DUO MODULES

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Abstract. Let R be a ring. An R-module M is called a *(weak) duo module* provided every (direct summand) submodule of M is fully invariant. It is proved that if R is a commutative domain with field of fractions K then a torsion-free uniform R-module is a duo module if and only if every element k in K such that kM is contained in M belongs to R. Moreover every non-zero finitely generated torsion-free duo R-module is uniform. In addition, if R is a Dedekind domain then a torsion R-module is a duo module if and only if it is a weak duo module and this occurs precisely when the P-primary component of M is uniform for every maximal ideal P of R.

2000 *Mathematics Subject Classification.* 16D99, 13C12 (13B20).

1. Introduction. Throughout all rings have an identity and all modules are unital. R is a ring and M a right R-module. A submodule N of M is called *fully invariant* if $f(N)$ is contained in N for every R-endomorphism f of M. Let $S = End(M_R)$, the ring of R-endomorphisms of M. Then M is a left S-, right R-bimodule and a submodule N of the right R-module M is fully invariant if and only if N is a sub-bimodule of M. Clearly 0 and M are fully invariant submodules of M. The right R-module M is called a *duo module* provided every submodule of M is fully invariant. For example, if U is a simple right R-module, then clearly U is a duo module but $U \oplus U$ is not duo. The ring R is called a *right duo ring* if the right R-module R is a duo module. Note that a ring R is a right duo ring if and only if every right ideal of R is a two-sided ideal; equivalently Ra is contained in aR for every element a in R. Clearly commutative rings and division rings are right (and left) duo rings but any 2×2 matrix ring over such a ring is not a right (or left) duo ring.

We begin with a simple observation.

LEMMA 1.1. *Let R be any ring. Then a right R-module M is a duo module if and only if for each endomorphism f of M and each element m of M there exists r in R such that $f(m) = mr$.*

Proof. The necessity follows because $f(mR)$ is contained in mR. Conversely, note that the stated condition implies that $f(N)$ is contained in N for every submodule N and endomorphism f of M. It follows that M is a duo module. □

A good source of duo modules is provided by multiplication modules. Let R be a commutative ring. An R-module M is called a *multiplication module* provided that for each submodule N of M there exists an ideal I of R such that $N = IM$. Among examples of multiplication R-modules we can mention projective ideals of R (see [**11**, Theorem 1]), ideals of R generated by idempotents (see [**3**, Corollary 1.3]) and finitely generated R-modules M such that every localization of R with respect to a maximal ideal of R is cyclic (see [**3**, Theorem 1.2]). Note further that if I is a multiplication ideal of R (i.e. I is an ideal of R and a multiplication R-module) and M is a multiplication R-module then the R-module IM is a multiplication R-module (see [**3**, Corollary 1.4]). A projective R-module M is a multiplication module if and only if M is a duo module (see [**10**, Corollary B]). Moreover, a finitely generated module M with annihilator A in R is a multiplication module if and only if M is a projective (R/A)-module and a duo R-module (see [**10**, Corollary C]).

Let R be any ring. Let M be an R-module and let f be an endomorphism of M. For any submodule N of M, we set $f^{-1}(N) = \{m \in M : f(m) \in N\}$. Note that $f^{-1}(N)$ is a submodule of M and that $f(f^{-1}(N)) \subseteq N$. Note further that $f(f^{-1}(N)) = N$ in case f is an epimorphism. Moreover, for any submodules $L \subseteq N$ of M, we have $f^{-1}(L) \subseteq f^{-1}(N)$. For any positive integer n, we shall denote $(f^n)^{-1}(N)$ by $f^{-n}(N)$; i.e. $f^{-n}(N) = \{m \in M : f^n(m) \in N\}$.

Another source of duo modules is provided by uniserial modules. A module M is called *uniserial* if, for all submodules L and N of M, either $L \subseteq N$ or $N \subseteq L$.

THEOREM 1.2. *Let M be a uniserial module which either satisfies the ascending chain condition on cyclic modules or is Artinian. Then M is a duo module.*

Proof. Suppose that M satisfies the acc on cyclic submodules. Let $0 \neq m \in M$ and let f be an endomorphism of M. Suppose that $f(m) \notin mR$. Then $m \in f(m)R$ and hence $m = f(m)r$ for some $r \in R$. It follows that $f^n(m) = f^{n+1}(m)r$ for every positive integer n. Thus $mR \subseteq f(m)R \subseteq f^2(m)R \subseteq \cdots$. By hypothesis, there exists a positive integer k such that $f^k(m)R = f^{k+1}(m)R$. There exists $s \in R$ such that $f^{k+1}(m) = f^k(m)s = f^k(ms)$. Hence $f(m) - ms \in ker(f^k)$. If $mR \subseteq ker(f^k)$, then $f^k(m) = 0$ and hence $m = f^k(m)r^k = 0$, a contradiction. Thus $ker(f^k) \subseteq mR$ and hence $f(m) - ms \in mR$, so that $f(m) \in mR$, a contradiction. It follows that $f(m) \in mR$. By Lemma 1.1, M is a duo module.

Now suppose that M is Artinian. Suppose that M is not a duo module. There exist a non-zero submodule L of M and an endomorphism g of M such that $g(L) \nsubseteq L$. Hence $L \subseteq g(L)$. It follows that $L \subseteq g(L) \subseteq g^2(L) \subseteq \cdots$. Consider the descending chain $M \supseteq g(M) \supseteq g^2(M) \supseteq \cdots$ of submodules of M. Because M is Artinian, there exists a positive integer n such that $g^n(M) = g^{n+1}(M)$, and hence $M = g(M) + ker(g^n)$. If $g(M) \subseteq ker(g^n)$, then $g^{n+1}(M) = 0$ and hence $L \subseteq g^{n+1}(L) = 0$, a contradiction. Thus $ker(g^n) \subseteq g(M)$ so that $M = g(M)$.

If $L \subseteq g^{-1}(L)$, then $g(L) \subseteq g(g^{-1}(L)) \subseteq L$, a contradiction. Thus $g^{-1}(L) \subseteq L$. It follows that $L \supseteq g^{-1}(L) \supseteq g^{-2}(L) \supseteq \cdots$ is a descending chain of submodules of M. There exists a positive integer k such that $g^{-k}(L) = g^{-(k+1)}(L)$. Because g^{k+1} is an epimorphism, we have $g(L) = g^{k+1}(g^{-k}(L)) = g^{k+1}(g^{-(k+1)}(L)) = L$, a contradiction. Thus M is a duo module. $\square$

We have already observed that the direct sum of duo modules need not be a duo module. Note the following fact.

PROPOSITION 1.3. *Any direct summand of a duo module is also a duo module.*

Proof. Let M be a duo module such that $M = M' \oplus M''$ is the direct sum of sub-modules M' and M''. Let $p : M \to M'$ be the canonical projection and let $i : M' \to M$ denote inclusion. Suppose that f is an endomorphism of M' and N is any submodule of M'. Then $g = ifp$ is an endomorphism of M and $f(N) = g(N)$, which is contained in N because M is duo. It follows that M' is a duo module. $\quad\square$

In general submodules of duo modules are not duo modules, so that the next result is of some interest.

PROPOSITION 1.4. *Let M be a duo module.*

(i) *If M is quasi-injective, then every submodule of M is a duo module.*
(ii) *If M is quasi-projective, then every homomorphic image of M is duo.*

Proof. (i) Let L be any submodule of M. Let N be a submodule of L and let f be an endomorphism of L. Because M is quasi-injective, f can be lifted to an endomorphism f^* of M. Then $f(N) = f^*(N)$, which is contained in N because M is a duo module. It follows that L is a duo module.

(ii) Let K be a submodule of M. Let H be a submodule of M containing K and let g be an endomorphism of the module M/K. Because M is quasi-projective, there exists an endomorphism g^* of M such that $g(m + K) = g^*(m) + K$ for all m in M. Because M is a duo module, $g^*(H)$ is contained in H and hence $g(H/K)$ is contained in H/K. It follows that M/K is a duo module. $\quad\square$

To see that submodules of duo modules are not duo modules consider the following example. Let K be a field and let V be a two-dimensional vector space over K. Let the ring R be the trivial extension of V by K. Thus R is the K-vector space $K \oplus V$ and multiplication is defined in R as follows: $(a, u)(b, v) = (ab, av + bu)$ for all a, b in K and u, v in V. Because R is a commutative ring, the R-module R is a duo module. However, the submodule $0 \oplus V$ is not a duo module, being the direct sum of two isomorphic simple modules. We do not know an example of a duo module M and a submodule N of M such that M/N is not a duo module.

Here is another simple observation.

PROPOSITION 1.5. *Let M be a module such that every countably generated submodule is a duo module. Then M is a duo module.*

Proof. Let m be any element of M and let f be any endomorphism of M. Let $N = mR + f(m)R + f^2(m)R + \cdots$. Then N is a countably generated submodule of M and the restriction of f to N is an endomorphism of N. By Lemma 1.1 and hypothesis, $f(m) = mr$ for some r in R. Again by Lemma 1.1, M is a duo module. $\quad\square$

Of course, many modules are not duo modules. Note the following fact.

PROPOSITION 1.6. *Let R be a proper subring of a ring S. Then the right R-module S is not a duo module.*

Proof. Let s be any element of the ring S such that s does not belong to R. Then the mapping $f : S \to S$ defined by $f(a) = sa$, for all a in S, is an R-homomorphism. Note that $s = f(1)$, so that R is not a fully invariant submodule of the R-module S. $\quad\square$

Next we consider some simple properties of duo modules. Let R be a ring. An R-module M is called *Hopfian* provided that every surjective endomorphism of M is an isomorphism. For example, every Noetherian module is Hopfian (see [**1**, Lemma 11.6]). The module M is called *generalized Hopfian* if every surjective endomorphism of M has a small kernel (see [**4**]). On the other hand, the module M is called *co-Hopfian* if every injective endomorphism is an isomorphism. By [**1**, Lemma 11.6], Artinian modules are co-Hopfian. Next, in [**5**], the module M is called *weakly co-Hopfian* if every injective endomorphism has an essential image.

PROPOSITION 1.7. *Every duo module is a generalized Hopfian and a weakly co-Hopfian module.*

Proof. Let f be any surjective endomorphism of M. Let L be any submodule of M such that $M = ker f + L$. Then $M = f(M) = f(ker f + L) = f(L)$, which is contained in L. It follows that $ker f$ is a small submodule of M. Next let g be an injective endomorphism of M. Let N be any submodule of M such that $N \cap g(M) = 0$. Because N is fully invariant, we obtain $g(N) = 0$ and hence $N = 0$. It follows that $g(N)$ is an essential submodule of M. $\square$

In contrast to Proposition 1.7, duo modules need not be Hopfian or co-Hopfian in general. Moreover, Hopfian modules need not be duo and neither need co-Hopfian modules be duo. For example, let R denote the ring $\mathbb{Z}$ of integers. For any prime p, the Prüfer p-group $\mathbb{Z}(p^\infty)$ is an Artinian uniserial R-module and hence is a duo module (Theorem 1.2) but the mapping $f : \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty)$ defined by $f(x) = px$ for all x in $\mathbb{Z}(p^\infty)$ is a surjective endomorphism which is not an isomorphism. Moreover, the R-module R is a duo module but the mapping $g : R \to R$ defined by $g(a) = 2a$, for all a in R, is an injective endomorphism which is not an isomorphism.

Let R be a ring and let M be a right R-module with endomorphism ring S. It is easy to check that every idempotent of S is central in S if and only if every direct summand of M is fully invariant, and modules M with this property will be called *weak duo modules*. It is proved in [**2**] that any weak duo module which satisfies the finite exchange property satisfies the (unrestricted) exchange property.

It is easy to give examples of weak duo modules. Clearly any indecomposable module is a weak duo module, so that in particular any uniform module is a weak duo module. If R is a domain, then the right R-module R is indecomposable and hence a weak duo module. However, if the right R-module R is a duo module then R is right Ore. Thus for any domain R which is not right Ore, the right R-module R is a weak duo module which is not a duo module. The next result is an analogue of Proposition 1.3.

PROPOSITION 1.8. *Any direct summand of a weak duo module is a weak duo module.*

Proof. This is clear. $\square$

The next result is a well-known fact about direct sum decompositions that is proved for completeness.

LEMMA 1.9. *Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2. Then M_1 is a fully invariant submodule of M if and only if $Hom(M_1, M_2) = 0$.*

Proof. Suppose first that M_1 is a fully invariant submodule of M. Let $f : M_1 \to M_2$ be any homomorphism. Let $p_1 : M \to M_1$ denote the canonical projection and let $i_2 : M_2 \to M$ denote inclusion. Then $f^* = i_2 f p_1$ is an endomorphism of M. By hypothesis, $f^*(M_1) \subseteq M_1$, so that $f(M_1) \subseteq M_1 \cap M_2 = 0$. It follows that $f = 0$.

Conversely, suppose that $Hom(M_1, M_2) = 0$. Let $p_2 : M \to M_2$ denote the canonical projection and $i_1 : M_1 \to M$ denote inclusion. For any endomorphism g of M, $g(M_1) \subseteq p_1g(M_1) + p_2gi_1(M_1) = p_1g(M_1) \subseteq M_1$, because $p_2gi_1 \in Hom(M_1, M_2) = 0$. It follows that M_1 is a fully invariant submodule of M. $\square$

COROLLARY 1.10. *Let M be a weak duo module such that $M = M_1 \oplus M_2$ is a direct sum of submodules M_1, M_2. Then $Hom(M_1, M_2) = 0$.*

Proof. This follows from Lemma 1.9. $\square$

A module M is called *Dedekind finite* or *directly finite* if whenever N is a submodule of M such that M is isomorphic to the module $M \oplus N$, then $N = 0$. Note that the module M is Dedekind finite if and only, if for any endomorphisms f and g of M such that $fg = 1$, we have $gf = 1$ (see, for example, [**8**, Proposition 1.25]).

COROLLARY 1.11. *Weak duo modules are directly finite.*

Proof. This follows from Corollary 1.10. $\square$

Generalized Hopfian modules and weakly co-Hopfian modules are directly finite. However, note that Hopfian modules need not be weak duo modules and co-Hopfian modules need not be weak duo modules. For example, let F be any field and let R denote the ring of upper triangular 2×2 matrices with entries in F. The right R-module R is both Noetherian and Artinian and hence is both Hopfian and co-Hopfian. However, the right R-module R is not a weak duo module since R contains non-central idempotents.

2. Direct sums of duo modules.   In this section we investigate conditions under which a direct sum of duo modules is also duo. Note that if R is any ring and M any non-zero R-module, then the R-module $M \oplus M$ is not a duo module. On the other hand, if R is a commutative ring, U and V are non-isomorphic simple R-modules, then the R-module $U \oplus V$ is cyclic and hence a duo module. We begin this section with a well-known fact about direct sum decompositions that is proved for completeness

LEMMA 2.1. *Let a module $M = \oplus_{i \in I} M_i$ be a direct sum of submodules M_i $(i \in I)$ and let N be a fully invariant submodule of M. Then $N = \oplus_{i \in I}(N \cap M_i)$.*

Proof. For each $j \in I$, let $p_j : M \to M_j$ denote the canonical projection and let $i_j : M_j \to M$ denote inclusion. Then i_jp_j is an endomorphism of M and hence $i_jp_j(N) \subseteq N$, for each $j \in I$. It follows that $N \subseteq \oplus_{j \in I} i_jp_j(N) \subseteq \oplus_{j \in I}(N \cap M_j) \subseteq N$, so that $N = \oplus_{j \in I}(N \cap M_j)$. $\square$

A module M is said to satisfy the *summand sum property* if $K + L$ is a direct summand of M whenever K and L are direct summands of M. Also, M satisfies the *summand intersection property* if $K \cap L$ is a direct summand of M whenever K and L are direct summands of M.

COROLLARY 2.2. *Weak duo modules satisfy the summand sum property and the summand intersection property.*

Proof. Let K and L be direct summands of a weak duo module M. There exists a submodule L' of M such that $M = L \oplus L'$. By Lemma 2.1, $K = (K \cap L) \oplus (K \cap L')$. Thus $K \cap L$ is a direct summand of K and hence also of M. Moreover,

$K + L = L \oplus (K \cap L')$. Because $K \cap L'$ is a direct summand of M and hence also of L', we deduce that $K + L$ is a direct summand of M. $\qquad\square$

LEMMA 2.3. *Let a module* $M = \oplus_{i \in I} M_i$ *be a direct sum of submodules* M_i $(i \in I)$. *Then the following statements are equivalent for a submodule* N *of* M.

(i) $N = \oplus_{i \in I}(N \cap M_i)$.
(ii) $N = \cap_{i \in I}\{N + (\oplus_{j \neq i} M_j)\}$.

Proof. (i) $\Rightarrow$ (ii). For each i in I, let $M_i' = \oplus_{j \neq i} M_j$. Let $i \in I$. Then $N + M_i' = \{\oplus_{j \in I}(N \cap M_j)\} + \{\oplus_{j \neq i} M_j\} = (N \cap M_i) + (\oplus_{j \neq i} M_j)$. It follows that $\cap_{i \in I}(N + M_i') = \oplus_{i \in I}(N \cap M_i) = N$.

(ii) $\Rightarrow$ (i). Let $x \in N$. Then there exist a positive integer k, distinct elements $i_j \in I$ $(1 \leq j \leq k)$ and elements $m_i \in M_{i_j}$ $(1 \leq j \leq k)$ such that $x = m_1 + \cdots + m_k$. Let $1 \leq j \leq k$. Clearly $m_j \in M_i'$ for all $i \neq i_j$. Moreover, $m_j = x - m_1 - \cdots - m_{j-1} - m_{j+1} - \cdots - m_k$, so that $m_j \in N + M_{i_j}'$. Thus $m_j \in \cap_{i \in I}(N + M_i') = N$, by (ii). It follows that $x \in (N \cap M_{i_1}) \oplus \cdots \oplus (N \cap M_{i_k})$. Thus $N = \oplus_{i \in I}(N \cap M_i)$. $\qquad\square$

Let R be a ring and let M be a right R-module. For any non-empty subset S of M, the *annihilator* of S (in R) will be denoted by $ann(S)$; i.e. $ann(S) = \{r \in R : sr = 0$ for all s in $S\}$. In case $S = \{m\}$, then we write $ann(m)$ for $ann(\{m\})$. We now prove another basic fact about direct sum decompositions.

LEMMA 2.4. *Let a module* $M = \oplus_{i \in I} M_i$ *be a direct sum of submodules* M_i $(i \in I)$. *Then the following statements are equivalent.*

(i) $R = ann(m_i) + ann(m_j)$ *for all* $m_i \in M_i$, $m_j \in M_j$, *for all* $i \neq j$ *in* I.
(ii) $N = \oplus_{i \in I}(N \cap M_i)$ *for every (cyclic) submodule* N *of* M.
Moreover, in this case $Hom(M_i, M_j) = 0$ *for all distinct* i, j *in* I.

Proof. (i) $\Rightarrow$ (ii). Let N be any submodule of M. Let $m \in N$. Then there exists a positive integer n, distinct elements $i_j \in I$ $(1 \leq j \leq n)$ and elements $m_j \in M_{i_j}$ $(1 \leq j \leq n)$ such that $m = m_1 + \cdots + m_n$. If $n = 1$, then $m = m_1 \in N \cap M_{i_1}$. Suppose that $n \geq 2$. By (i) there exist elements r, s in R such that $m_1 r = 0$, $m_n s = 0$ and $1 = r + s$. Then $ms = m_1 + m_2 s + \cdots m_{n-1} s$. Note that $m_j s \in M_{i_j}$ $(2 \leq j \leq n-1)$ and $ms \in N$. By induction on n, $m_1 \in N \cap M_{i_1}$. Similarly $m_j \in N \cap M_{i_j}$ $(2 \leq j \leq n)$. Now (ii) follows.

(ii) $\Rightarrow$ (i). Let i, j be distinct elements of I, let $x \in M_i$ and let $y \in M_j$. If $L = (x + y)R$, then $L = \oplus_{i \in I}(L \cap M_i)$ and hence $x + y \in (L \cap M_i) \oplus (L \cap M_j)$. There exist $a, b \in R$ such that $x + y = (x + y)a + (x + y)b$, where $(x + y)a \in M_i$ and $(x + y)b \in M_j$. Then $x = (x + y)a$, so that $x(1 - a) = 0$ and $ya = 0$. Thus $1 = (1 - a) + a \in ann(x) + ann(y)$. Now (i) follows.

Finally, let i, j be distinct elements of I. Let $f : M_i \to M_j$ be any homomorphism. Let $n \in M_i$. By (i), $R = ann(n) + ann(f(n))$ so that $1 = c + d$ for some c, d in R with $nc = 0$, $f(n)d = 0$. It follows that $f(n) = f(n)c + f(n)d = f(nc) + f(n)d = 0$. Thus $f = 0$. $\qquad\square$

Note that in Lemma 2.4, $Hom(M_i, M_j) = 0$ for all distinct i, j in I does not imply (i) (or (ii)). For let R denote the ring $\mathbb{Z}$ of integers and let M_1 and M_2 denote the R-modules $\mathbb{Q}$ and $\mathbb{Z}/\mathbb{Z}p$, respectively, for some prime p. Note that $Hom(M_1, M_2) = 0$ and $Hom(M_2, M_1) = 0$. Let N denote the submodule $R(1, 1 + \mathbb{Z}p)$ of the R-module $M_1 \oplus M_2$. It is easy to check that if $L_1 = M_1 \oplus 0$ and $L_2 = 0 \oplus M_2$, then $N \cap L_2 = 0$ and hence $N \neq (N \cap L_1) \oplus (N \cap L_2)$.

Let R be a ring and M a right R-module. By a *complementary collection of submodules* of M we mean a collection of submodules M_i $(i \in I)$ such that $M = \oplus_{i \in I} M_i$. Also by a *complementary pair of submodules* of M we mean a pair of submodules K, K' of M such that $M = K \oplus K'$. Thus a complementary pair of submodules of M is a complementary collection of submodules of M containing two submodules. Note further that $\{0, M\}$ is a complementary pair of submodules of M, for any module M.

LEMMA 2.5. *The following statements are equivalent for any submodule N of a module M.*

(i) $N = \oplus_{i \in I}(N \cap M_i)$ *for every complementary collection of submodules M_i $(i \in I)$ of M.*

(ii) $N = (N \cap K) \oplus (N \cap K')$ *for every complementary pair of submodules K, K' of M.*

Proof. (i) $\Rightarrow$ (ii). This is clear.

(ii) $\Rightarrow$ (i). Let M_i $(i \in I)$ be any complementary collection of submodules of M. Let $x \in N$. Then there exist a positive integer k, distinct elements $i_j \in I$ $(1 \le j \le k)$ and elements $m_j \in M_{i_j}$ $(1 \le j \le k)$ such that $x = m_1 + \cdots + m_k$. Let $1 \le j \le k$. By (ii), $N = (N \cap M_{i_j}) \oplus \{N \cap (\oplus_{i \ne i_j} M_i)\}$, so that $m_j \in N \cap M_{i_j}$. Thus $x \in \oplus_{i \in I}(N \cap M_i)$. It follows that $N = \oplus_{i \in I}(N \cap M_i)$. $\square$

COROLLARY 2.6. *The following statements are equivalent for a submodule N of a module M.*

(i) $N = \cap_{i \in I}\{N + (\oplus_{j \ne i} M_j)\}$ *for every complementary collection of submodules M_i $(i \in I)$ of M.*

(ii) $N = (N + K) \cap (N + K')$ *for every complementary pair of submodules K, K' of M.*

Proof. This follows from Lemmas 2.3 and 2.5. $\square$

THEOREM 2.7. *Let a module $M = \oplus_{i \in I} M_i$ be a direct sum of submodules M_i $(i \in I)$. Then M is a weak duo module if and only if*

(i) M_i *is a weak duo module for all $i \in I$,*

(ii) $Hom(M_i, M_j) = 0$ *for all distinct $i, j \in I$, and*

(iii) $N = \oplus_{i \in I}(N \cap M_i)$ *for every direct summand N of M.*

Proof. The necessity follows from Proposition 1.8, Corollary 1.10 and Lemma 2.1. Conversely, suppose that M satisfies (i), (ii) and (iii). Let K be any direct summand of M and let f be any endomorphism of M. For each j in I let $p_j : M \to M_j$ denote the canonical projection and let $i_j : M_j \to M$ denote inclusion. By (i), $p_j f i_j(K \cap M_j) \subseteq K \cap M_j$ for all $j \in I$. Moreover, by (ii) $p_k f i_j(K \cap M_j) = 0$ for all distinct $j, k \in I$. Now (iii) gives $f(K) = \sum_{j \in I} f(K \cap M_j) \subseteq \sum_{j \in I} p_j f i_j(K \cap M_j) \subseteq \sum_{j \in I}(K \cap M_j) \subseteq K$. Thus M is a weak duo module. $\square$

COROLLARY 2.8. *Let a module $M = \oplus_{i \in I} M_i$ be a direct sum of submodules M_i $(i \in I)$. Then M is a weak duo module if and only if*

(i) $\oplus_{\in I'} M_i$ *is a weak duo module for every finite subset I' of I, and*

(ii) M *satisfies the summand intersection property.*

Proof. The necessity follows from Proposition 1.8 and Corollary 2.2. Conversely, suppose that M satisfies (i) and (ii). In particular, note that M_i is a weak duo module for all $i \in I$. Moreover, for all distinct elements $i, j \in I$, $M_i \oplus M_j$ being a weak duo module implies that $Hom(M_i, M_j) = 0$, by Corollary 1.10. Next let N be any direct summand

of M. Let $x \in N$. There exists a finite subset I' of I such that $x \in \oplus_{i\in I'} M_i = L$ (say). By (ii), $N \cap L$ is a direct summand of L. By (i) and Lemma 2.1, $N \cap L = \oplus_{i\in I'}(N \cap M_i)$. It follows that $x \in \oplus_{i\in I}(N \cap M_i)$. Hence $N = \oplus_{i\in I}(N \cap M_i)$. By Theorem 2.7, M is a weak duo module. $\square$

COROLLARY 2.9. *Let a module $M = \oplus_{i\in I} M_i$ be a direct sum of indecomposable submodules M_i ($i \in I$). Then M is a weak duo module if and only if*
  (i) *$Hom(M_i, M_j) = 0$ for all distinct $i, j \in I$, and*
  (ii) *for every direct summand N of M there exists a subset I' of I such that $N = \oplus_{i\in I'} M_i$.*

Proof. The necessity follows from Theorem 2.7 because weak duo modules have the summand intersection property (Corollary 2.2). The sufficiency follows by Theorem 2.7. $\square$

Using Lemma 2.4, the proof of Theorem 2.7 can be adapted to prove the following result.

THEOREM 2.10. *Let a module $M = \oplus_{i\in I} M_i$ be a direct sum of submodules M_i ($i \in I$). Then M is a duo module if and only if*
  (i) *M_i is a duo module for all $i \in I$, and*
  (ii) *$N = \oplus_{i\in I}(N \cap M_i)$ for every submodule N of M.*

Note that Lemma 2.4 shows that (ii) in Theorem 2.10 can be replaced by the equivalent condition "(ii) $R = ann(m_i) + ann(m_j)$ for all $m_i \in M_i$, $m_j \in M_j$ for all distinct elements $i, j \in I$." Compare the following result with Corollary 2.8.

COROLLARY 2.11. *Let a module $M = \oplus_{i\in I} M_i$ be a direct sum of submodules M_i ($i \in I$). Then M is a duo module if and only if $M_i \oplus M_j$ is a duo module for all distinct i, j in I.*

Proof. The necessity follows by Proposition 1.3. Conversely, suppose that $M_i \oplus M_j$ is a duo module for all distinct $i \neq j$ in I. Then M_i is a duo module for all $i \in I$, by Proposition 1.3. Moreover, for all $i \neq j$ in I, $R = ann(m_i) + ann(m_j)$ for all $m_i \in M_i$, $m_j \in M_j$ by Lemmas 2.1 and 2.4. Combining Lemma 2.4 and Theorem 2.10, we see that M is a duo module. $\square$

3. Modules over commutative domains.
Throughout this section R will denote a commutative domain with field of fractions $K \neq R$. We investigate when modules over R are duo modules. We begin with an elementary fact.

LEMMA 3.1. *Let an R-module $M = M_1 \oplus M_2$ be a direct sum of a non-zero torsion-free submodule M_1 and a non-zero submodule M_2. Then M is not a duo module.*

Proof. Let m_1 and m_2 be non-zero elements of M_1 and M_2, respectively. Then $ann(m_1) = 0$ and hence $ann(m_1) + ann(m_2) = ann(m_2) \neq R$. By Lemma 2.4 and Theorem 2.10, M is not a duo module. $\square$

We first consider torsion-free modules. Note that the R-module R is a duo module but the R-module K is not a duo module (Proposition 1.6). Let M be a torsion-free R-module. Without loss of generality we can suppose that M is an R-submodule of the K-vector space $K \otimes_R M$. Let $\mathcal{O}(M) = \{k \in K : kM \subseteq M\}$. Note that $\mathcal{O}(M)$ is a subring of K containing R. For example, if M is the R-module R, then $\mathcal{O}(M) = R$. On

the other hand, if S is any subring of K containing R and M is the R-module S, then $\mathcal{O}(M) = S$. In particular, $\mathcal{O}(_R K) = K$.

Although the next result is well known we give an elementary proof for completeness.

LEMMA 3.2. *Let U be a torsion-free uniform R-module. Then a mapping $f : U \to U$ is an endomorphism of U if and only if there exists $k \in \mathcal{O}(U)$ such that $f(u) = ku$ for all $u \in U$.*

Proof. The sufficiency is clear. Conversely, suppose that f is an endomorphism of U. Let $0 \neq u \in U$. Because U is uniform, there exist $a, b \in R$ with $b \neq 0$ such that $bf(u) = au$. Then $k = a/b \in K$. Let $x \in U$. There exist $c, d \in R$ with $d \neq 0$ such that $dx = cu$. Then $df(x) = f(dx) = f(cu) = cf(u)$, so that $bdf(x) = bcf(u) = acu = adx$. Because U is torsion-free, $bf(x) = ax$. It follows that $kx = f(x) \in U$ for all $x \in U$. Hence $k \in \mathcal{O}(U)$ and $f(x) = kx$ for all $x \in U$. □

THEOREM 3.3. *Let R be a commutative domain. Then the following statements are equivalent for a torsion-free uniform R-module U.*

(i) *U is a duo module.*
(ii) *U contains a non-zero cyclic fully invariant submodule.*
(iii) *$\mathcal{O}(U) = R$.*

Proof. (i) ⇒ (ii) is clear.

(ii) ⇒ (iii) Let $k \in \mathcal{O}(U)$. Then the mapping $f : U \to U$ defined by $f(u) = ku$, for all $u \in U$, is an endomorphism of U. Let $0 \neq x \in U$ such that Rx is a fully invariant submodule of U. Then $f(x) = rx$ for some $r \in R$ and hence $kx = f(x) = rx$, so that $k = r \in R$ because U is torsion-free. It follows that $\mathcal{O}(U) = R$.

(iii) ⇒ (i) This follows from Lemma 3.2. □

In contrast to Theorem 3.3, note that any uniform R-module is a weak duo module.

COROLLARY 3.4. *A commutative domain R is integrally closed if and only if every finitely generated torsion-free uniform R-module is a duo module.*

Proof. Suppose first that R is integrally closed. Let U be any finitely generated torsion-free uniform R-module. Let $k \in \mathcal{O}(U)$. Because $kU \subseteq U$, k is integral over R (see, for example, [7, Theorem 12]) and hence $k \in R$. Thus $\mathcal{O}(U) = R$. By Theorem 3.3, U is a duo module. Conversely, suppose that every finitely generated torsion-free uniform R-module is a duo module. Let $q \in K$ such that q is integral over R. There exist a positive integer n and elements $r_i \in R$ ($0 \leq i \leq n-1$) such that $r_0 + r_1 q + \cdots + r_{n-1} q^{n-1} + q^n = 0$. Let $V = R + Rq + \cdots + Rq^{n-1}$. Then V is a finitely generated submodule of the R-module K, so that V is a torsion-free uniform R-module. Clearly $qV \subseteq V$ so that $q \in \mathcal{O}(V) = R$, by Theorem 3.3. It follows that R is integrally closed. □

Note that, in general, not every torsion-free uniform R-module is finitely generated. Recall that an element p of the domain R is called *prime* if Rp is a non-zero prime ideal of R. Equivalently, p is a non-zero non-unit of R such that whenever $a, b \in R$ and $ab \in Rp$ then $a \in Rp$ or $b \in Rp$. Recall that R is a UFD if and only if every non-zero prime ideal contains a prime element (see [7, Theorem 5]). Prime elements p and q in R are called *non-associated* if $Rp \neq Rq$. It is easy to check that prime elements p, q are non-associated if and only if $Rp \not\subseteq Rq$ and $Rq \not\subseteq Rp$.

EXAMPLE 3.5. Suppose that R contains an infinite collection of non-associated prime elements p_i ($i \geq 1$). Then the submodule $U = \sum_{i \geq 1} R(1/p_i)$ of K is a torsion-free uniform duo R-module that is not finitely generated.

Proof. Because it is a submodule of the R-module K, U is a torsion-free uniform module. Let $k \in \mathcal{O}(U)$. Then $k(1/p_1) = (a_1/p_1) + \cdots + (a_n/p_n)$ for some positive integer n and elements $a_i \in R$ ($1 \leq i \leq n$). If $n = 1$, then $k = a_1 \in R$. Suppose that $n > 1$. Then $kp_2 \ldots p_n = b$ where $b = a_1 p_2 \ldots p_n + \cdots + a_n p_1 \ldots p_{n-1} \in R$. Next $k(1/p_2) \in U$ so that $k(1/p_2) = (b_1/p_1) + \cdots + (b_t/p_t)$ for some positive integer t and elements $b_i \in R$ ($1 \leq i \leq t$). It follows that $kp_1 p_3 \ldots p_n = c \in R$. Hence $bp_1 = kp_1 \ldots p_n = cp_2$, so that $b \in Rp_2$. Similarly $b \in Rp_i$ ($3 \leq i \leq n$). Then $k = b/(p_2 \ldots p_n) \in R$. It follows that $\mathcal{O}(U) = R$. By Theorem 3.3, U is a duo module.

Finally note that U is not finitely generated because $1/p_{n+1}$ does not belong to $R(1/p_1) + \cdots + R(1/p_n)$ for every positive integer n. □

Using Example 3.5 we next give an example of an R-module M which is a direct sum $M_1 \oplus M_2$ of duo modules M_1 and M_2 with $Hom(M_1, M_2) = Hom(M_2, M_1) = 0$ but M is not a duo module. (See Theorem 2.10.)

EXAMPLE 3.6. Suppose that R is a UFD such that R contains an infinite collection of non-associated prime elements p_i ($i \geq 1$). Let $U_1 = \sum_{i \geq 1} R(1/p_{2i})$ and $U_2 = \sum_{i \geq 1} R(1/p_{2i-1})$. Then U_1 and U_2 are both torsion-free uniform duo submodules of K such that $Hom(U_1, U_2) = 0$ and $Hom(U_2, U_1) = 0$ but the R-module $M = U_1 \oplus U_2$ is not a duo module.

Proof. By Example 3.5, U_1 and U_2 are both torsion-free uniform duo submodules of K. Let $f : U_1 \to U_2$ be any R-homomorphism. Because K is an injective R-module, f induces an R-homomorphism $g : K \to K$. Then $g(1) = k$ for some $k \in K$. It is easy to check that $f(x) = kx$ for all $x \in U_1$. Thus $kU_1 \subseteq U_2$. Now $k(1/p_2) = (a_1/p_1) + (a_3/p_3) + \cdots + (a_{2n-1}/p_{2n-1})$ for some positive integer n and elements $a_{2i-1} \in R$ ($1 \leq i \leq n$). It follows that $k = b/(p_1 p_3 \ldots p_{2n-1})$ for some $b \in R$. Let $i \geq 1$. Then $k(1/p_{2i}) \in U_2$ implies that $b \in Rp_{2i}$. Thus $b \in \cap_{i \geq 1} Rp_{2i}$. Because R is a UFD, it follows that $b = 0$. Hence $k = 0$ and $f = 0$. Thus $Hom(U_1, U_2) = 0$. Similarly $Hom(U_2, U_1) = 0$. Finally $M = U_1 \oplus U_2$ is not a duo module by Lemma 3.1. □

We now consider finitely generated torsion-free duo R-modules.

THEOREM 3.7. *Let R be a commutative domain. Then the following statements are equivalent for a non-zero finitely generated torsion-free R-module M.*

    (i) *M is a duo module.*
    (ii) *M contains a non-zero cyclic fully invariant submodule.*
    (iii) *M is a uniform module and $\mathcal{O}(M) = R$.*

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii). Let $0 \neq m \in M$ such that Rm is a fully invariant submodule of M. By Zorn's Lemma there exists a submodule L of M maximal such that $Rm \cap L = 0$. It is easy to check that $Rm \oplus L$ is an essential submodule of M and hence $M/(Rm \oplus L)$ is a finitely generated torsion R-module. It follows that $aM \subseteq Rm \oplus L$ for some non-zero element a of R. Define a mapping $g : M \to Rm \oplus L$ by $g(z) = az$ for all $z \in M$. Let $u \in L$. Define a mapping $f : Rm \oplus L \to M$ by $f(rm + x) = ru$ for all $r \in R$, $x \in L$. Then f and g are both homomorphisms and fg is an endomorphism of M. Now $au = f(am) = fg(m) \in Rm \cap L$, because Rm is a fully invariant submodule of M. Thus

$au = 0$ and hence $u = 0$. It follows that $L = 0$, so that Rm is an essential submodule of M. But $Rm \cong R$ which is a uniform R-module. Thus M is uniform and $\mathcal{O}(M) = R$ by Theorem 3.3.

(iii) $\Rightarrow$ (i) This follows from Theorem 3.3. □

We do not know an example of a torsion-free duo R-module which is not uniform. Next we prove a result involving localization. For any maximal ideal P of the ring R, R_P will denote the localization of R with respect to P. That is, R_P is the subring of K consisting of all elements r/c with $r \in R$ and c in $R \setminus P$. Let M be any R-module. Let

$$N = \{m \in M : cm = 0 \text{ for some } c \in R \setminus P\}.$$

Then N is a submodule of M such that $c(m + N) = 0$ for any $m \in M$, $c \in R \setminus P$ implies $m + N = 0$. Thus we can form the localization of $\overline{M} = M/N$ with respect to P, which we denote by M_P. Note that the elements of M_P are of the form $\overline{m}/c$, where $\overline{m} = m + N \in \overline{M}$, $m \in M$ and $c \in R \setminus P$. Also note that M_P is an R_P-module.

PROPOSITION 3.8. *Let M be a module such that the R_P-module M_P is a duo module, for every maximal ideal P of R. Then M is a duo module.*

Proof. Let $m \in M$ and let f be an endomorphism of M. Let $A = \{r \in R : rf(m) \in Rm\}$. Suppose that $A \neq R$. There exists a maximal ideal P of R such that $A \subseteq P$. Define $f^* : M_P \to M_P$ by $f^*(\overline{x}/c) = \overline{f(x)}/c$ for all $x \in M$, $c \in R \setminus P$. It is easy to check that f^* is well-defined and is an endomorphism of M_P. Because M_P is a duo R_P-module, $f^*(\overline{m}/1) = (r/d)(\overline{m}/1)$ for some $r \in R$, $d \in R \setminus P$. Thus $\overline{f(m)}/1 = (\overline{rm})/d$ and it follows that $d'f(m) = rm \in Rm$ for some $d' \in R \setminus P$. Hence $d' \in A \subseteq P$, a contradiction. It follows that $A = R$ and hence $f(m) \in Rm$. By Lemma 1.1, M is a duo R-module. □

For any module M, $E(M)$ will denote the injective envelope of M.

LEMMA 3.9. *Let R be a DVR with unique maximal ideal $P = Rp$ for some element p. Then the following statements are equivalent for a torsion R-module M.*

(i) *M is a uniform module.*
(ii) *$M \cong E(R/Rp)$ or $M \cong R/Rp^n$ for some positive integer n.*
(iii) *M is a duo module.*
(iv) *M is a weak duo module.*

Proof. (i) $\Rightarrow$ (ii) follows from [**6**, Theorem 9].

(ii) $\Rightarrow$ (iii). If (ii) holds, then M is an Artinian uniserial module and hence M is duo by Theorem 1.2.

(iii) $\Rightarrow$ (iv). This is clear.

(iv) $\Rightarrow$ (i). Suppose that M is a weak duo module. Suppose that X is a non-zero injective submodule of M. Then without loss of generality we can suppose that $X \cong E(R/Rp)$ and that $M = X \oplus X'$ for some submodule X'. Suppose that $X' \neq 0$. Then there exists a submodule Y' of X' such that $Y' \cong R/Rp$. Thus there exists an embedding $f : Y' \to X$ that lifts to a non-zero homomorphism $g : X' \to X$, because X is injective. This contradicts Corollary 1.10. It follows that $X' = 0$ and hence that M is uniform.

Now suppose that M is reduced. By [**6**, Theorem 9], $M = N_1 \oplus N_1'$ for some non-zero cyclic submodule N_1 and submodule N_1'. Suppose that $N_1' \neq 0$. Again using [**6**, Theorem 9] we have $N_1' = N_2 \oplus N_2'$ for some non-zero cyclic submodule N_2 and submodule N_2'. For $i = 1, 2$, there exists a positive integer n_i such that $N_i \cong R/Rp^{n_i}$.

Thus $Hom(N_1, N_2) \neq 0$, which contradicts Corollary 1.10. It follows that $N_1' = 0$ and again that M is uniform. $\square$

Now let R be a Dedekind domain and let M be an R-module. For each maximal ideal P of R let

$$T_P(M) = \{m \in M : P^n m = 0 \text{ for some positive integer } n\}.$$

Note that $T_P(M)$ is a submodule of M and is called the *P-primary component* of M. Note further that M is a torsion R-module if and only if $M = \oplus_P T_P(M)$, where the direct sum runs over all maximal ideals P of R.

THEOREM 3.10. *Let R be a Dedekind domain. Then the following statements are equivalent for a non-zero torsion R-module M.*

(i) *M is a duo module.*

(ii) *M is a weak duo module.*

(iii) *There exist distinct maximal ideals P_i ($i \in I$) of R and submodules M_i ($i \in I$) of M such that $M = \oplus_{i \in I} M_i$ and, for each $i \in I$, either $M_i \cong E(R/P_i)$ or $M \cong R/P_i^{n_i}$ for some positive integer n_i.*

(iv) *The module $T_P(M)$ is zero or uniform, for every maximal ideal P of R.*

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii). Suppose that M is a weak duo R-module. There exist distinct maximal ideals P_i ($i \in I$) of R such that $T_{P_i}(M) \neq 0$, for all $i \in I$, and $T_P(M) = 0$, for every, maximal ideal $P \notin \{P_i : i \in I\}$. Let $M_i = T_{P_i}(M)$ for each $i \in I$. Then $M = \oplus_{i \in I} M_i$. Let $i \in I$. By Proposition 1.8, M_i is a weak duo module. It is well known that M_i is an R_{P_i}-module and that a non-empty subset X of M_i is an R-submodule of M_i if and only if X is an R_{P_i}-submodule of M_i. Moreover, a mapping $f : M_i \to M_i$ is an R-homomorphism if and only if f is an R_{P_i}-homomorphism. It follows that the R_{P_i}-module M_i is a weak duo module. Let S denote the ring R_{P_i}. By Lemma 3.9, $M_i \cong E(S/SP_i)$ or $M_i \cong S/SP_i^{n_i}$, for some positive integer n_i. Now [9, Proposition 5.6] gives $M_i \cong E(R/P_i)$ or $M_i \cong R/P_i^{n_i}$.

(iii) $\Rightarrow$ (iv). This is clear.

(iv) $\Rightarrow$ (i). Let P be any maximal ideal of R. Then $M_P \cong T_P(M)$ as R_P-modules. By (iv), $M_P = 0$ or M_P is a uniform R_P-module and hence M_P is a duo R_P-module by Lemma 3.9. Thus M_P is a duo R_P-module for every maximal ideal P of R. By Proposition 3.8, M is a duo R-module. $\square$

COROLLARY 3.11. *Let R be a Dedekind domain. Then the following statements are equivalent for a finitely generated R-module M.*

(i) *M is a duo module.*

(ii) *M is a weak duo module.*

(iii) *$M \cong I$ for some ideal I of R or $M \cong (R/P_1^{n_1}) \oplus \cdots \oplus (R/P_k^{n_k})$ for some positive integers $k, n_1, \ldots, n_k$ and distinct maximal ideals P_i ($1 \leq i \leq k$) of R.*

Proof. (i) $\Rightarrow$ (ii). This is clear.

(ii) $\Rightarrow$ (iii). Note first that $M = M_1 \oplus M_2$, for some torsion-free submodule M_1 and torsion submodule M_2. If $M_1 \neq 0$, then $M_2 = 0$, by Lemma 3.1. Thus M is torsion-free or torsion. If M is non-zero torsion, then $M \cong (R/P_1^{n_1}) \oplus \cdots \oplus (R/P_k^{n_k})$, for some positive integers $k, n_1, \ldots, n_k$ and distinct maximal ideals P_i ($1 \leq i \leq k$), by Theorem 3.10. Suppose that M is torsion-free. Then $M \cong F \oplus A$ for some finitely

generated free R-module F and ideal A of R. By Corollary 1.10, $M \cong I$ for some ideal I of R.

(iii) $\Rightarrow$ (i). This follows from Corollary 3.4 and Theorem 3.10. $\square$

Note finally that if R is a commutative domain which is not integrally closed then there exists a finitely generated uniform R-module U that is not a duo module (Corollary 3.4) but which is a weak duo module.

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