote

$$\mathrm{Z}^*_M(N) = \{n \in N : nR \quad \text{is } M\text{-small}\}$$

for an $R$-module $N \in \sigma[M]$. In case $M = R$, we write $\mathrm{Z}^*(N)$ instead of $\mathrm{Z}^*_R(N)$.

Let $N \in \sigma[M]$. Then it can be easily seen that

$$\mathrm{Rad}N \leq \mathrm{Z}^*_M(N) \leq \mathrm{Z}^*(N).$$

If $N$ is $M$-small, then $\mathrm{Z}^*_M(N) = N$. Since $\sigma[N] \subseteq \sigma[M]$, we also have $\mathrm{Z}^*_N(X) \leq \mathrm{Z}^*_M(X)$ for any module $X \in \sigma[M]$.

**Lemma 1.1** *Let $M$ be a module. Then*
*a) $Z^*_M(N) = Rad\widehat{N} \cap N$ for any $N \in \sigma[M]$.*
*b) Let $N \in \sigma[M]$. For any submodule $K$ of $N$, $Z^*_M(K) = K \cap Z^*_M(N)$.*
*c) Let $f : N \to K$ be a homomorphism of modules $N, K$ where $N, K \in \sigma[M]$. Then $f(Z^*_M(N)) \leq Z^*_M(K)$.*
*d) Let $N_i$ $(i \in I)$ be any collection of modules in $\sigma[M]$ and let $N = \oplus_{i \in I} N_i$. Then $Z^*_M(N) = \oplus_{i \in I} Z^*_M(N_i)$.*

**Proof** (a) and (b) are clear. (c) and (d) can be obtained by the similar techniques of [10, Lemma 2.1 and 2.3]. □

Now we give a lemma showing some properties of $\mathrm{Z}^*_M(.)$ in case it is zero.

**Lemma 1.2** *Let $N \in \sigma[M]$. Then*
*a) $Z^*_M(N) = 0$ if and only if $Rad\widehat{N} = 0$.*
*b) $Z^*_M(N) = 0$ if and only if $Z^*_K(N) = 0$ for every $K \in \sigma[M]$ with $N \in \sigma[K]$.*

**Proof** a) By Lemma 1.1 and, since $N \leq_e \widehat{N}$.
b) Suppose that $\mathrm{Z}^*_M(N) = 0$, and let $K \in \sigma[M]$ with $N \in \sigma[K]$ and $x \in Z^*_K(N)$. Then $xR$ is $K$-small, i.e. $xR \cong L \ll T$ for some $L, T \in \sigma[K]$. Since $K \in \sigma[M]$, $L, T \in \sigma[M]$. This implies that $xR$ is $M$-small. Thus $x \in Z^*_M(N) = 0$. Converse is open. □

Since $\mathrm{Z}^*_M(.)$ is related with the radical of a module then one may think whether the results hold for radicals of modules are true for $\mathrm{Z}^*_M(.)$. Therefore here we consider V-modules and GCO-modules by being encouraged from [12, 23.1] and [3, 16.4].

**Theorem 1.3** *The following are equivalent for a module $M$.*
*a) $M$ is a V-module,*
*b) $Z^*_M(N) = 0$ for every module $N \in \sigma[M]$,*
*c) $Z^*_M(N) = 0$ for every factor module $N$ of $M$.*

**Proof** Since $Z^*_M(N) = \text{Rad}\widehat{N} \cap N$ for $N \in \sigma[M]$, it is clear from [12, 23.1]. $\square$

Let $N \in \sigma[M]$. $N$ is called *cogenerator in* $\sigma[M]$ if there exists a monomorphism $N \to \prod_\Lambda M_\lambda$ with modules $M_\lambda \in \sigma[M]$ [12]. A module $M$ is called *locally noetherian* if every finitely generated submodule of $M$ is noetherian.

**Theorem 1.4** *Let $M$ be a locally noetherian module. The following are equivalent.*
*a) $M$ is a V-module,*
*b) $\sigma[M]$ has a semisimple $M$-injective cogenerator,*
*c) $\sigma[M]$ has a cogenerator $Q$ with $Z^*_M(Q) = 0$.*

**Proof** It is clear from [12, 23.1]. $\square$

**Theorem 1.5** *The following are equivalent for a module $M$.*
*a) $M$ is a GCO-module,*
*b) For every module $N \in \sigma[M]$, $Z^*_M(N)$ is $M$-projective,*
*c) Every $M$-small module in $\sigma[M]$ is $M$-projective,*
*d) For every module $N \in \sigma[M]$, $Z_M(N) \cap Z^*_M(N) = 0$,*
*e) For every simple module $E \in \sigma[M]$, $Z_M(\widehat{E}) \cap Z^*_M(\widehat{E}) = 0$,*
*f) $M/Soc(M)$ is a V-module and $Z_M(M) \cap Z^*_M(M) = 0$,*
*g) $Z^*_M(M/K) = 0$ for every $K \leq_e M$ and $Z_M(M) \cap Z^*_M(M) = 0$,*
*h) Every non-zero module $N$ with $Z^*_M(N) = N$ contains a non-zero $M$-projective submodule,*
*i) For every module $N \in \sigma[M]$ with $Z_M(N) \leq_e N$ (i.e. $N \in \mathcal{G}(M)$), $Z^*_M(N) = 0$.*

**Proof** $(a) \Rightarrow (b)$ Since simple modules in $\sigma[M]$ splits into four disjoint classes by combining the exclusive choices [$M$-projective or $M$-singular] and [$M$-injective or $M$-small], one deduces that $M$ is a GCO-module if and only if every $M$-small simple module is $M$-projective. So, let $n \in Z^*_M(N)$ for $N \in \sigma[M]$ and $K$ be a maximal submodule of $nR$. Then $nR/K$ is simple and $M$-projective. This implies that $K \leq_d nR$. Hence $nR$ and then $Z^*_M(N)$ is semisimple. By [7, Proposition 4.32], $Z^*_M(N)$ is $M$-projective.
$(b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$ It is clear.
$(e) \Rightarrow (a)$ It follows from [3, 16.4 (d)$\Rightarrow$ (a)].
$(d) \Rightarrow (g)$ Let $K \leq_e M$. Then $M/K$ is $M$-singular. This implies that $Z_M(M/K) = M/K$. By hypothesis, $Z^*_M(M/K) = 0$.
$(g) \Leftrightarrow (f)$ It follows from [3, 16.1 (a)$\Leftrightarrow$ (d)].
$(f) \Rightarrow (a)$ It follows from [3, 16.4 (e)$\Rightarrow$ (a)].
$(b) \Rightarrow (h)$ It is clear.
$(h) \Rightarrow (a)$ Let $N$ be an $M$-singular simple module in $\sigma[M]$. If $N$ is $M$-small then

$N$ contains a non-zero $M$-projective module $P$ in $\sigma[M]$. Since $N$ is simple $N = P$ and then $N$ is projective and $M$-singular in $\sigma[M]$, a contradiction. Hence $N$ is $M$-injective.
$(d) \Rightarrow (i)$ It is clear.
$(i) \Rightarrow (d)$ Let $0 \neq n \in \mathrm{Z}_M(N) \cap \mathrm{Z}^*_M(N)$. Then $nR$ is $M$-singular and $M$-small. Since $nR = \mathrm{Z}_M(nR) \leq_e nR$, $\mathrm{Z}^*_M(nR) = 0$ by hypothesis, a contradiction. □

If we consider the GCO-modules with ascending (descending) chain condition on essential submodules we have the following corollaries. First one is a generalization of [3, 16.13 (1)].

**Corollary 1.6** *The following are equivalent for a module $M$.*
*a) $M$ is a GCO-module with ascending chain condition on essential submodules,*
*b) $M/SocM$ is a V-module and Noetherian, $Z_M(M) \cap Z^*_M(M) = 0$.*

**Proof** By Theorem 1.5 and [3, 5.15]. □

**Corollary 1.7** *For a module $M$ with $M/SocM$ finitely generated, the following are equivalent.*
*a) $M$ is a GCO-module with descending chain condition on essential submodules,*
*b) $M/SocM$ is semisimple, $Z_M(M) \cap Z^*_M(M) = 0$.*

**Proof** By Theorem 1.5, [3, 5.15] and [1, Proposition 10.15]. □

GV-modules can be characterized by replacing $\mathrm{Z}_M(N)$ by the singular submodule $\mathrm{Z}(N)$ and $M$-projectivity by projectivity in Theorem 1.5.

**Theorem 1.8** *The following are equivalent for a module $M$.*
*a) $M$ is a GV-module,*
*b) For every module $N \in \sigma[M]$, $Z^*_M(N)$ is projective,*
*c) Every $M$-small module in $\sigma[M]$ is projective,*
*d) For every module $N \in \sigma[M]$, $Z(N) \cap Z^*_M(N) = 0$,*
*e) For every simple module $E \in \sigma[M]$, $Z(\widehat{E}) \cap Z^*_M(\widehat{E}) = 0$,*
*f) $M/Soc(M)$ is a V-module and $Z(M) \cap Z^*_M(M) = 0$,*
*g) $Z^*_M(M/K) = 0$ for every $K \leq_e M$ and $Z(M) \cap Z^*_M(M) = 0$,*
*h) Every non-zero module $N$ with $Z^*_M(N) = N$ contains a non-zero projective submodule,*
*i) For every module $N \in \sigma[M]$ with $Z(N) \leq_e N$, $Z^*_M(N) = 0$.*

**Example 1.9** *If $M$ is a GV-module, $Z(M) \cap RadM = 0$ but $Z(M) \cap Z^*(M)$ need not be zero in general.*

**Proof** Let $M = \mathrm{Z}/2\mathrm{Z}$. $M$ is simple and hence a GV-module. Also $\mathrm{Z}(M) \cap \mathrm{Rad}M = 0$. But $\mathrm{Z}(M) \cap \mathrm{Z}^*(M) = M$ since $M$ is singular and small Z-module. □

Applying Theorem 1.8 to $M = R$, we immediately have the following corollary which is a generalization of [8, Theorem 10].

**Corollary 1.10** *The following are equivalent for a ring $R$.*
*a) $R$ is a right GV-ring,*
*b) For every $R$-module $M$, $Z^*(M)$ is projective,*
*c) Every small module is projective,*
*d) For every $R$-module $M$, $Z(M) \cap Z^*(M) = 0$,*
*e) For every simple module $S$, $Z(E(S)) \cap Z^*(E(S)) = 0$.*
*f) $R/Soc(R)$ is a V-module and $Z(R_R) \cap Z^*(R_R) = 0$,*
*g) $Z^*(R/K) = 0$ for every essential right ideal $K$ of $R$ and $Z(R_R) \cap Z^*(R_R) = 0$,*
*h) Every non-zero $R$-module $M$ with $Z^*(M) = M$ contains a non-zero projective submodule,*
*i) For every $R$-module $M$ with $Z(M) \leq_e M$, $Z^*(M) = 0$.*

**Theorem 1.11** *Let $M$ be a module with $M/Z^*_M(M)$ a V-module. Then the following are equivalent.*
*a) $M$ is a GCO-module,*
*b) $Z^*_M(M)$ is semisimple $M$-projective.*

**Proof** $(a) \Rightarrow (b)$ By Theorem 1.5.
$(b) \Rightarrow (a)$ Since $\mathrm{Z}^*_M(M)$ is semisimple, $\mathrm{Z}^*_M(M) \leq \mathrm{Soc}(M)$. Then by hypothesis, $M/\mathrm{Soc}(M)$ is a V-module. $\mathrm{Z}_M(M) \cap \mathrm{Rad}M$ is a direct summand of $\mathrm{Z}^*_M(M)$. Since $\mathrm{Z}^*_M(M)$ is $M$-projective, we have $\mathrm{Z}_M(M) \cap \mathrm{Rad}M = 0$. By [3, 16.4], $M$ is a GCO-module. □

In [3, 17.5], we do not need the condition that $M$ is self-projective.

**Theorem 1.12** *Let $M$ be a module with $M/Z^*_M(M)$ semisimple. Then the following are equivalent.*
*a) $M$ is a GCO-module,*
*b) $M$ is an SI-module,*
*c) $Z^*_M(M)$ is semisimple $M$-projective.*

**Proof** $(a) \Leftrightarrow (c)$ By Theorem 1.11.
$(b) \Rightarrow (a)$ Clear.
$(c) \Rightarrow (b)$ Since $\mathrm{Z}^*_M(M) \leq \mathrm{Soc}(M)$, $M/\mathrm{Soc}M$ is semisimple. Let $K \leq_e M$. Then $\mathrm{Soc}M \leq K$. This implies that $M/K$ is semisimple. On the other hand, since finitely generated $M$-singular modules can not be $M$-projective, we have $\mathrm{Z}_M(M) \cap \mathrm{Rad}(M) = 0$. Thus $M$ is an SI-module by [3, 17.2]. □

## 2 $\delta$-$M$-small Modules

In this section, we define $\delta$-$M$-small modules and use them to characterize GCO-modules.

Zhou [14] introduced the concept "$\delta$-small submodule" as a generalization of small submodule. Let $N$ be a submodule of a module $M$. $N$ is called *$\delta$-small in $M$* if whenever $M = N + K$ and $M/K$ is singular for any $K \leq M$ we have $M = K$, denoted by $N \ll_\delta M$. Here we consider this definition in the category $\sigma[M]$ for a module $M$.

**Definition 2.1** Let $N \leq K \in \sigma[M]$. $N$ is called a *$\delta$-$M$-small submodule of $K$* in $\sigma[M]$ if whenever $K = N + X$ and $K/X$ is $M$-singular for $X \leq K$ we have $K = X$, we denoted by $N \ll_{\delta_M} K$.

For modules $N, K \in \sigma[M]$, $N \ll_\delta K \Rightarrow N \ll_{\delta_M} K$. The properties of $\delta$-small submodules that are listed in Lemma 1.3 in [14] also hold in $\sigma[M]$. We write them for convenience. Note that the class of $M$-singular modules is closed under submodules, homomorphic images and direct sums [3].

**Lemma 2.2** *Let $N \in \sigma[M]$.*
*a) For modules $K, L \in \sigma[M]$ with $K \leq L \leq N$ we have*
*$L \ll_{\delta_M} N$ if and only if $K \ll_{\delta_M} N$ and $L/K \ll_{\delta_M} N/K$.*
*b) For $K, L \in \sigma[M]$,*
*$K + L \ll_{\delta_M} N$ if and only if $K \ll_{\delta_M} N$ and $L \ll_{\delta_M} N$.*
*c) If $K \ll_{\delta_M} N$ and $f : N \to L$ is a homomorphism, then $f(K) \ll_{\delta_M} L$.*
*In particular, if $K \ll_{\delta_M} N \leq L$ then $K \ll_{\delta_M} L$.*
*d) If $K \leq L \leq_d N \in \sigma[M]$ and $K \ll_{\delta_M} N$ then $K \ll_{\delta_M} L$.*

As a generalization of $M$-small module we define $\delta$-$M$-small module.

**Definition 2.3** Let $N \in \sigma[M]$. $N$ is called a *$\delta$-$M$-small module in $\sigma[M]$* if $N \cong K \ll_{\delta_M} L \in \sigma[M]$.

The following equivalence can be seen similarly as it is for $M$-small modules. For $M$-small modules it is proved in [6].

**Lemma 2.4** *$N$ is a $\delta$-$M$-small module in $\sigma[M]$ if and only if $N \ll_{\delta_M} \widehat{N}$.*

**Proof** It is enough to show that if $N$ is $\delta$-$M$-small then $N \ll_{\delta_M} \widehat{N}$. Let $K, L \in \sigma[M]$ be such that $N \cong K \ll_{\delta_M} L$. Since $\widehat{K}$ is injective in $\sigma[M]$, there exists a homomorphism $f : L \to \widehat{K}$ such that $fi = g$ where $i : K \to L$ and $g : K \to \widehat{K}$ are inclusion maps. Since $K \ll_{\delta_M} L$, $K = f(K) \ll_{\delta_M} \widehat{K}$. This implies that $N \ll_{\delta_M} \widehat{N}$. □

If $N$ is an $M$-small module then it is $\delta$-$M$-small. The class of $\delta$-$M$-small modules is closed under submodules, homomorphic images and finite direct sums.

**Definition 2.5** *Let* $N \in \sigma[M]$. *We define*
$\delta_M(N) := \{n \in N : nR \ll_{\delta_M} N\}$
$\delta^*_M(N) := \{n \in N : nR \ll_{\delta_M} \widehat{nR}\} = \{n \in N : nR \ll_{\delta_M} \widehat{N}\} = \delta_M(\widehat{N}) \cap N$.

In case $M = R$, we write $\delta_R(N) = \delta(N)$ and $\delta^*_R(N) = \delta^*(N)$. Then

$$\mathrm{Rad}(N) \leq \delta_M(N) \leq \delta^*_M(N)$$
$$\mathrm{Rad}(N) \leq \mathrm{Z}^*_M(N) \leq \delta^*_M(N).$$

If $N$ is a $\delta$-$M$-small module then $\delta^*_M(N) = N$. Also by definition for $N \leq K \in \sigma[M]$ $\delta^*_M(N) = N \cap \delta^*_M(K)$. In particular, $\delta^*_M(\delta^*_M(N)) = \delta^*_M(N)$. $\delta(N)$ is defined by Zhou [14]. Note that for any ring $R$, $\mathrm{Soc}(R_R) \leq \delta(R_R)$ by [14, Theorem 1.6].

If for every $N \in \sigma[M]$, $\delta^*_M(N) = 0$, then $M$ is a V-module. But the converse is not true in general:

**Example 2.6** Let $F$ be any field and $R$ be the direct product of any infinite number of copies of $F$. Then $R$ is a commutative V-ring and $\mathrm{Soc}(R)$ is the ideal of $R$ consisting of all elements which have at most a finite number of non-zero components. Then by [14, Theorem 1.6], $\mathrm{Soc}(R) \leq \delta(R) \leq \delta^*(R)$ implies that $\delta^*(R) \neq 0$. Hence $R$ is a V-ring but $\delta^*(R) \neq 0$. Actually, by Corollary 2.9 $\mathrm{Soc}(R) = \delta^*(R)$.

But Theorem 1.5 still holds when $\mathrm{Z}^*_M(.)$ is replaced by $\delta^*_M(.)$.

**Theorem 2.7** *The following are equivalent for a module* $M$.
*a)* $M$ *is a GCO-module,*
*b) For every module* $N \in \sigma[M]$, $\delta^*_M(N)$ *is* $M$*-projective,*
*c) Every* $\delta$-$M$*-small module in* $\sigma[M]$ *is* $M$*-projective,*
*d) For every module* $N \in \sigma[M]$, $Z_M(N) \cap \delta^*_M(N) = 0$,
*e) For every simple module* $E \in \sigma[M]$, $Z_M(\widehat{E}) \cap \delta^*_M(\widehat{E}) = 0$,
*f)* $M/Soc(M)$ *is a V-module and* $Z_M(M) \cap \delta^*_M(M) = 0$,
*g)* $\delta^*_M(M/K) = 0$ *for every* $K \leq_e M$ *and* $Z_M(M) \cap \delta^*_M(M) = 0$,
*h) Every non-zero module* $N$ *with* $\delta^*_M(N) = N$ *contains a non-zero* $M$*-projective submodule,*
*i) For every module* $N \in \sigma[M]$ *with* $Z_M(N) \leq_e N$, $\delta^*_M(N) = 0$.

**Proof** (a) implies (b), since $M$-singular $M$-injective and $\delta$-$M$-small modules are zero. Then $\delta^*_M(N)$ is semisimple and then $M$-projective. The others can be seen by definitions and Theorem 1.5. □

Replacing $\mathrm{Z}_M(N)$ by the singular submodule $\mathrm{Z}(N)$ and $M$-projectivity by projectivity in Theorem 2.7 we have the following.

**Theorem 2.8** *The following are equivalent for a module $M$.*
*a) $M$ is a GV-module,*
*b) For every module $N \in \sigma[M]$, $\delta^*(N)$ is projective,*
*c) Every $\delta$-$M$-small module in $\sigma[M]$ is projective,*
*d) For every module $N \in \sigma[M]$, $Z(N) \cap \delta^*(N) = 0$,*
*e) For every simple module $E \in \sigma[M]$, $Z(\widehat{E}) \cap \delta^*(\widehat{E}) = 0$,*
*f) $M/Soc(M)$ is a V-module and $Z(M) \cap \delta^*(M) = 0$,*
*g) $\delta^*(M/K) = 0$ for every $K \leq_e M$ and $Z(M) \cap \delta^*(M) = 0$,*
*h) Every non-zero module $N$ with $\delta^*(N) = N$ contains a non-zero projective submodule,*
*i) For every module $N \in \sigma[M]$ with $Z(N) \leq_e N$, $\delta^*(N) = 0$.*

Applying the above theorem to a ring we have the following corollary.

**Corollary 2.9** *The following are equivalent for a ring $R$.*
*a) $R$ is a right GV-ring,*
*b) For every $R$-module $M$, $\delta^*(M)$ is projective,*
*c) Every $\delta$-small module is projective,*
*d) For every $R$-module $M$, $Z(M) \cap \delta^*(M) = 0$,*
*e) For every simple module $S$, $Z(E(S)) \cap \delta^*(E(S)) = 0$.*
*f) $R/Soc(R)$ is a V-module and $Z(R_R) \cap \delta^*(R_R) = 0$,*
*g) $\delta^*(R/K) = 0$ for every essential right ideal $K$ of $R$ and $Z(R_R) \cap \delta^*(R_R) = 0$,*
*h) Every non-zero $R$-module $M$ with $\delta^*(M) = M$ contains a non-zero projective submodule,*
*i) For every $R$-module $M$ with $Z(M) \leq_e M$, $\delta^*(M) = 0$.*

*In this case $Soc(R_R) = \delta(R_R) = \delta^*(R_R)$.*

**Proof** The last part is because of that $\delta^*(R_R)$ is semisimple. □

If $M/Z^*_M(M)$ is a V-module (semisimple) then $M/\delta^*_M(M)$ is a V-module (respectively semisimple). Then Theorem 1.11 and 1.12 still hold for $\delta^*_M(.)$.

**Theorem 2.10** *Let $M$ be a module with $M/\delta^*_M(M)$ a V-module. Then the following are equivalent.*
*a) $M$ is a GCO-module,*
*b) $\delta^*_M(M)$ is semisimple $M$-projective.*

**Theorem 2.11** *Let $M$ be a module with $M/\delta^*_M(M)$ semisimple. Then the following are equivalent.*
*a) $M$ is a GCO-module,*
*b) $M$ is an SI-module,*
*c) $\delta^*_M(M)$ is semisimple $M$-projective.*

Also under the assumption "$M/\mathrm{Z}^*_M(M)$ is V-module (semisimple)" the conditions of Theorem 1.11 (respectively 1.12) are equivalent to "$\delta^*_M(M)$ is semisimple $M$-projective".

Consider some examples.

**Examples 2.12** 1) Let $R$ be the $2\times 2$ upper triangular matrix $\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where $F$ is a field. $R$ is a right GV-ring but not a right V-ring by [2]. Then $\mathrm{Soc}(R_R) = \delta(R_R) = \delta^*(R_R) = \mathrm{Z}^*(R_R) = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ ([8, Example 11]), $\mathrm{J}(R) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$.

2) Let $R = \mathrm{Z}/4\mathrm{Z}$. Then $\mathrm{Soc}(R) = \mathrm{Z}(R) = 2R$. Since $R/\mathrm{Soc}(R) \cong \mathrm{Z}/2\mathrm{Z}$, $\mathrm{Soc}(R) = \delta(R)$. Z is a small module. This implies that for every $R$-module $M$, $\mathrm{Z}^*(M) = M$ [8, Lemma 8] and hence for every $R$-module $M$, $\delta^*(M) = M$. On the other hand $R$ is not an SI-ring but every singular $R$-module is semisimple by [13, Example 8].

If $R$ is a right SI-ring, then $\mathrm{Soc}(R_R) = \delta^*(R_R)$ is projective. But the second example above says that if every singular right $R$-module is semisimple and $\delta^*(R_R)$ is projective then $R$ need not be a right SI-ring, compare with [3, 17.4 (a)⇔(c)].

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