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ALMOST-PERFECT MODULES

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Abstract. We call a module M *almost perfect* if every M-generated flat module is M-projective. Any perfect module is almost perfect. We characterize almost-perfect modules and investigate some of their properties. It is proved that a ring R is a left almost-perfect ring if and only if every finitely generated left R-module is almost perfect. R is left perfect if and only if every (projective) left R-module is almost perfect.

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1. Introduction. Throughout this paper, R denotes an associative ring with unit and all modules are unitary left R-modules. The notation $\ll$ will be used for small submodules of modules. We refer the reader to [3, 7, 11] for the definitions used but not defined in the paper.

Amini et al. [**2**] call a ring R *left almost perfect* (A-perfect) if every flat left R-module is R-projective. In this paper, we are motivated to study a module theoretic version of almost-perfect rings. We see that any perfect module is almost perfect, and any projective almost-perfect module satisfying $(*)$ is semi-perfect (the definitions are given in the text). We notice that the class of non-zero almost-perfect abelian groups coincide with the class of non-zero torsion abelian groups. Some basic properties of the class of almost-perfect modules are also investigated. We obtain some necessary and sufficient conditions for a module to be almost perfect, and a ring to be left almost-perfect or left perfect in terms of almost-perfect modules. In the final part of this paper, we consider the endomorphism ring of almost-perfect modules.

2. Results. DEFINITION 1. A module M is called *almost perfect* (A-perfect)[1] if every M-generated flat module is M-projective.

By definitions, R is a left A-perfect ring if and only if ${}_RR$ is an A-perfect module.

EXAMPLE 2. It is obvious that if M is a semi-simple module, then it is A-perfect. Moreover, an A-perfect module over a (von Neumann) regular ring is semi-simple. Indeed, let M be an A-perfect module over a regular ring and N a submodule of M. Since the factor module M/N is M-generated flat, it is M-projective. It follows that N is a direct direct summand of M. Thus, M is semi-simple.

EXAMPLE 3. Torsion modules over an integral domain are A-*perfect*.

Dedicated to Professor Patrick F. Smith on his 65th birthday
[1]See Remark 26

Proof. Let R be an integral domain, M a torsion R-module and K an M-generated flat R-module. Then K is torsion-free and there exists an epimorphism $g : M^{(\Lambda)} \to K$ for an index set Λ. Since $M^{(\Lambda)}$ is torsion, we have that $Img \subseteq T(K) = 0$, where $T(K)$ is the torsion submodule of K. Hence, $K = 0$ and so K is M-projective. □

The set of rational numbers $\mathbb{Q}$ is not A-perfect as a $\mathbb{Z}$-module because $\mathbb{Q}_{\mathbb{Z}}$ is flat $\mathbb{Q}$-generated but not $\mathbb{Q}$-projective.

Note that A-perfect flat modules are quasi-projective.

Recall some definitions: An epimorphism $f : P \to M$ is called a *projective cover* of the module M in case P is a projective module and kernel of f is a small submodule. An epimorphism $f : F \to M$ with F flat is called a *flat cover* of the module M if, for each homomorphism $g : H \to M$ with H flat, there exists a homomorphism $h : H \to F$ such that $fh = g$ and every endomorphism k of F with $fk = f$ is an automorphism of F. Due to [**4**], every module has a flat cover.

Semi-perfect and perfect modules are defined by Mares [**8**] as a generalization of Bass' notion of semi-perfect and perfect rings. Perfect modules are studied by a few authors, for example, Cunningham-Rutter [**5**], Varadarajan [**9**] and Wisbauer [**11**]. A module M is called *semi-perfect* if every factor module of M has a projective cover. It is known that M is semi-perfect if and only if every finitely M-generated module has a projective cover. It is also obvious that if M is semi-perfect, then every finitely M-generated flat module is projective. A module M is called *perfect* if any direct sum of copies of M are semi-perfect.

It can be easily seen that projective covers of M-generated modules are M-generated for a projective module M. But flat covers of M-generated modules need not be M-generated for any module M (see Example 7). We donot know whether flat covers of M-generated modules are M-generated or not for a projective module M.

In this paper, a module M is said to satisfy $(*)$ if flat covers of M-generated modules are M-generated. Note that any free module, in particular, any ring satisfies $(*)$.

The following well-known lemma will be used in this paper (see [**2**, Lemma 3.6]).

LEMMA 4. *Let $f : F \to M$ be a flat cover of the module M. If F is projective, then $f : F \to M$ is a projective cover of M.*

The following result may be known but we donot have a reference. We give a proof for completeness' sake.

PROPOSITION 5. *Let M be a module. Consider the following statements:*

(1) *M is perfect.*

(2) *Every M-generated module has a projective cover.*

(3) *Every M-generated flat module is projective.*

(4) *Flat covers of M-generated modules are projective.*

Then (4) $\Rightarrow$ (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3)*;* (3) $\Rightarrow$ (4) *if M satisfies* $(*)$.

Proof. The implication (4) $\Rightarrow$ (1) follows from the fact that if a flat cover of a module is projective, then it is a projective cover of the module by Lemma 4. The equivalency (1) $\Leftrightarrow$ (2) is obvious. The implication (2) $\Rightarrow$ (3) follows from the fact that any flat module which has a projective cover is projective. For (3) $\Rightarrow$ (4), suppose that M satisfies $(*)$. Then the flat cover of any M-generated module is projective by hypothesis. □

We conclude from Proposition 5 that the following implication holds for modules.

$$\text{perfect} \Rightarrow A\text{-perfect}.$$

The following theorem characterizes A-perfect modules.

THEOREM 6. *Let M be a module. Consider the following statements:*

(1) *M is semi-perfect and flat covers of finitely M-generated modules are finitely M-generated.*

(2) *Finitely M-generated flat modules are projective and flat covers of finitely M-generated modules are finitely M-generated.*

(3) *Flat covers of finitely M-generated modules are projective.*

(4) *Flat covers of M-cyclic modules are projective.*

(5) *Finitely M-generated flat modules are M-projective and flat covers of finitely M-generated modules are finitely M-generated.*

(6) *Flat covers of finitely M-generated modules are M-projective.*

(7) *Flat covers of M-cyclic modules are M-projective.*

(8) *Every flat module is M-projective.*

(9) *M is A-perfect.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) $\Rightarrow$ (9)*;* (5) $\Rightarrow$ (6)*;* (3) $\Rightarrow$ (2) *and* (4) $\Rightarrow$ (5) *if M is flat;* (9) $\Rightarrow$ (8) *if M satisfies* $(*)$*;* (6) $\Rightarrow$ (4) *if M is projective.*

Proof. (1) $\Rightarrow$ (2) Let N be a finitely M-generated flat module. Then there exists an epimorphism $M^n \to N$ for some positive integer n. Since M is semi-perfect, M^n is semi-perfect ([**7**, 11.3.4]) and so N has a projective cover. Let the projective module be P and the epimorphism $f : P \to N$ with $Ker f \ll P$. Since $P/Ker f \cong N$ is flat, $Ker f = 0$ [**7**, 10.5.3]. Hence, $P \cong N$ is projective.

(2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (3) Let X be a finitely M-generated module. Then flat covers of X-cyclic modules are projective by [**1**, Corollary 3.4 and Proposition 3.2]. Hence, flat cover of X is projective.

(4) $\Rightarrow$ (6) $\Rightarrow$ (7) are obvious.

(7) $\Rightarrow$ (8) Let N be a flat module, $g : N \to M/K$ a homomorphism and $f : F \to M/K$ a flat cover of M/K. Since N is flat and f is a flat cover, there exists a homomorphism $h : N \to F$ such that $fh = g$. By assumption, F is M-projective. So there exists a homomorphism $k : F \to M$ such that $\pi k = f$, where $\pi : M \to M/K$ is the canonical epimorphism. Define $\alpha = kh$. Then $\pi\alpha = g$, and so N is M-projective. So (8) holds.

(8) $\Rightarrow$ (6) and (8) $\Rightarrow$ (9) are obvious.

(9) $\Rightarrow$ (8) Assume that M satisfies $(*)$. Let F be a flat cover of an M-cyclic module. By $(*)$, F is M-generated. By hypothesis, F is M-projective. Hence (7), and so (8) holds.

(3) $\Rightarrow$ (1) Assume that M is flat. By hypothesis and Lemma 4, every finitely M-generated module has a projective cover which is equivalent to the fact that M is semi-perfect. Now, let X be a finitely M-generated module. Then there exists an epimorphism $f : M^n \to X$ for some positive integer n. Let $g : F \to X$ be a flat cover of X. By assumption, F is projective and so g is also a projective cover of X. M^n being flat implies that there exists a homomorphism $h : M^n \to F$ such that $gh = f$. Then $F = Imh + Ker g$. Since $Ker g \ll F$, we have $F = Imh$. Hence, F is finitely M-generated.

(6) $\Rightarrow$ (4) By [**1**, Proposition 3.2].

Consequently, the statements above are all equivalent if M is projective and satisfies $(*)$. □

We obtain the following implication for modules by Theorem 6:

$$\text{projective } A\text{-perfect with } (*) \Rightarrow \text{semi-perfect}.$$

The following example shows that $(*)$ does not hold in general.

EXAMPLE 7. Let $R = \mathbb{Z}$ and the $\mathbb{Z}$-module $M = \mathbb{Z}/(p)$ for a prime p. The flat cover of M is the ring of p-adic integers which is not (finitely) M-generated. Hence M does not satisfy $(*)$. Moreover, since M is simple, it is A-perfect but not semi-perfect.

The projectivity condition on M in Theorem 6 ($9 \Rightarrow 4$) can not be removed and even replaced by flatness:

EXAMPLE 8. Let R be a regular ring and M a semi-simple left R-module which is not projective. We claim that M satisfies $(*)$ and is A-perfect flat but is not semi-perfect.

Since R is regular, every left R-module is flat and so M satisfies $(*)$. Since M is semi-simple, it is A-perfect. If M has a projective cover, $f : P \to M$, then $P/ker f \cong M$ is flat. Since $ker f \ll P$, $ker f = 0$ (see [**7**, 10.5.3]). This gives that $P \cong M$ is projective, which is a contradiction. It follows that M is not semi-perfect.

To be specific, we can take the ring $R = \{(x_1, \ldots, x_n, x, x, \ldots) \mid x_i, x \in \mathbb{Z}_2, i = 1, \ldots, n\}$. Then R is regular and $M := R/ \oplus_{i=1}^{\infty} F_i$ is simple singular (so it is not projective) R-module, where $F_i = \mathbb{Z}_2$, $i = 1, 2, \ldots$.

PROPOSITION 9. *Let M be a flat module. If flat covers of M-generated modules are projective, then M satisfies* $(*)$.

Proof. Let X be an M-generated module and $f : F \to X$ be a flat cover of X. By hypothesis, F is projective and then by Lemma 4, f is a projective cover of X. Let g be the epimorphism $M^{(\Lambda)} \to X$ for some index set Λ. Since $M^{(\Lambda)}$ is flat, there exists a homomorphism $h : M^{(\Lambda)} \to F$ such that $fh = g$. Since $ker f \ll F$, h is an epimorphism. So F is M-generated. □

Recall that an ideal I of a ring R is called *left t-nilpotent* if, for any sequence $a_1, a_2, \ldots$ in I, there exists an n such that $a_1 a_2 \ldots a_n = 0$. A module M is called a *progenerator* if M is a finitely generated projective generator.

Mares [**8**, Theorem 7.6] prove that if M is a progenerator, then M is perfect if and only if M is semi-perfect and the Jacobson radical $J(R)$ is left t-nilpotent. After Mares, in [**5**, Theorem 1], it is proved that a projective module M is perfect if and only if M is semi-perfect and $J(Tr(M))$ is left t-nilpotent, where $Tr(M)$ is the trace ideal $\sum\{f(M) \mid f \in Hom_R(M, R)\}$ of M. This gives the following result via Theorem 6.

THEOREM 10. *If M is a projective module which satisfies* $(*)$, *then the following are equivalent.*

(1) *M is perfect.*

(2) *M is A-perfect and $J(Tr(M))$ is left t-nilpotent.*

If M is a generator, then the trace ideal of M is R.

COROLLARY 11. *If M is a projective generator, then the following are equivalent.*

(1) *M is perfect.*

(2) *M is A-perfect and $J(R)$ is left t-nilpotent.*

PROPOSITION 12. *The class of A-perfect modules is closed under factor modules.*

Proof. Let N be a submodule of an A-perfect module M and K an M/N-generated flat module. Then K is M-generated flat and by assumption, it is M-projective. Hence, K is M/N-projective. Thus, M/N is A-perfect. □

We know from [**6**] that an abelian group is quasi-projective if and only if it is free or a torsion group such that every p-component A_p is a direct sum of cyclic groups of the same order p^n. If G is a non-zero A-perfect flat(= torsion-free) abelian group, then it is quasi-projective and hence it is free. But this leads to a contradiction because $\mathbb{Z}$ is not an A-perfect $\mathbb{Z}$-module. As a consequence we obtain the result below:

PROPOSITION 13. *A non-zero abelian group G is torsion if and only if it is A-perfect.*

Proof. The necessity follows from Example 3. For the sufficiency, let G be A-perfect and consider the torsion subgroup $T(G)$ of G. If $T(G) \neq G$, then $G/T(G)$ is a non-zero torsion-free A-perfect abelian group by Proposition 12, but this is impossible. Thus, $G = T(G)$. □

It can be easily seen that a principal ideal domain R is A-perfect if and only if there exists a finitely generated torsion-free A-perfect R-module.

The class of A-perfect modules need not be closed under direct sums.

EXAMPLE 14. If R is a left A-perfect ring which is not left perfect (see [**2**] for such a ring), then $R^{(\mathbb{N})}$ is not A-perfect as a left R-module.

Proof. Since ${}_RR^{(\mathbb{N})}$ is free, it is a generator for left R-modules, and so it satisfies $(*)$. If ${}_RR^{(\mathbb{N})}$ was A-perfect, then it would be semi-perfect by Theorem 6. Thus, R would be left perfect by [**11**, 43.9], which is a contradiction. □

PROPOSITION 15. *Let* $M = \oplus_{i=1}^{n} M_i$ *be a module. Suppose that* $\oplus_{k=1}^{i-1} M_k$ *is* M_i-*generated and* M_i *is* $\oplus_{k=1}^{i-1} M_k$-*generated for each* $i = 2, \ldots, n$. *Then each* M_i *is A-perfect if and only if M is A-perfect.*

Proof. The sufficiency is clear by Proposition 12. For the necessity it is enough to prove the statement for $n = 2$. The rest of the proof follows from induction. Let M_1 and M_2 be A-perfect and suppose that M_1 is M_2-generated and M_2 is M_1-generated. If K is an $M_1 \oplus M_2$-generated flat module, then K is both M_1- and M_2-generated by hypothesis. Hence, K is both M_1- and M_2-projective which implies that K is $M_1 \oplus M_2$-projective. □

COROLLARY 16. *A module M is A-perfect if and only if* M^n *is A-perfect for any positive integer n.*

PROPOSITION 17. *If* M_1 *is an A-perfect generator and* M_2 *is semi-simple, then* $M_1 \oplus M_2$ *is A-perfect.*

Proof. Let X be an $M_1 \oplus M_2$-generated flat module. Since M_1 is a generator, X is M_1-generated. By hypothesis, it is M_1-projective. X is also M_2-projective because M_2 is semi-simple. Hence, X is $M_1 \oplus M_2$-projective and thus $M_1 \oplus M_2$ is A-perfect. □

The next two theorems characterize left A-perfect and left perfect rings in terms of A-perfect modules, respectively.

THEOREM 18. *The following are equivalent for a ring R.*
(1) *R is a left A-perfect ring.*
(2) *Every finitely generated left R-module is A-perfect.*
(3) *Every finitely generated projective left R-module is A-perfect.*

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are obvious. For (1) $\Rightarrow$ (2), let M be a finitely generated R-module and F an M-generated flat R-module. Then consider the epimorphism $g : R^n \to M$ for some n and the canonical epimorphism $\pi : M \to M/N$ for any submodule N of M. Since F is R-projective, there exists $h : F \to R^n$ such that $\pi g h = f$, for any homomorphism $f : F \to M/N$. Define $h' = gh$. Then we obtain that $\pi h' = f$ which means that F is M-projective. □

Note that a ring R is left perfect if and only if every left R-module is semi-perfect, if and only if every projective left R-module is semi-perfect (see [**11**, 42.3; 43.9]).

THEOREM 19. *The following are equivalent for a ring R.*
(1) *R is left perfect.*
(2) *Every left R-module is A-perfect.*
(3) *Every projective left R-module is A-perfect.*
(4) *Every free left R-module is A-perfect.*
(5) *${}_RR^{(\mathbb{N})}$ is A-perfect.*

Proof. (1) $\Rightarrow$ (2) is obvious because every flat left module is projective over a left perfect ring. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious. For (5) $\Rightarrow$ (1), ${}_RR^{(\mathbb{N})}$ is semi-perfect by Theorem 6 and hence R is left perfect by [**11**, 43.9]. □

In [**2**], it is proved that the polynomial ring $R[x]$, in one indeterminate x, is not an (left or right) A-perfect ring for any ring R. However, by Theorem 19, we see that $R[x]$ is A-perfect as a left R-module if R is left perfect.

THEOREM 20. *Let ${}_RM$ be a progenerator and $S = End_R(M)$. The following are equivalent.*
(1) *${}_RM$ is A-perfect.*
(2) *S is left A-perfect.*
(3) *R is left A-perfect.*

Proof. (1) $\Rightarrow$ (2) We will use the notation $\otimes$ instead of $\otimes_S$ in this proof. Let X be a flat left S-module. We claim that X is S-projective, that is,

$$Hom_S(X, S) \longrightarrow Hom_S(X, S/I) \longrightarrow 0$$

is exact for any exact sequence $S \longrightarrow S/I \longrightarrow 0$, where I is a left ideal of S. Since M is an R-S-bimodule and ${}_RM$ is flat, $M \otimes X$ is a flat left R-module, so it is M-projective by hypothesis. Note that $M \cong M \otimes S$ as an R-module. So $M \otimes X$ is $M \otimes S$-projective. This gives the following exact sequences, where vertical maps are isomorphisms by

[**3**, Propositions 20.6 and 20.10] and this completes the proof.

$$\begin{array}{ccccc}
Hom_R(M \otimes X, M \otimes S) & \longrightarrow & Hom_R(M \otimes X, M \otimes S/I) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
Hom_S(X, Hom_R(M, M \otimes S)) & \longrightarrow & Hom_S(X, Hom_R(M, M \otimes S/I)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
Hom_S(X, S) & \longrightarrow & Hom_S(X, S/I) & \longrightarrow & 0
\end{array}$$

(2) $\Rightarrow$ (3) Since ${}_RM$ is a progenerator, R is Morita equivalent to S (see [**3**, Corollary 22.5]). By [**2**, Proposition 3.4], R is left A-perfect.

(3) $\Rightarrow$ (1) Since ${}_RM$ is finitely generated, ${}_RM$ is A-perfect by Theorem 18. □

COROLLARY 21. *Let $e^2 = e \in R$ such that $ReR = R$. Then Re is an A-perfect left R-module if and only if $End_R(Re) \cong eRe$ is a left A-perfect ring, if and only if R is a left A-perfect ring.*

Proof. $Tr(Re) = ReR = R$ and so ${}_RRe$ is a progenerator. So the proof follows from Theorem 20. □

If ${}_RM$ is a progenerator, then M_S is a progenerator, where $S = End_R(M)$ and $R \cong End_S(M_S)$ (see [**11**, 18.8]). Then by Theorem 20, M_S is A-perfect if and only if S is right A-perfect, if and only if R is right A-perfect. Note that the notion of A-perfect rings is not left–right symmetric [**2**, Example 3.3].

In Theorem 20, (1) $\not\Rightarrow$ (2) and (3) if M is not a generator:

EXAMPLE 22. Let K be a field and I an infinite index set. Let $R = \prod_{i \in I} K_i$ such that for each $i \in I$, $K_i = K$. Then $M := \bigoplus_{i \in I} K_i$ is a non-finitely generated projective R-module which is not a generator. $End_R(M) \cong R$ is not A-perfect since R is not semi-perfect. But M is A-perfect since it is semi-simple.

In Theorem 20, (3) $\not\Rightarrow$ (1) and (2) if M is not finitely generated:

EXAMPLE 23. Consider an A-perfect ring R that is not left perfect. Let ${}_RM = R^{(\mathbb{N})}$. Then M is a non-finitely generated projective generator. ${}_RM$ and $End({}_RM)$ are not A-perfect by Example 14 and [**11**, 43.9].

In Theorem 20, (2) $\not\Rightarrow$ (1):

EXAMPLE 24. As we mentioned before, the abelian group $\mathbb{Q}$ is not A-perfect. On the other hand, since $End(\mathbb{Q}_{\mathbb{Z}}) \cong \mathbb{Q}_{\mathbb{Q}}$, $End(\mathbb{Q}_{\mathbb{Z}})$ is an A-perfect ring.

In Theorem 20, (2) $\not\Rightarrow$ (3) if M is not a generator:

EXAMPLE 25. Let R be a ring with a simple projective module M and not right A-perfect (e.g. any ring with non-zero projective socle which is not semi-perfect). Then $End(M)$ is a division ring and so a right A-perfect ring. But M is not a generator.

REMARK 26. After the submission of our paper, the paper [**1**] is appeared and Amini–Amini–Ershad call any module M almost-perfect if flat covers of M-cyclic modules are projective. This is the condition (4) in Theorem 6 and so almost-perfect in the sense of [**1**] implies almost-perfect in our sense. But the converse need not be true. For example, any semi-simple module is almost-perfect in our sense but need not be almost-perfect in the sense of [**1**]. We should also note that eRe is left A-perfect if and only if R is left A-perfect for any non-zero idempotent e in R by [**1**, Proposition 2.24] (cf. Corollary 21).

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