CHARACTERIZATION OF MODULES AND RINGS BY THE SUMMAND INTERSECTION PROPERTY AND THE SUMMAND SUM PROPERTY

A. Hamdouni*, A. Ç. Özcan†, A. Harmancı‡

September 16, 2005

Abstract

R will be a ring with identity and modules M will be unital right $R-$modules. In this paper, properties of modules having the summand intersection property (SIP) and the summand sum property (SSP) are studied. We study the direct sum of modules, the SIP and the SSP. We add some results concerning characterization of some rings by means of modules having the SIP or the SSP.

2000 Mathematics Subject Classification: 16D10
Key words and phrases: semisimple module, injective module, projective module, injective hull, SIP modules, SSP modules, prime module, divisible module, torsion free module, (semi-) hereditary ring, semisimple ring, V-ring, finitely cogenerated module, finitely copresented module.

1 Introduction

Let R be a ring with identity and let M be a right $R-$module. M is said to have *the summand intersection property* (briefly SIP) if the intersection

*Faculté des Sciences, Département de Mathématiques 5000 Monastir, TUNISIE, hamdouniabdelkader2002@yahoo.fr

†Hacettepe University Department of Mathematics 06532 Beytepe Ankara TURKEY, ozcan@hacettepe.edu.tr

‡Hacettepe University Department of Mathematics 06532 Beytepe Ankara TURKEY, harmanci@hacettepe.edu.tr

of any two direct summands is again a direct summand. This definition is given by Wilson [9] and later studied by the authors Garcia [4] and Hausen [5]. Also Garcia consider the dual of the SIP, namely, M has *the summand sum property* (briefly SSP) if the sum of any two direct summands is again a direct summand.

This paper is divided into three different parts. In the second section of the paper, we prove that a projective module M has the SIP if and only if for any two direct summands A and B of M, $A+B$ is projective.

In Section 3, we deal with the SIP(resp. the SSP) property of modules which are of the form $M \oplus N$ where M (resp. N) is indecomposable. Also we prove that if $R \oplus M$ has the SIP (resp. the SSP), then every cyclic submodule of M is projective (resp. direct summand of M). Over a Noetherian domain R, an injective $R-$module M is torsion free if and only if $M \oplus M$ has the SIP. We show that for a projective $R-$module M such that $\oplus_{\Lambda} M$ has the SSP for every index set Λ, then, M is semisimple. As an answer to the question that when a direct sum of two modules having the SIP(resp. the SSP) has the SIP(resp. the SSP), we prove the following. If M and N are two $R-$modules having the SIP(resp. the SSP) such that $r(M)+r(N)=R$, then $M \oplus N$ has the SIP(resp. the SSP).

In the last section some new characterizations of hereditary rings, semisimple rings and V-rings are given. Hence rings satisfying "(*) Any direct sum of modules having the SIP has the SIP" and "(**) Any direct sum of modules having the SSP has the SSP" are considered at the end of the paper.

A ring R is called right (semi–) hereditary if each of its (finitely generated) right ideals is projective. A ring R is called a right V–ring if every simple right $R-$module is injective. The right annihilator of a subset X (resp. an element x) in an $R-$module M will be denoted by $r(X)$ (resp. $r(x)$). $N \leq M$ means that N is a submodule of M.

2 Characterization of modules with the SIP and the SSP

In this section, we give some properties of modules having the summand intersection property and modules having the summand sum property.

Recall that an $R-$module M is called a prime $R-$module if $r(x)=r(y)$ for every non-zero elements x and y in M.

Before we state our next result, one can easily show that for any submodule N of an $R-$module M, if N is injective, then N is a direct summand of M.

In the following proposition, we give a condition under which an $R-$module has the SIP.

Proposition 2.1 *Let M be an injective and a prime $R-$module. Then M has the SIP.*

Proof Let $M = A \oplus A_1$ and $M = B \oplus B_1$. Since M is an injective $R-$module, then A and B are injective $R-$modules. Let I be an ideal of R and f a nonzero $R-$homomorphism from I to $A \cap B$. Let $i_1 : A \cap B \to A$ and $i_2 : A \cap B \to B$ be the inclusion homomorphisms. Consider the $R-$homomorphisms $i_1 \circ f$ from I to A and $i_2 \circ f$ from I to B. By Baer's Criterion, there exist $a \in A$ and $b \in B$ such that $i_1 \circ f(w) = aw$ and $i_2 \circ f(w) = bw$ for each $w \in I$. Since each of i_1 and i_2 is a monomorphism, $i_1 \circ f(w) = f(w)$ and $i_2 \circ f(w) = f(w)$. Therefore, $aw = bw$. Thus, $(a - b)w = 0$. Assume that $a \neq b$. This implies $a \neq 0$ or $b \neq 0$. Suppose that $a \neq 0$. Since $w \in r(a-b)$ and M is prime, $w \in r(a)$ and so $f = 0$. This is a contradiction. Therefore, $a = b \in A \cap B$ and hence, $A \cap B$ is injective. Thus, $A \cap B$ is a direct summand of M. $\square$

The converse of the above proposition is not always true.

Example 2.2 *There exist modules with the SIP that are not either injective or prime.*

Proof Consider $M = \mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}-$module. Then M is semisimple and hence M has the SIP. But M is not injective [3]. Let $\bar{2}, \bar{3} \in M$. Then $r(\bar{2}) = 3\mathbb{Z}$ and $r(\bar{3}) = 2\mathbb{Z}$. Hence M is not prime. $\square$

The following theorem gives a characterization for modules with the SIP and the SSP. The first part of this theorem appears in [5] and the second part appears in [4, 2] . We give its proof for the sake of completeness.

Theorem 2.3 *Let M be an $R-$module. Then*
(1) M has the SIP if and only if for every decomposition $M = A \oplus B$ and every $R-$homomorphism f from A to B, the kernel of f is a direct summand of M.
(2) M has the SSP if and only if for every decomposition $M = A \oplus B$ and every $R-$homomorphism f from A to B, the image of f is a direct summand of M.

Proof (1) Assume M is a module with the SIP. Let $M = A \oplus B$ and f an $R-$homomorphism from A to B. Let $T = \{a + f(a) | a \in A\}$. To show that $M = T \oplus B$, let $x \in M$, then, $x = a + b$ where $a \in A$and $b \in B$. Now, $x = a + f(a) - f(a) + b$. But $a + f(a) \in T$ and $-f(a) + b \in B$, so

$M = T + B$. Now, let $x \in T \cap B$. Hence, $x = a + f(a)$, where $a \in A$, and so $a = x - f(a) \in A \cap B = 0$. Therefore, $f(a) = 0$. Thus, $x = 0$. Since M has the SIP, then $T \cap A$ is a direct summand in M. It is easy to show that $T \cap A = Ker\ f$, so $Ker\ f$ is a direct summand in M.
The converse, assume that for every decomposition $M = A \oplus B$ and every $R-$homomorphism f from A to B, the kernel of f is a direct summand of M. Let $M = N \oplus N_1$, $M = K \oplus K_1$ and let $\pi_{N_1} : M \to N_1$ and $\pi_K : M \to K$ be the natural epimorphisms. Now, define $h = (\pi_{N_1} \circ \pi_K)/N$. Notice that h is defined from N to N_1. Thus, $Ker\ h$ is a direct summand of M. It is easy to check that $Ker\ h = (N \cap K) \oplus (N \cap K_1)$. Since $N \cap K$ is a direct summand of $Ker\ h$ and $Ker\ h$ is a direct summand of M, then $N \cap K$ is a direct summand of M.
(2) Assume that M is a module with the SSP. Let $M = A \oplus B$ and f an $R-$homomorphism from A to B. Let $T = \{a + f(a) | a \in A\}$. As in the first paragraph of the proof $M = T \oplus B$. By hypothesis, $M = (A + T) \oplus L$ for some submodule L of M. Since $A \cap Im\ f = 0$, it is easily checked that $A + T = A \oplus Im\ f$. Hence, $Im\ f$ is a direct summand of M.
For the converse, assume that for every decomposition $M = A \oplus B$ and every $R-$homomorphism f from A to B, the image of f is a direct summand of M. Let $M = N \oplus N_1$, $M = K \oplus K_1$ and let $\pi_{N_1} : M \to N_1$ and $\pi_K : M \to K$ be the natural epimorphisms. Now, define $h = (\pi_{N_1} \circ \pi_K)/N$. Notice that h is defined from N to N_1. Thus, $Im\ h$ is a direct summand of M. Let $T = Im\ h$. Then

$$T = (N + K_1) \cap (N + K) \cap N_1 \text{ and } M = L \leq M.$$

Hence, $N_1 = T \oplus (N_1 \cap L)$. Then,

$$\begin{aligned} (N + K) \cap [(N + K_1) \cap (N_1 \cap L)] &= [(N + K) \cap (N + K_1) \cap N_1] \cap (N_1 \cap L) \\ &= T \cap (N_1 \cap L) = 0. \end{aligned}$$

To show that $N + K$ is a direct summand of M, it is enough to prove that

$$M = (N + K) + [(N + K_1) \cap (N_1 \cap L)].$$

Since $T \leq N + K$ and $T \leq N + K_1$, the modular law and

$$M = N \oplus N_1 = N \oplus [T \oplus (N_1 \cap L)] = (N \oplus T) \oplus (N_1 \cap L)$$

imply

$$\begin{aligned} N + K &= (N + K) \cap [(N \oplus T) \oplus (N_1 \cap L)] \\ &= (N \oplus T) \oplus [(N + K) \cap (N_1 \cap L)] \end{aligned}$$

and,

$$\begin{aligned} N+K_1 &= (N+K_1)\cap[(N\oplus T)\oplus(N_1\cap L)] \\ &= (N\oplus T)\oplus[(N+K_1)\cap(N_1\cap L)]. \end{aligned}$$

Hence,

$$\begin{aligned} M &= N+K+K_1 \\ &= (N\oplus T)+[(N+K)\cap(N_1\cap L)]+[(N+K_1)\cap(N_1\cap L)] \\ &\subseteq (N+K)+[(N+K_1)\cap(N_1\cap L)]. \end{aligned}$$

Thus, $N+K$ is a direct summand of M and so M has the SSP. □

Using the first part of the previous theorem, it is easy to show that every free $R-$module over a principal ideal domain has the SIP.

The following examples show that the direct sum of two modules having the SIP (resp. SSP), may not have the SIP (resp. SSP).

Example 2.4 *There are modules with the SIP (resp. the SSP) such that their direct sum need not have the SIP (resp. the SSP).*

Proof (1) Consider $\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$ -module. It is clear that $\mathbb{Z}_{p^\infty}$ is indecomposable and hence, has the SIP. Now, define a $\mathbb{Z}-$homomorphism f from $\mathbb{Z}_{p^\infty}$ to $\mathbb{Z}_{p^\infty}$ as follows

$$f(n/p^t+\mathbb{Z}) = n/p^{t-1}+\mathbb{Z} \text{ with } n\in\mathbb{Z} \text{ and } t\in\mathbb{N}.$$

It is clear that $Ker\ f=(1/p+\mathbb{Z})$. But $\mathbb{Z}_{p^\infty}$ is indecomposable and hence, $Ker\ f$ is not a direct summand of $\mathbb{Z}_{p^\infty}$. By Theorem 2.3(1), $\mathbb{Z}_{p^\infty}\oplus\mathbb{Z}_{p^\infty}$ does not have the SIP.

(2) Consider $\mathbb{Z}_4$ and $\mathbb{Z}_2$ as $\mathbb{Z}$ -modules. It is clear that each of $\mathbb{Z}_4$ and $\mathbb{Z}_2$ is indecomposable and hence, has the SIP and the SSP. Define f from $\mathbb{Z}_4$ to $\mathbb{Z}_2$ by $f(\overline{x})=\overline{x}$. Then, $Ker\ f=\{\overline{0},\overline{2}\}$ is not a direct summand of $\mathbb{Z}_4$. By Theorem 2.3(1), $\mathbb{Z}_4\oplus\mathbb{Z}_2$ does not have the SIP. Now define f from $\mathbb{Z}_2$ to $\mathbb{Z}_4$ by $f(\overline{x})=2\overline{x}$. Then $Im\ f=2\mathbb{Z}_4$ which is not a direct summand. Hence $\mathbb{Z}_4\oplus\mathbb{Z}_2$ does not have the SSP by Theorem 2.3(2).

(3) Consider $\mathbb{Z}\oplus\mathbb{Z}$ as a $\mathbb{Z}-$module. Since $\mathbb{Z}$ is indecomposable, it has the SSP. But $\mathbb{Z}\oplus\mathbb{Z}$ has not the SSP by Theorem 2.3(2). (see also [2] or [4]). □

A module M is said to have (D3) if for any direct summands A and B of M with $A+B=M$, $A\cap B$ is a direct summand of M. Note that projective modules have (D3), and any direct summand of a module with (D3) has (D3). Dually, a module M has (C3) if for any direct summands A and B with $A\cap B=0$ then $A\oplus B$ is a direct summand of M. Any injective module satisfies (C3), and any direct summand of a module with (C3) has (C3). (see [7])

Theorem 2.5 *Let M be an $R-$module.*
(1) If for any two direct summands A and B of M, $A+B$ has (D3), then M has the SIP.
(2) If for any two direct summands A and B of M, $A \cap B$ has (C3), then M need not have the SSP.

Proof (1) Assume that for any two direct summands A and B of the module M, $A+B$ has (D3). Let $M' = A+B$. Then A and B are also direct summands of M'. By assumption M' has (D3). It follows that $A \cap B$ is a direct summand of M'. Let $M' = (A \cap B) \oplus L$ for some $L \leq M'$. Then $A = (A \cap B) \oplus (A \cap L)$. Hence $A \cap B$ is a direct summand of M since A is a direct summand of M.
(2) It follows from Example 2.6. $\square$

Example 2.6 *There are (injective) modules M such that, for any direct summands A and B of M, $A \cap B$ has C3 but M need not have the SSP.*

Proof Let F be a field and R denote the following ring $R = \{\begin{pmatrix} a & y & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & x \\ 0 & 0 & 0 & a \end{pmatrix} : a, b, x, y \in F\}$, and M the right $R-$module R_R with matrix operations. All proper submodules of M are

$$K = \{\begin{pmatrix} 0 & b & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & x \\ 0 & 0 & 0 & 0 \end{pmatrix} : b, x \in F\},\ L = \{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & x \\ 0 & 0 & 0 & 0 \end{pmatrix} : b, x \in F\},$$

$$N = \{\begin{pmatrix} a & y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix} : a, y \in F\},\ U = \{\begin{pmatrix} 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : y \in F\},$$

$$V = \{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix} : x \in F\},\ K+L = \{\begin{pmatrix} 0 & y & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & x \\ 0 & 0 & 0 & 0 \end{pmatrix} : b, x, y \in F\}.$$

and, (0), M, K, L and N are the only submodules that are direct summands, $M = N \oplus K = N \oplus L$ and $K \cap L = V$ and $SocM = U \oplus V$. It follows that M has (C3) and for any direct summands A and B of M, $A \cap B$ has (C3). But $K+L$ is clearly an essential submodule. Hence M does not have the SSP. M is an injective right $R-$module (See [6, Examples 16.19]). $\square$

Proposition 2.7 *Let M be an $R-$module. Then*
(1) If for any two direct summands A and B of M, $A+B$ is projective $R-$module, then M has the SIP.
(2) If for any two direct summands A and B of M, $A \cap B$ is injective $R-$module, then M has the SSP.

Proof (1) By Theorem 2.5(1).
(2) Let A and B be direct summands of M. By hypothesis M is injective since $M \cap M = M$. Then any direct summand of M is injective, in particular A and B are injective. By hypothesis $M = (A \cap B) \oplus K$ for some $K \leq M$, and so

$$A = (A \cap B) \oplus (A \cap K) \text{ and } B = (A \cap B) \oplus (B \cap K).$$

Therefore $A \cap B$, $A \cap K$ and $B \cap K$ are injective. Since

$$A + B = (A \cap B) \oplus (A \cap K) \oplus (B \cap K),$$

it follows that $A + B$ is injective and then it is a direct summand of M. $\square$

Corollary 2.8 *[2, Lemma 19] Let M be an $R-$module.*
(1) If M has (D3) and the SSP then M has the SIP.
(2) If M has (C3) and the SIP then M has the SSP.

Proof (1) It follows from Theorem 2.5(1).
(2) Let M be a (C3)module. Assume M has the SIP. Let N and T be direct summands of M. Since M has the SIP, then there exists $L \leq M$ such that $(N \cap T) \oplus L = M$. By modularity law, we get that

$$N = (N \cap T) \oplus (L \cap N) \text{ and } T = (N \cap T) \oplus (L \cap T).$$

Then, we have

$$N + T = (N \cap T) + [(L \cap N) \oplus (L \cap T)].$$

Next, we prove that

$$(N \cap T) \cap [(L \cap N) \oplus (L \cap T)] = 0.$$

For if, $x \in (N \cap T) \cap [(L \cap N) \oplus (L \cap T)]$, then $x = n_1 + n_2$ where $n_1 \in L \cap N$ and $n_2 \in L \cap T$. We have

$$n_2 = x - n_1 \in [(N \cap T) + (L \cap N)] \cap (L \cap T) \leq N \cap (L \cap T) = 0.$$

Hence, $n_2 = 0$ and $x = n_1$. Now,

$$x = n_1 \in (N \cap T) \cap (L \cap N) = N \cap T \cap L = 0.$$

Thus,

$$N + T = (N \cap T) \oplus (L \cap N) \oplus (L \cap T) = T \oplus (L \cap N).$$

Since M has the SIP and L, N are direct summands, then $L \cap N$ is a direct summand and so by (C3) it follows that $N + T = T \oplus (L \cap N)$ is a direct summand of M. Thus, M has the SSP. □

The converse of (1) of Theorem 2.7 is not true in general. Let M denote the $\mathbb{Z}-$module $\mathbb{Z}_6$, and let the submodules $A = \{\overline{0}, \overline{2}, \overline{4}\}$ and $B = \{\overline{0}, \overline{3}\}$. It is clear that A and B are direct summands of M. But $A + B = M$ is not a projective $\mathbb{Z}-$module. However, we have the following

Theorem 2.9 *Let M be a projective $R-$module. If M has the SIP, then for any direct summands A and B of M, $A + B$ is a projective $R-$module.*

Proof Suppose that M is projective and has the SIP and let A and B be any direct summands of M. Let $M = (A \cap B) \oplus K$ for some $K \leq M$. Then $A = (A \cap B) \oplus (A \cap K)$, $B = (A \cap B) \oplus (B \cap K)$ and $A + B = (A \cap B) \oplus (A \cap K) \oplus (B \cap K)$. By hypothesis $A \cap B$, $A \cap K$ and $B \cap K$ are direct summands of M and so they are projective. Hence, $A + B$ is projective as a direct sum of projective modules $A \cap B$, $A \cap K$, and $B \cap K$. □

Combining Theorem 2.5(1) and Theorem 2.9 we have the following.

Corollary 2.10 *Let M be a projective $R-$module. Then M has the SIP if and only if for any direct summands A and B of M, $A + B$ is a projective $R-$module.*

The dual of Theorem 2.9 is not true in general because otherwise every injective module with the SSP has the SIP. If it is true then any right hereditary ring must be semisimple by [4, Proposition 1.6] and [9, Proposition 3], this is a contradiction.

3 The direct sum of modules, the SIP and the SSP

It's known that each direct summand of an $R-$module with the SIP (resp. the SSP) has the SIP(resp. the SSP), but the direct sum of modules with the SIP(resp. the SSP) may not have the SIP (resp. the SSP)(see Example 2.4).

In this section we give a necessary condition under which $M \oplus N$ has the SIP (resp. the SSP).

Now, we start by the following proposition.

Proposition 3.1 *(1) Let M be an indecomposable $R-$module and let N be any $R-$module. If $M \oplus N$ has the SIP, then every nonzero $R-$homomorphism from M to N is a monomorphism.*
(2) Let M be any $R-$module and let N be an indecomposable $R-$module. If $M \oplus N$ has the SSP, then every nonzero $R-$homomorphism from M to N is an epimorphism.

Proof (1) Assume $Hom(M, N) \neq 0$ and let f be a nonzero $R-$homomorphism from M to N. Since $M \oplus N$ has the SIP, then $Ker\ f$ is a direct summand of M. But M is indecomposable, so $Ker\ f = 0$ and f is a monomorphism.
(2) Assume $Hom(M, N) \neq 0$ and let f be a nonzero $R-$homomorphism from M to N. Since $M \oplus N$ has the SSP, then $Im\ f$ is a direct summand of N. But N is indecomposable and $f \neq 0$, so $Im\ f = N$ and f is an epimorphism. $\square$

Recall that an $R-$module M is Quasi-Dedekind if every nonzero endomorphism of M is a monomorphism.

Corollary 3.2 *Let M be an indecomposable $R-$module and let N be any $R-$module such that $Hom(M, N) \neq 0$. If $M \oplus N$ has the SIP, then M is Quasi-Dedekind. In particular, if $M \oplus M$ has the SIP, then M is Quasi-Dedekind.*

Proof By Proposition 3.1(1), there is a monomorphism f from M to N. Assume M is not a Quasi-Dedekind $R-$module. Thus, there exists a nonzero endomorphism g of M such that $Ker\ g \neq 0$. Since f is a monomorphism, then $Ker\ f \circ g = Ker\ g \neq 0$. This is a contradiction. Thus, M is a Quasi-Dedekind R-module. $\square$

Corollary 3.3 *Let M be any $R-$module and let N be an indecomposable $R-$module such that $Hom(M, N) \neq 0$. If $M \oplus N$ has the SSP, then every nonzero $R-$endomorphism of N is an epimorphism. In particular, if $N \oplus N$ has the SSP, then every nonzero $R-$endomorphism of N is an epimorphism.*

Proof By Proposition 3.1(2), there is an epimorphism f from M to N. Assume that there exists a nonzero endomorphism g of N such that $Im\ g \neq N$.

Since f is an epimorphism, then $Im\ g \circ f = Im\ g \neq N$. This is a contradiction. Thus, every nonzero $R-$endomorphism of N is an epimorphism. $\square$

Proposition 3.4 *Let M be an $R-$module. If $R \oplus M$ has the SIP(resp. the SSP), then every cyclic submodule of M is projective (resp. direct summand of M). In particular, if $R \oplus R$ has the SIP, then every cyclic ideal is projective as an $R-$module.*

Proof Let $m \in M$, consider the following short exact sequence:

$$0 \to Ker\ f \xrightarrow{i_1} R \xrightarrow{f} mR \to 0$$

where i_1 is the inclusion homomorphism and f is defined as follows $f(r) = mr$, for $r \in R$. Let i_2 be the inclusion homomorphism from mR to M. Now, consider $i_2 \circ f$ from R to M.
If $R \oplus M$ has the SIP, then by Theorem 2.3(1), $Ker\ i_2 \circ f$ is a direct summand of R. But i_2 is a monomorphism, therefore,

$$Ker\ f = Ker\ i_2 \circ f.$$

Thus, the sequence splits. Since R is projective, then mR is projective.
If $R \oplus M$ has the SSP, then by Theorem 2.3(2), $Im\ i_2 \circ f$ is a direct summand of M. But i_2 is a monomorphism, therefore,

$$Im\ i_2 \circ f = Im\ f = mR.$$

Thus, mR is a direct summand of M. $\square$

Before, we state our next result, let us recall that an $R-$module M is called dualizable if $Hom(M, R) \neq 0$.

Corollary 3.5 *Let M be a dualizable indecomposable $R-$module and $M \oplus R$ has the SIP. Then, M is isomorphic to an ideal of R.*

Proof Since $Hom(M, R) \neq 0$, then by Proposition 3.1, M is isomorphic to an ideal of R. $\square$

Proposition 3.6 generalizes [9, Lemma 2] and gives a description of injective modules with the SIP.

Proposition 3.6 *Let M and N be indecomposable R-modules such that M is injective and $Hom(M, N) \neq 0$. If $M \oplus N$ has the SIP, then M is isomorphic to N and M is a Quasi-Dedekind $R-$module.*

Proof By Proposition 3.1, M is isomorphic to a submodule of N and M is a Quasi-Dedekind $R-$module. Since M is injective, there is an injective submodule N_1 of N. By the injectivity of N_1, N_1 is a direct summand of N. Since N is indecomposable, $N_1 = N$. Thus M is isomorphic to N. $\square$

The following can be found in [9] with its proof.

Theorem 3.7 *Let R be a noetherian domain and M an injective $R-$module. If M has the SIP, then either*
(1) M is torsion free, or
(2) M is torsion and for any two distinct indecomposable direct summands A and B of M, $Hom(A, B) = 0$.

Now, we prove:

Theorem 3.8 *Let R be a noetherian domain and M an injective $R-$module. The following are equivalent.*
(1) $M \oplus M$ has the SIP.
(2) M is torsion free.
(3) $\oplus_\Lambda M$ has the SIP for any index set Λ.

Proof (1) $\Rightarrow$ (2) Since M is a direct summand of $M \oplus M$, then M has the SIP. By Theorem 3.7, M is either torsion or torsion free. Suppose that M is torsion so $M \oplus M$ is torsion. Since R is noetherian domain, then by [3, Theorem 25.6], $M = \oplus_{\alpha\in\Lambda} M_\alpha$ where M_α is an indecomposable submodule of M. Now, let $\beta \in \Lambda$ and hence,

$$M \oplus M = M_\beta \oplus M_\beta \oplus (\oplus_{\alpha\in\Lambda} M_\alpha) \oplus (\oplus_{\alpha\in\Lambda} M_\alpha), \text{ with } \alpha \neq \beta.$$

Now, $M_\beta \oplus M_\beta$ has the SIP and injective, thus by Corollary 3.2, M_β is Quasi-Dedekind and hence, M_β is prime which is a contradiction. To verify this, suppose M_β is prime. Since M is torsion, then M_β is torsion. But M_β is injective over integral domain, therefore, M_β is divisible. Now, let $0 \neq x \in M_\beta$ and $0 \neq r \in r(x)$. Since M_β is divisible, then $x = yr$ for some $y \in M_\beta$. Thus, M_β is not prime.
(2) $\Rightarrow$ (1) Since M is torsion free, then $M \oplus M$ is torsion free. Thus, $M \oplus M$ is torsion free and injective hence has the SIP, Proposition 2.1.
(2) $\Rightarrow$ (3) Since M is torsion free, then $\oplus_\Lambda M$ is torsion free for any index set Λ. But $\oplus_\Lambda M$ is injective, then by Proposition 2.1, $\oplus_\Lambda M$ has the SIP. $\square$

Before starting the next result, we need the following proposition.

Proposition 3.9 *Let $M = M_1 \oplus M_2$ be an $R-$module. If $r(M_1) + r(M_2) = R$, then every submodule N of M can be written as $N = N_1 \oplus N_2$, where $N_1 \leq M_1$ and $N_2 \leq M_2$.*

Proof Let N be a submodule of M. Let $N_1 = Nr(M_2)$ and $N_2 = Nr(M_1)$. It is easy to show that $N_1 \leq M_1$ and $N_2 \leq M_2$. Therefore, $N_1 \cap N_2 = 0$. It is enough to show that $N \leq N_1 \oplus N_2$. Let $n \in N$, there exists $a \in r(M_1)$ and $b \in r(M_2)$ such that $a + b = 1$. Therefore, we have

$$n = n.1 = n.(a + b) = n.a + n.b.$$

But $n.b \in N_1$ and $n.a \in N_2$. Hence, $N = N_1 \oplus N_2$, where $N_1 \leq M_1$ and $N_2 \leq M_2$. $\square$

We end this section by the following theorem giving condition under which the direct sum of modules with the SIP(resp. the SSP) has the SIP(resp. the SSP).

Theorem 3.10 *Let M and N be two $R-$modules having the SIP(resp. the SSP), such that $r(M) + r(N) = R$. Then, $M \oplus N$ has the SIP (resp. the SSP).*

Proof Let A and B two direct summands of $M \oplus N$. By the previous proposition, $A = M_1 \oplus N_1$ and $B = M_2 \oplus N_2$, where M_1 and M_2 are two submodules of M, N_1 and N_2 are two submodules of N. It is easy to show that M_1 and M_2 are two direct summands of M, N_1 and N_2 are two direct summands of N.
If M and N have the SIP, then $M_1 \cap M_2$ is a direct summand of M and $N_1 \cap N_2$ is a direct summand of N. Therefore, $(M_1 \cap M_2) \oplus (N_1 \cap N_2)$ is a direct summand of $M \oplus N$. Now,

$$(M_1 \cap M_2) \oplus (N_1 \cap N_2) = (M_1 \oplus N_1) \cap (M_2 \oplus N_2) = A \cap B.$$

Thus, $A \cap B$ is a direct summand of $M \oplus N$ and hence, $M \oplus N$ has the SIP.
If M and N have the SSP, then $M_1 + M_2$ is a direct summand of M and $N_1 + N_2$ is a direct summand of N. Therefore, $(M_1 + M_2) \oplus (N_1 + N_2)$ is a direct summand of $M \oplus N$. Now,

$$(M_1 + M_2) \oplus (N_1 + N_2) = (M_1 + N_1) + (M_2 + N_2) = A + B.$$

Thus, $A + B$ is a direct summand of $M \oplus N$ and hence, $M \oplus N$ has the SSP. $\square$

4 Characterization of rings by means of the SIP and the SSP

In this section, we give some theorems that classify right hereditary, right semihereditary, semisimple and right V- rings in terms of modules that have the SIP or the SSP.

The following two theorems give a characterization of right hereditary and right semihereditary rings. For the proof, see [1].

Theorem 4.1 *The following statements are equivalent for a ring R.*
(1) R is right hereditary.
(2) Every submodule of a projective $R-$module is projective.
(3) Every submodule of a free $R-$module is projective.

Theorem 4.2 *The following statements are equivalent for a ring R.*
(1) R is right semihereditary.
(2) Every finitely generated submodule of a projective $R-$module is projective.
(3) Every finitely generated submodule of a free $R-$module is projective.

The following theorem appears in [9]. However, we give it with a different proof.

Theorem 4.3 *The following statements are equivalent for a ring R.*
(1) R is right hereditary.
(2) Every projective $R-$module has the SIP.

Proof (1) $\Rightarrow$ (2) Suppose that R is right hereditary and M is any projective $R-$module, then by Theorem 4.1, every submodule of M is projective. Thus, by Corollary 2.10, M has the SIP.
(2) $\Rightarrow$ (1) Let M be any projective $R-$module and let N be any submodule of M. Choose a free module F and an epimorphism σ from F to N. Let i be the inclusion map from N to M . Consider $i \circ \sigma$ from F to M. By hypothesis $F \oplus M$ has the SIP. By Theorem 2.3(1), $Ker\ i \circ \sigma$ is a direct summand. Since i is a monomorphism, so $Ker\ i \circ \sigma = Ker\ \sigma$. Hence, N is isomorphic to a direct summand of F. Thus, N is projective. $\square$

It is known that a ring R is right hereditary if and only if every factor module of every injective $R-$module is injective.

The following theorem, gives a characterization of right hereditary ring in terms of modules having the SSP.

Theorem 4.4 *([2, Theorem 10]) The following statements are equivalent for a ring R.*
(1) R is right hereditary.
(2) Every injective $R-$module has the SSP.

Proof (1) $\Rightarrow$ (2) Suppose that R is right hereditary. Then every factor of an an injective $R-$module is injective. Let M be an injective $R-$module which has decomposition $M = A \oplus B$ and f be an $R-$homomorphism from A to B. Then A is injective. By assumption $Im\ f \cong A/Ker\ f$ is injective. Hence, $Im\ f$ is direct summand of B. Thus, M has the SSP, by Theorem 2.3(2).
(2) $\Rightarrow$ (1) Assume that every injective $R-$module has the SSP. Let M be an injective $R-$module and N be a submodule of M. By assumption the injective hull $E(M/N)$ of M/N and the injective module $M \oplus E(M/N)$ have the SSP. Let ϕ denote the canonical map from M to M/N and ι the injection of M/N into $E(M/N)$ and f the composition of $\iota\phi$. Then $Im\ f = M/N$. By Theorem 2.3(2), M/N is direct summand of $E(M/N)$. Hence, M/N is injective. Thus, R is right hereditary ring. $\square$

Theorem 4.5 *The following statements are equivalent for a ring R.*
(1) R is right hereditary.
(2) Every free $R-$module has the SIP.

Proof (1) $\Rightarrow$ (2) Follows from Theorem 4.3.
(2) $\Rightarrow$ (1) Let I be an ideal in R. Choose a free $R-$module F and an epimorphism σ from F to I. Let i be the inclusion map from I to R . Consider $i \circ \sigma$ from F to R. Since $F \oplus R$ is free, then $F \oplus R$ has the SIP and hence, $Ker\ i \circ \sigma$ is a direct summand. But i is a monomorphism, then $Ker\ i \circ \sigma = Ker\ \sigma$. Thus, $Ker\ \sigma$ is a direct summand of F and hence, I is projective. $\square$

Hausen in [5], shows that a ring R is right semihereditary if and only if every finitely generated projective $R-$module has the SIP. The proof of Theorem 4.6 follows from the same argument of the proof of Theorem 4.5.

Theorem 4.6 *The following statements are equivalent for a ring R.*
(1) R is right semihereditary.
(2) Every finitely generated free $R-$module has the SIP.

Theorem 4.7 *Let R be a commutative ring. The following statements are equivalent.*
(1) R is right semihereditary.
(2) The $R-$module $R \oplus R \oplus R$ has the SIP.

Proof (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (1) Let $J = aR + bR$ be two generated right ideal in R. Define f from $R \oplus R$ to J by

$$f(r_1 + r_2) = ar_1 + br_2.$$

It is clear that f is onto. Let i be the inclusion map from J to R. Since $i \circ f$ is defined from $R \oplus R$ to R and $R \oplus R \oplus R$ has the SIP, then $Ker\ i \circ f$ is a direct summand by Theorem 2.3(1). But i is a monomorphism, therefore $Ker\ i \circ f = Ker\ f$ is a direct summand. Hence, $aR + bR$ is projective $R-$module. Thus R is right semihereditary by [8, Corollary 4.4]. $\square$

The following theorems give characterization of semisimple rings in terms of modules having the SIP [9, Proposition 3] and of modules having the SSP [2, Theorem 9].

Theorem 4.8 *The following statements are equivalent for a ring R.*
(1) R is semisimple.
(2) Every $R-$module has the SIP.
(3) Every injective $R-$module has the SIP.

Theorem 4.9 *The following statements are equivalent for a ring R.*
(1) R is semisimple.
(2) Every $R-$module has the SSP.
(3) Every projective $R-$module has the SSP.

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Trivial.
(3) $\Rightarrow$ (1) Assume that every projective $R-$module has the SSP. We show that R is semisimple. Let K be a submodule of R. Choose a free module F and an epimorphism τ from F onto K. By assumption, the projective module $F \oplus R$ has the SSP.Let ι denote the injective map from K to R and $f = \iota\tau$ the homomorphism from F to R. Then $Im\ f = K$ is direct summand of R, by Theorem 2.3(2). Hence, R is semisimple ring. $\square$

Now we give some other conditions to characterize semisimple rings.

Theorem 4.10 *The following statements are equivalent for a ring R.*
(1) R is semisimple.
(2) Every injective $R-$module has the SIP.
(3) Every injective $R-$module is semisimple.
(4) Every quasi-injective $R-$module has the SIP.
(5) Every quasi-injective $R-$module is semisimple.

Proof (1) $\Rightarrow$ (2) Clear since every $R-$module is semisimple.
(2) $\Rightarrow$ (3) Let M be an injective $R-$module and let N be a submodule of M. Let f be the natural epimorphism from M to M/N. If M/N is injective, then $M \oplus M/N$ is injective and hence, $M \oplus M/N$ has the SIP. Thus, $Ker\ f$ is a summand of M. Thus, M is semisimple. Assume that M/N is not injective. Let $E(M/N)$ be the injective hull of M/N and i be the inclusion map from M/N to $E(M/N)$. Now, consider $i \circ f$ from M to $E(M/N)$. Since $M \oplus E(M/N)$ has the SIP, $Ker\ i \circ f = Ker\ f = N$ is a direct summand of M. Thus, M is semisimple.
(3) $\Rightarrow$ (4) Let M be any quasi-injective $R-$module. Let $E(M)$ be the injective hull of M. By (3), $E(M)$ is semisimple. As a submodule of the semisimple module $E(M)$, M is semisimple and therefore M has the SIP.
(4) $\Leftrightarrow$ (5) Let M be any quasi-injective module. Let N be any submodule of M. By (4) $M \oplus E(M/N)$ has the SIP. Let π denote the canonical epimorphism from M to M/N and i the inclusion map from M/N into $E(M/N)$. Then $Ker\ i \circ f = N$. By Theorem 2.3, N is direct summand of M. Hence, M is semisimple.
(5) $\Leftrightarrow$ (1) Let M be any $R-$module. By (5) $E(M)$ is is semisimple. Then M is semisimple. □

Dually we have the following characterization of semisimple rings.

Theorem 4.11 *The following are equivalent for a ring R.*
(1) R is semisimple.
(2) Every free $R-$module has the SSP.
(3) Every free $R-$module is semisimple.
(4) Every quasi-projective $R-$module has the SSP.
(5) Every quasi-projective $R-$module is semisimple.

Proof (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (3) Let M be a free $R-$module and $N \leq M$. There exists a free $R-$module F and an epimorphism f from F to N. Then $F \oplus M$ is free and hence has the SSP. Let i be the inclusion map from N to M. Then $Im\ i \circ f = N$ is a direct summand of M. Hence M is semisimple.
(3) $\Rightarrow$ (4) Since every $R-$module is an epimorphic image of a free $R-$module, every $R-$module is semisimple.
(4) $\Rightarrow$ (5) Let M be a quasi-projective $R-$module and $N \leq M$. Let F be a free $R-$module and f an epimorphism from F to N. Since $F \oplus M$ is quasi-projective, $Im\ i \circ f = N$ is a direct summand of M where i is the inclusion map from N to M. Hence M is semisimple.
(5) $\Rightarrow$ (1) Clear. □

Let R be a ring. An $R-$module X is called finitely copresented if i) X is finitely cogenerated and ii) in every exact sequence $0 \to X \to L \to N \to 0$ in Mod -R with L finitely cogenerated, N is also finitely cogenerated. R is a right V-ring if and only if every finitely cogenerated $R-$module is semisimple [10, 23.1] if and only if every finitely copresented $R-$module is injective [10, 31.7].

Theorem 4.12 *The following are equivalent for a ring R.*
(1) R is a right V-ring.
(2) Every finitely cogenerated $R-$module has the SIP.
(3) Every finitely copresented $R-$module has the SIP.
(4) Every finitely cogenerated $R-$module has the SSP.
(5) Every finitely copresented $R-$module has the SSP.

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) They are clear by [10, 23.1] and definitions.
(3) $\Rightarrow$ (1) Let M be a finitely copresented $R-$module. By [10, 30.1], $E(M)$ and $E(M)/M$ are finitely cogenerated. Since $E(M)/M$ is finitely cogenerated, $E(E(M)/M)$ is finitely cogenerated. Since any finitely cogenerated injective modules are finitely copresented, by (3) and [10, 21.4], $E(M) \oplus E(E(M)/M)$ has the SIP. This implies that $Ker\ (i \circ f) = Ker\ f = M$ is a direct summand of $E(M)$ where f is the canonical epimorphism from $E(M)$ to $E(M)/M$ and i is the inclusion homomorphism from $E(M)/M$ to $E(E(M)/M)$. Hence, M is injective. By [10, 31.7], R is a right V-ring.
(1) $\Rightarrow$ (4) $\Rightarrow$ (5) They are clear.
(5) $\Rightarrow$ (1) Let M be a finitely copresented $R-$module. Then $E(M)$ and $E(M)/M$ is finitely copresented. Then $M \oplus E(M)$ is finitely copresented by [10, 30.2(3)]. Consider the inclusion map $i : M \to E(M)$. By (5), $Im\ i = M$ is a direct summand of $E(M)$. Hence, M is injective. By [10, 31.7], R is a right V-ring. $\square$

We consider the following conditions for modules over the ring R.
(*) Any direct sum of modules with the SIP has the SIP.
(**) Any direct sum of modules with the SSP has the SSP.

Proposition 4.13 *If a ring R satisfies (*) or (**), then R is a right V-ring.*

Proof Let M be a finitely cogenerated $R-$module. Then M is a direct sum of indecomposable $R-$modules by [10, 21.3]. Since indecomposable modules have the SIP and the SSP, then M has the SIP or the SSP by hypothesis. Then by Theorem 4.12, R is a right V-ring. $\square$

Proposition 4.14 *The following are equivalent for a ring R.*
(1) R is a right noetherian ring with (*).
(2) R is semisimple.

Proof (2) $\Rightarrow$ (1) If R is semisimple, then every $R-$module has SIP by [9]. Hence, (1) holds.
(1) $\Rightarrow$ (2) Let M be an injective $R-$module. Since R is right noetherian ring, M is a direct sum of indecomposable modules. Since indecomposable modules have the SIP, M has the SIP. By [9], R is semisimple. $\square$

Proposition 4.15 *Let R be a ring. If R is a right noetherian ring with (**) then R is a right hereditary right V-ring.*

Proof Let M be an injective $R-$module. Then M is a direct sum of indecomposable modules. Hence, M has the SSP. By Theorem 4.4, R is right hereditary. $\square$

References

[1] **W. K. Al-Aubaidy**, The Ring of Endomorphisms of Multiplication Modules, M Sc. Thesis, University of Baghdad, 1993.

[2] **M. Alkan** and **A. Harmanci**, On Summand Sum and Summand Intersection Property of Modules, Turkish J. Math, 26(2002), 131-147.

[3] **F. W. Anderson** and **K. R. Fuller**, Rings and Categories of Modules, Springer-Verlag, New York 1974.

[4] **J. L. Garcia**, Properties of Direct Summands of Modules, Comm. Algebra, 17(1989), 73-92.

[5] **J. Hausen**, Modules with the Summand Intersection Property, Comm. Algebra 17(1989), 135-148.

[6] **T. Y. Lam**, Lectures on Modules and Rings, Springer-Verlag, New-York 1999.

[7] **S. H. Mohamed** and **B. J. Muller**, Continuous and Discrete Modules, London Math. Soc. LNS 147 Cambridge Univ. Press, Cambridge, 1990.

[8] **A. G. Naoum** and **M. A. K. Hasan**, On Finitely Generated Projective and Ohm Condition, Perodica Mathematica Hungaria 16(1) (1985), 51-60.

[9] **G. V. Wilson**, Modules with the Direct Summand Intersection Property, Comm. Algebra 14 (1986), 21-38.

[10] **R. Wisbauer**, Foundations of Module and Ring Theory, Gordon and Breach, Reading, 1991.