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ON JACOBSON AND NIL RADICALS RELATED TO POLYNOMIAL RINGS

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Abstract. This note is concerned with examining nilradicals and Jacobson radicals of polynomial rings when related factor rings are Armendariz. Especially we elaborate upon a well-known structural property of Armendariz rings, bringing into focus the Armendariz property of factor rings by Jacobson radicals. We show that $J(R[x]) = J(R)[x]$ if and only if $J(R)$ is nil when a given ring R is Armendariz, where $J(A)$ means the Jacobson radical of a ring A. A ring will be called *feckly Armendariz* if the factor ring by the Jacobson radical is an Armendariz ring. It is shown that the polynomial ring over an Armendariz ring is feckly Armendariz, in spite of Armendariz rings being not feckly Armendariz in general. It is also shown that the feckly Armendariz property does not go up to polynomial rings.

1. On radicals when factor rings are Armendariz

Throughout this note every ring is associative with identity unless otherwise stated. For a ring R, $J(R)$, $N_*(R)$, $N^*(R)$, $N_0(R)$ and $N(R)$ denote the Jacobson radical, the prime radical, the upper nilradical (i.e., sum of all nil ideals), the Wedderburn radical (i.e., the sum of all nilpotent ideals), and the set of all nilpotent elements in R, respectively. Following [1, p. 130], a subset of R is said to be *locally nilpotent* if its finitely generated subrings are nilpotent. Also due to [1, p. 130], the Levitzki radical of R, written by $s\sigma(R)$, means the sum of all locally nilpotent ideals of R. It is well-known that $N^*(R) \subseteq J(R)$ and $N_0(R) \subseteq N_*(R) \subseteq s\sigma(R) \subseteq N^*(R) \subseteq N(R)$. We use $R[x]$ (resp., $R[[x]]$) to denote the polynomial (resp., power series) ring with an indeterminate x over R. For $f(x) \in R[x]$, let $C_{f(x)}$ denote the set of all coefficients of $f(x)$. Let $\mathrm{Mat}_n(R)$ (resp., $U_n(R)$) be the n by n full (resp., upper triangular) matrix ring over R. Let $D_n(R)$ denote the subring

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$\{m \in U_n(R) \mid$ the diagonal entries of m are all equal$\}$ of $U_n(R)$. Let E_{ij} denote the matrix in $\mathrm{Mat}_n(R)$ with (i,j)-entry 1 and elsewhere 0. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n).

A ring is called *reduced* if it has no nonzero nilpotent elements, i.e., $N(R) = 0$. For a reduced ring R and $f(x), g(x) \in R[x]$, Armendariz [6, Lemma 1] proved that

$$\text{if } f(x)g(x) = 0, \text{ then } ab = 0 \text{ for all } a \in C_{f(x)}, b \in C_{g(x)}.$$

Based on this result, Rege and Chhawchharia [27] called a ring (possibly without identity) *Armendariz* if it satisfies this property. So reduced rings are clearly Armendariz and the class of Armendariz rings is closed under subrings obviously. These facts will be used freely in this note. A ring is called *Abelian* if every idempotent is central. Armendariz rings are Abelian by the proof of [3, Theorem 6] (or [20, Lemma 7]).

Antoine called a ring R *nil-Armendariz* [4, Definition 2.3] if $ab \in N(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever $f(x)g(x) \in N(R)[x]$ for $f(x), g(x) \in R[x]$, and showed that Armendariz rings are nil-Armendariz in [4, Proposition 2.7] but not conversely by help of [4, Proposition 2.8] since Armendariz rings are Abelian. Antoine also showed that if R is a nil-Armendariz ring, then $N(R)$ forms a subring of R, in [4, Theorem 3.2]. While, Jung et al. proved that a ring R is nil-Armendariz if and only if $ab \in N^*(R)$ for all $a \in C_{f(x)}, b \in C_{g(x)}$ whenever $f(x)g(x) \in N^*(R)[x]$ for $f(x), g(x) \in R[x]$ in [17, Theorem 11].

Lemma 1.1. (1) [18, Lemma 2.3(5)] *If a ring R is Armendariz, then*

$$N_0(R) = N_*(R) = N^*(R).$$

(2) [4, Proposition 2.7] *Armendariz rings are nil-Armendariz.*

(3) [4, Theorem 3.5] *A ring R is nil-Armendariz if and only if $R/N^*(R)$ is Armendariz.*

(4) [3, Theorem 2] *A ring R is Armendariz if and only if so is $R[x]$.*

(5) [12, Theorem 1.4(2)] *If a ring R is Armendariz, then so is $R/N_*(R)$.*

(6) [2, Theorem 3] $N_*(R[x]) = N_*(R)[x]$ *for any ring R.*

(7) [2, Theorem 1] $J(R[x]) = N[x]$ *for any ring R, where $N = J(R[x]) \cap R$ is a nil ideal of R which contains $s\sigma(R)$.*

If a given ring R is Armendariz, then both $R/N^*(R)$ and $R/N_*(R)$ are Armendariz by Lemma 1.1(3, 5). So it is natural to ask whether $R/J(R)$ is Armendariz if R is an Armendariz ring. However the answer of this question is negative by the following. But we will see an affirmative answer when Jacobson radicals are nil.

Example 1.2. We apply the ring in [14, Example 3]. Let R_0 be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$, where p is an odd prime. We next set R be the quaternions over R_0. Then R is clearly a domain (hence Armendariz) and $J(R) = pR$. But $R/J(R)$ is isomorphic to $\mathrm{Mat}_2(\mathbb{Z}_p)$ by the argument in [11, Exercise 2A] but $\mathrm{Mat}_2(\mathbb{Z}_p)$ is not Armendariz by [27, Remark 3.1]. Thus $R/J(R)$ is not Armendariz.

Considering Lemma 1.1(3, 5), one may naturally ask whether the converse of Lemma 1.1(5) also holds. But the answer is negative as can be seen by $R = U_n(A)$ $(n \geq 2)$ over a reduced ring A. In fact, $N_*(R) = \{M \in U_n(A) \mid$ the diagonals of M are all zero$\}$ and $R/N_*(R)$ is isomorphic to a direct sum of n-copies of A. So $R/N_*(R)$ is reduced (hence Armendariz), but R is not Armendariz since R is not Abelian.

The proof of the following theorem can be obtained by the results of A. R. Nasr-Isfahani and A. Moussavi [25]. But we provide another proof as follows, using the well-known fact of Amitsur for Jacobson radicals of polynomial rings.

Theorem 1.3. *For an Armendariz ring R we have*

$$J(R[x])=N_0(R[x])=N_*(R[x])=N^*(R[x])=N_0(R)[x]=N^*(R)[x]=N_*(R)[x].$$

Proof. Let R be an Armendariz ring. Then $R[x]$ is also Armendariz by Lemma 1.1(4), and so we have

$$N_0(R[x]) = N_*(R[x]) = N^*(R[x])$$

by Lemma 1.1(1). Next we get

$$J(R[x]) \subseteq N^*(R)[x]$$

by help of [2, Theorem 1]. We already have $N_*(R)[x] = N_*(R[x])$ and $N_0(R)[x] = N_0(R[x])$ by Lemma 1.1(6) and [7, Corollary 4], respectively. It is clear that $N_*(R[x]) \subseteq N^*(R[x])$ and $N^*(R[x]) \subseteq J(R[x])$. These results are combined to obtain

$$J(R[x]) \subseteq N^*(R)[x]=N_0(R)[x]=N_*(R)[x]=N_*(R[x]) \subseteq N^*(R[x]) \subseteq J(R[x]),$$

and therefore

$$J(R[x])=N_0(R)[x]=N_0(R[x])=N_*(R[x])=N^*(R[x])=N^*(R)[x]=N_*(R)[x].$$

□

In the light of Theorem 1.3, one may conjecture that $J(R)$ is nil for an Armendariz ring R. But this fails in general as can be seen by the Jacobson radical $x\mathbb{Z}[[x]]$ of the domain (hence Armendariz) $\mathbb{Z}[[x]]$.

Recall that a ring is called *right Goldie* if it has no infinite direct sum of right ideals and has the ascending chain condition on right annihilators.

Corollary 1.4. *For a right Goldie Armendariz ring R we have*

$$J(R[x]) = N_0(R)[x] = N[x],$$

where N is the nilpotent radical of R.

Proof. Let R be a right Goldie Armendariz ring. Then $N^*(R)$ is nilpotent by [22], N say. It then follows that

$$J(R[x]) = N_0(R)[x] = N[x]$$

by Theorem 1.3, noting that $N_0(R) = N^*(R) = N$. □

Considering $N_*(R)[x] = N_*(R[x])$, one may hope the equalities $J(R)[x] = J(R[x])$ and $N^*(R)[x] = N^*(R[x])$. However the following examples provide negative situations. Recall the ring $R = \mathbb{Z}[[x]]$. Note that $J(R) = x\mathbb{Z}[[x]]$ but $J(R[t]) = 0$ by Theorem 1.3 or Corollary 1.4, where t is an indeterminate over R. Thus $J(R)[t] \neq J(R[t])$ in general. Next we have $N^*(R)[t] \neq N^*(R[t])$ by help of Smoktunowicz [28].

Proposition 1.5. (1) *If R is an Armendariz ring and $J(R)$ is nil, then $R/J(R)$ is Armendariz.*

(2) *If R is an Armendariz ring, then $R[x]/J(R[x])$ is Armendariz.*

(3) *If R is an Armendariz ring which satisfies a polynomial identity, then $R[x]/J(R[x])$ is Armendariz and $J(R[x])$ is nil.*

Proof. (1) Armendariz rings are nil-Armendariz by Lemma 1.1(2). This yields $R/J(R)$ being Armendariz by Lemma 1.1(3) when $J(R)$ is nil.

(2) Let R be an Armendariz ring. Then $J(R[x]) = N_*(R[x]) = N_*(R)[x]$ by Theorem 1.3. This yields

$$\frac{R[x]}{J(R[x])} = \frac{R[x]}{N_*(R[x])} = \frac{R}{N_*(R)}[x] = \frac{R}{N^*(R)}[x]$$

by help of Lemma 1.1(6). Moreover $R/N^*(R)$ is Armendariz by Lemma 1.1(2, 3), and hence $\frac{R}{N^*(R)}[x]$ is Armendariz by Lemma 1.1(4). Therefore $R[x]/J(R[x])$ is Armendariz.

(3) If R is an Armendariz ring, then $R[x]/J(R[x])$ is Armendariz by (2). If a given ring R satisfies a polynomial identity, then every nil ideal of R is locally nilpotent by [24]. So we get $N \subseteq s\sigma(R)$ where $N = J(R[x]) \cap R$ in Lemma 1.1(7). Further, $N[x]$ is nil since N is locally nilpotent. This entails that $J(R[x])$ is nil. □

Recall that there exists a ring R with $J(R)[x] \neq J(R[x])$, and note that $J(R[x]) \subseteq J(R)[x]$ by Lemma 1.1(7). We see a condition under which $J(R)[x] = J(R[x])$ holds as follows.

Proposition 1.6. *For an Armendariz ring R the following conditions are equivalent*:

(1) $J(R[x]) = J(R)[x]$.

(2) $J(R)$ *is nil.*

(3) $J(R)$ *is nil and* $R/J(R)$ *is Armendariz.*

Proof. (1)⇒(2). If $J(R[x]) = J(R)[x]$, then $J(R)$ is nil by Lemma 1.1(7).

(2)⇒(3). Assume that $J(R)$ is nil. Then $R/J(R)$ is Armendariz by Proposition 1.5(1).

(3)⇒(1). Since R is Armendariz, $R[x]/J(R[x])$ is Armendariz by Proposition 1.5(2). Assume that $J(R)$ is nil. Then $J(R) = N^*(R)$, and so $J(R[x]) = N^*(R)[x]$ by Theorem 1.3. Thus we have $J(R[x]) = J(R)[x]$. □

In Proposition 1.6, we get

$$\frac{R[x]}{J(R[x])} = \frac{R[x]}{J(R)[x]} \cong \frac{R}{J(R)}[x]$$

which are Armendariz rings.

As regards Proposition 1.5(1), one may ask whether a ring R is Armendariz if $J(R)$ is nil and $R/J(R)$ is Armendariz. However the answer is negative by the following.

Example 1.7. We use the ring construction and computation in [15, Example 1.2]. Let S be a simple domain and $R_n = D_{2^n}(S)$, where n is a positive integer. Define a map

$$\sigma : R_n \to R_{n+1} \text{ by } A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Then R_n can be considered as a subring of R_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in R_n$). Set $R = \varinjlim R_n$ be the direct limit of $\{R_n, \sigma_{nm}\}$, noting that $\sigma_{nm} = \sigma^{m-n}$ (when $n \leq m$) is a direct system over $I = \{1, 2, \ldots\}$. Then $N_*(R) = 0$ by [16, Theorem 2.2] (or applying the computation in [15, Example 1.2]), but it is easily checked that

$$J(R) = N^*(R) = \{A = (a_{ij}) \in R \mid a_{ii} = 0 \text{ for all } i\}.$$

So $\frac{R}{J(R)} \cong S$ is a domain (hence Armendariz) and $J(R)$ is nil. However R is not Armendariz by help of [20, Example 3]. In fact, let

$$f(x) = E_{1(2^n+2)} + (E_{1(2^n+2)} - E_{1(2^n+3)})x$$

and

$$g(x) = E_{(2^n+3)(2^n+4)} + (E_{(2^n+2)(2^n+4)} + E_{(2^n+3)(2^n+4)})x$$

in $D_{2^{n+1}}(S)[x] \subset R[x]$ for $n \geq 2$. Then $f(x)g(x) = 0$, but we have

$$E_{1(2^n+2)}(E_{(2^n+2)(2^n+4)} + E_{(2^n+3)(2^n+4)}) = E_{1(2^n+4)} \neq 0.$$

From Proposition 1.5(1), one may also ask whether $J(R)$ is nil if both R and $R/J(R)$ are Armendariz. However the answer is also negative as can be seen by the ring $R = D[[x]]$ over a division ring D. Note that $J(R) = xD[[x]]$ is not nil. But both R and $R/J(R)(\cong D)$ are reduced (hence Armendariz).

A ring R is usually called *right* (*left*) *weakly* π*-regular* if for each $a \in R$ there exists a positive integer n such that $a^n \in a^nRa^nR$ (resp., $a^n \in Ra^nRa^n$). Jacobson radicals of left or right weakly π-regular rings are nil by [19, Lemma 5]. So if an Armendariz ring R is right weakly π-regular, then $R/J(R)$ is Armendariz by Proposition 1.5.

Due to Antoine [5], a ring R is called *1-Armendariz* if for given $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x$ in $R[x]$, $f(x)g(x) = 0$ implies that $a_ib_j = 0$ for each i, j. These rings are also called *weak Armendariz* by Lee and Wong [23]. It is obvious that Armendariz rings are 1-Armendariz, but the converse need not be true by the examples in [5] and [23]. It is also clear that the class of 1-Armendariz rings is closed under subrings. We will use this fact freely.

Proposition 1.8. *For a ring R, the following conditions are equivalent*:

(1) *R is reduced.*
(2) *$D_3(R)$ is Armendariz.*
(3) *$D_3(R)$ is 1-Armendariz.*
(4) *$D_2(R)$ is Armendariz.*
(5) *$D_2(R)$ is 1-Armendariz.*

Proof. The equivalences of (1), (2), and (3) are shown in [16, Proposition 2.8]. (2)⇒(4) and (4)⇒(5) are obvious. We apply the method in the proof of [16, Proposition 2.8] to prove (5)⇒(1). Let $D_2(R)$ be 1-Armendariz, and assume on the contrary that R is not reduced. Take a nonzero $a \in R$ with $a^2 = 0$. Put

$$u = \begin{pmatrix} a & a \\ 0 & a \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in $D_2(R)$. Then $u^2 = 0 = v^2$ and $uv = vu = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) \neq 0$. This yields $(u + vx)(u - vx) = 0$ for

$$u + vx, u - vx \in D_2(R)[x].$$

So $D_2(R)$ is not 1-Armendariz, a contradiction. Therefore R is reduced. □

In connection with Proposition 1.8, one can conclude that the factor rings of Armendariz rings need not be Armendariz. Let R be a reduced ring and $n \geq 2$. Then $R[x]/R[x]x^nR[x]$ is not reduced and so $D_2(R[x]/R[x]x^nR[x])$ is not (1-)Armendariz by Proposition 1.8. Note that $D_2(R[x])$ is Armendariz by Proposition 1.8 and

$$D_2\left(\frac{R[x]}{R[x]x^nR[x]}\right) \cong \frac{D_2(R[x])}{D_2(R[x]x^nR[x])}.$$

But $R[x]/R[x]x^nR[x]$ is Armendariz by Anderson and Camillo [3, Theorem 5]. This argument also provides a conclusion that $D_2(R)$ need not be Armendariz over Armendariz rings R.

Note. Proposition 1.8 also provides the following machine as a byproduct. Any given non-reduced commutative ring A (e.g., $\mathbb{Z}_{m^k}$ with $m \geq 2$ and $k \geq 2$), one can always construct

commutative rings which are not Armendariz

by help of $D_2(A)$.

Observing Proposition 1.8, one may suspect that if $R/J(R)$ is Armendariz, then $R/J(R)$ is reduced. But the following erases the possibility.

Example 1.9. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^2, and set $R = A/I$. We identity a, b with their images in R for simplicity. Then R is Armendariz by [4, Example 4.8], and so $R[x]$ is also Armendariz by Lemma 1.1(4).

We claim $J(R[x]) = 0$. To see that, let $0 \neq f \in N(R)$. Then f is of the form bf_0b for some $f_0 \in R$ by the argument in [17, pages 4–5]. So every nilpotent element in R is of index 2 of nilpotency (i.e., $c^2 = 0$ for all $c \in N(R)$). However $fa = bf_0ba$ is not nilpotent, forcing $b \notin N^*(R)$. This implies $N^*(R) = 0$. But since $J(R[x]) = N[x]$ for some nil ideal N by Lemma 1.1(7), $J(R[x])$ must be zero.

Therefore $\frac{R[x]}{J(R[x])} = \frac{R[x]}{0} \cong R[x]$ is Armendariz, but $R[x]$ is not reduced by the existence of nonzero b with $b^2 = 0$.

2. On feckly Armendariz rings

Recently, Ungor et al. called a ring *feckly reduced* [30, Definition 2.1] if its factor ring by the Jacobson radical is reduced. They investigate the structure of such a ring and its extensions. As a generalization of a feckly reduced ring, in this section, we study the structure of rings whose factors by Jacobson radicals are Armendariz.

A ring R will be called *feckly Armendariz* if $R/J(R)$ is Armendariz. Feckly reduced rings are clearly feckly Armendariz, but not conversely by Example 1.9. Any local ring, i.e., a ring satisfying the condition that for any $r \in R$, either r or $1 - r$ is invertible, is feckly Armendariz.

Notice that the class of feckly Armendariz rings is neither closed under subrings nor homomorphic images. For example, let E be the Hamilton quaternions over real numbers and R be the ring in Example 1.2. Then R is a subring of E which is not feckly Armendariz. But E is clearly feckly Armendariz. As another example, consider $R[x]$ over the R in Example 1.2. Then $J(R[x]) = 0$ by Lemma 1.1(7), entailing that the over-ring $R[x]$ is reduced (hence feckly Armendariz). $R[x]/xR[x] \cong R$ is not feckly Armendariz by the computation in Example 1.2, in spite of $R[x]$ being feckly Armendariz.

Proposition 2.1. (1) *If a ring R is feckly Armendariz, then eRe is also a feckly Armendariz ring for any nonzero idempotent $e \in R$.*

(2) *Let $R = \oplus_{\gamma\in\Gamma} R_\gamma$ be a direct sum of rings R_γ and Γ an indexed set. Then R is a feckly Armendariz ring if and only if R_γ is a feckly Armendariz ring for each $\gamma \in \Gamma$.*

(3) *Let R be a feckly Armendariz ring. If $f_1(x) \cdots f_n(x) \in J(R)[x]$ for $f_1(x), \ldots, f_n(x) \in R[x]$, then $a_1 \cdots a_n \in J(R)[x]$ for all $a_i \in C_{f_i(x)}$.*

Proof. (1) Let R be a feckly Armendariz ring and $0 \neq e^2 = e \in R$. Then $R/J(R)$ is Armendariz and $e \notin J(R)$. Write $\bar{R} = R/J(R)$ and $\bar{r} = r + J(R)$ for $r \in R$. Since Armendariz rings are Abelian, $\bar{e}$ is a nonzero central idempotent in $\bar{R}$. Note that

$$\frac{eRe}{J(eRe)} = \frac{eRe}{eJ(R)e} \cong \bar{e}\bar{R}\bar{e} = \bar{e}\bar{R}.$$

But the class of Armendariz rings is closed under subrings. Since $\bar{R} = R/J(R)$ is Armendariz, the subring $\bar{e}\bar{R}\bar{e}$ of $\bar{R}$ is also Armendariz.

(2) Write $\bar{R} = R/J(R)$ and $\bar{R}_\gamma = R_\gamma/J(R_\gamma)$ for each $\gamma \in \Gamma$. Note that $\bar{R} \cong \oplus_{\gamma\in\Gamma}\bar{R}_\gamma$ and $J(R) = \oplus_{\gamma\in\Gamma}J(R_\gamma)$. If R_γ is feckly Armendariz for each $\gamma \in \Gamma$, then so is R since the class of Armendariz rings is closed under direct sums and $\bar{R} \cong \oplus_{\gamma\in\Gamma}\bar{R}_\gamma$.

Now, assume that R is feckly Armendariz and let for $\gamma \in \Gamma$,

$$f(x) = \sum_{i=0}^{m} \overline{a}(\gamma)_i x^i \text{ and } g(x) = \sum_{j=0}^{n} \overline{b}(\gamma)_j x^j \in \bar{R}_\gamma[x],$$

where $\overline{a}(\gamma)_i = a(\gamma)_i + J(R_\gamma)$ and $\overline{b}(\gamma)_j = b(\gamma)_j + J(R_\gamma)$ for each i, j and $f(x)g(x) = 0$. We let $F(x)$ $(G(x))$ be in $\bar{R}$ with γ-th component $f(x)$ $(g(x))$ and elsewhere 0, i.e.,

$$F(x) = \sum_{i=0}^{m}(0, 0, \ldots, \overline{a}(\gamma)_i, 0, \ldots)x^i \text{ and}$$
$$G(x) = \sum_{j=0}^{n}(0, 0, \ldots, \overline{b}(\gamma)_j, 0, \ldots)x^j \in \bar{R}[x].$$

Then we have $F(x)G(x) = 0$. Since R is feckly Armendariz,

$$(0, 0, \ldots, \overline{a}(\gamma)_i, 0, \ldots)(0, 0, \ldots, \overline{b}(\gamma)_j, 0, \ldots) \in J(R) = \oplus_{\gamma\in\Gamma}J(R_\gamma),$$

and so $a(\gamma)_i b(\gamma)_j \in J(R_\gamma)$ for all i, j. This shows that R_γ is feckly Armendariz for each $\gamma \in \Gamma$.

(3) It is obtained by the similar argument of the proof of [3, Proposition 1]. $\square$

In the proof of Proposition 2.1(1), one may conjecture that $\frac{eRe}{eJ(R)e} \cong \frac{R}{J(R)}$. However this need not hold as can be seen by $R = U_2(D)$ and $e = E_{11}$, where D is a division ring. In fact, $J(R) = \left(\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right)$, entailing $\frac{R}{J(R)} \cong D \oplus D$ and $eJ(R)e = 0$; and so $\frac{eRe}{eJ(R)e} \cong eRe \cong D$ and $\frac{R}{J(R)} \cong D \oplus D$. Note that R is feckly Armendariz but not Abelian, comparing with the fact that Armendariz rings are Abelian.

Corollary 2.2. (1) *If there exists a ring R such that* $\mathrm{Mat}_n(R)$ *is feckly Armendariz for all $n \geq 2$, then R is feckly Armendariz.*

(2) *For a central idempotent e of a ring R, eR and $(1-e)R$ are feckly Armendariz if and only if R is feckly Armendariz.*

Proof. (1) This comes from Proposition 2.1(1), since $E_{11}\mathrm{Mat}_n(R)E_{11} = RE_{11} \cong R$.

(2) It follows from Proposition 2.1(1,2), since $R \cong eR \oplus (1-e)R$. $\square$

Note that $J(R[x]) = (J(R[x]) \cap R)[x] \subseteq J(R)[x]$, i.e., $J(R[x]) \cap R \subseteq J(R)$ for a ring R.

Recall that for a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$ where $r \in R$, $m \in M$ and the usual matrix operations are used.

Proposition 2.3. (1) *Let R be a ring and $J(R) = J(R[x]) \cap R$. If R is feckly Armendariz, then so is $R[x]$.*

(2) *A ring R is feckly Armendariz if and only if the power series ring $R[[x]]$ is.*

(3) *Let $R = \left(\begin{smallmatrix} S & M \\ 0 & T \end{smallmatrix}\right)$, where S and T are rings and M is an (S,T)-bimodule. Then S and T are both feckly Armendariz if and only if so is R.*

(4) *Let R be a ring and M be an (R,R)-bimodule. Then $T(R,M)$ is feckly Armendariz if and only if so is R.*

Proof. (1) Let R be feckly Armendariz. We already have that $\frac{R[x]}{J(R[x])} = \frac{R[x]}{N[x]}$ for the nil ideal $N = J(R[x]) \cap R$ of R by Lemma 1.1(7). But $J(R) = N$ by hypothesis. Then

$$\frac{R[x]}{J(R[x])} = \frac{R[x]}{N[x]} = \frac{R[x]}{J(R)[x]} \cong \frac{R}{J(R)}[x]$$

is Armendariz by Lemma 1.1(4), noting that $R/J(R)$ is Armendariz. This implies that $R[x]$ is feckly Armendariz.

(2) It is clear that $R[[x]]/J(R[[x]]) \cong R/J(R)$. This proves the result.

(3) It is easily checked that $J(R) = \begin{pmatrix} J(S) & M \\ 0 & J(T) \end{pmatrix}$, and so $\frac{R}{J(R)} \cong \frac{S}{J(S)} \oplus \frac{T}{J(T)}$. Note that the class of Armendariz rings is closed under direct products and subrings in [16, Lemma 1.1]. Thus we obtain the result.

(4) The proof is almost similar to one of (3). □

The condition "$J(R) = J(R[x]) \cap R$" in Proposition 2.3(1) is not superfluous by the following, i.e., the feckly Armendariz property does not go up to polynomial rings.

Example 2.4. Let D be a division ring and $R = D + t\mathrm{Mat}_2(D)[[t]]$. Then $J(R) = t\mathrm{Mat}_2(D)[[t]]$ and so $R/J(R) \cong D$, showing that R is feckly reduced (hence feckly Armendariz). Write $M = \mathrm{Mat}_2(D)$.

We will show that $R[x]$ is not feckly Armendariz. Let $0 \neq a \in N(M)$ (e.g., E_{12}), and assume that $at^i \in N^*(R)$ for some i. Then

$$Mt^{i+2} = (MaM)t^{i+2} = (Mt)at^i(Mt) \subseteq N^*(M),$$

which is a contradiction since $t^{i+2} \in Mt^{i+2}$ is not nilpotent. So $at^i \notin N^*(R)$ for all $0 \neq a \in N(M)$ and $i \geq 1$. But since $J(R[x]) = N[x]$ for some nil ideal N by Lemma 1.1(7), $J(R[x])$ cannot contain $E_{12}t$ and $E_{21}t$. Set $S = \frac{R[x]}{J(R[x])} = \frac{R[x]}{N[x]} \cong \frac{R}{N}[x]$ and identify each polynomial in $R[x]$ with the images in S for simplicity.

Assume on the contrary that $\frac{R}{N}[x]$ is Armendariz. Then R/N is also Armendariz. Next consider two polynomials

$$f(y) = E_{11}t + E_{12}ty \text{ and } g(y) = E_{22}t - E_{12}ty$$

in $\frac{R}{N}[y]$, where y is an indeterminate over R/N. Then $f(y)$ and $g(y)$ are nonzero in $\frac{R}{N}[y]$. But $f(y)g(y) = 0$ and $E_{11}tE_{12}t \neq 0$, concluding that R/N is not Armendariz, which is a contradiction to $\frac{R}{N}[x]$ being Armendariz. Therefore $R[x]/J(R[x])$ is not Armendariz, i.e., $R[x]$ is not feckly Armendariz.

Example 2.4 also shows that the feckly reduced property does not go up to polynomial rings.

Observe that $U_2(A)$ is feckly Armendariz by Proposition 2.3(3) if A is any semiprimitive Armendariz ring, but $U_2(A)$ is not Armendariz. Moreover, we have the following.

Recall that for a ring R and $n \geq 2$, let $V_n(R)$ be the ring of all matrices (a_{ij}) in $D_n(R)$ such that $a_{st} = a_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1$. Note that $V_n(R) \cong \frac{R[x]}{x^n R[x]}$.

Proposition 2.5. (1) *Let I be an ideal of a ring R such that $I \subseteq J(R)$. Then R is feckly Armendariz if and only if R/I is feckly Armendariz.*

(2) *For a ring R, the following are equivalent*:

(i) *R is feckly Armendariz.*

(ii) *$U_n(R)$ is feckly Armendariz for all $n \geq 2$.*

(iii) *$D_n(R)$ is feckly Armendariz for all $n \geq 2$.*

(iv) *$V_n(R)$ is feckly Armendariz for all $n \geq 2$.*

Proof. (1) Since $I \subseteq J(R)$, we have $J(\frac{R}{I}) = \frac{J(R)}{I}$. So, from $\frac{\frac{R}{I}}{J(\frac{R}{I})} \cong \frac{R}{J(R)}$, we can obtain the result.

(2) (i)$\Leftrightarrow$(ii): Let

$$I = \{A \in U_n(R) \mid \text{ each diagonal entry of } A \text{ is zero}\}$$

of $U_n(R)$. Then $I \subseteq J(U_n(R))$ and $\frac{U_n(R)}{I} \cong \oplus_{i=1}^n R_i$, where $R_i = R$. The proof is completed by the condition (1) and Proposition 2.1(2).

(i)$\Leftrightarrow$(iii): Let

$$I' = \{B \in D_n(R) \mid \text{ each diagonal entry of } B \text{ is zero}\}$$

of $D_n(R)$. Then $I' \subseteq J(D_n(R))$ and $\frac{D_n(R)}{I'} \cong R$. By the condition (1), we have (i)$\Leftrightarrow$(iii).

(i)$\Leftrightarrow$(iv) is similar to the proof of (i)$\Leftrightarrow$(iii), noting that $I'' \subseteq J(V_n(R))$ and $\frac{V_n(R)}{I''} \cong R$ where

$$I'' = \{C \in V_n(R) \mid \text{ each diagonal entry of } C \text{ is zero}\}$$

of $V_n(R)$. □

Following the literature, a ring is called *directly finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. Abelian rings are easily shown to be directly finite. Recall that Armendariz rings are Abelian (hence directly finite). The concepts of Armendariz and feckly Armendariz are independent of each other as can be seen by the ring in Example 1.2 and by help of Proposition 2.3(3). But these concepts meet together in the class of directly finite rings as can be seen in the following.

Proposition 2.6. (1) *Let R be a ring and I be an ideal of R with $I \subseteq J(R)$. If R/I is directly finite, then so is R.*

(2) *Every feckly Armendariz ring is directly finite.*

Proof. (1) Let R/I be directly finite and $ab = 1$ for some $a, b \in R$. Assume on the contrary that $ba \neq 1$. Since R/I is directly finite, we have $\bar{b}\bar{a} = \bar{1}$, i.e., $1 - ba \in I \subseteq J(R)$. It then follows that $ba = 1 - (1 - ba)$ is invertible, a contradiction to ba being a nonzero idempotent. Thus $ba = 1$, showing that R is directly finite.

(2) It is proved by (1) since $R/J(R)$ is Armendariz (hence directly finite). □

We use $\oplus$ to denote the direct sum. Let R_0 be an algebra with identity over a commutative ring S. Due to Dorroh [8], the *Dorroh extension* of R by S is the Abelian group $S \oplus R_0$ with multiplication given by $(a_1, r_1)(a_2, r_2) = (a_1a_2, a_1r_2 + a_2r_1 + r_1r_2)$ for $a_i \in S$ and $r_i \in R_0$.

Proposition 2.7. *Let R_0 be an algebra over a field F. Then R_0 is feckly Armendariz if and only if the Dorroh extension R of R_0 by F is feckly Armendariz.*

Proof. Since R_0 has the identity, $a \in F$ is identified with $a1 \in R_0$ and so $R_0 = \{a + r \mid (a, r) \in R\}$.

We first claim that $J(R) = 0 \oplus J(R_0)$. Assume that $(a, r) \in J(R)$ with $a \neq 0$. Then $(1, a^{-1}r) = (a^{-1}, 0)(a, r) \in J(R)$, and so $(0, -a^{-1}r) = (1, 0) - (1, a^{-1}r)$ is invertible, which is a contradiction. This implies $J(R) \subseteq 0 \oplus J(R_0)$. Let $(0, r') \in J(R)$. Then $(0, r'(a + r)) = (0, r')(a, r) \in J(R)$ for all $(a, r) \in R$. Recall that $a + r$ runs over all elements in R_0. So we obtain that $(0, r') \in J(R)$ if and only if $r' \in J(R_0)$, from the fact that $(0, r'(a + r))$ is right quasi-regular in R if and only if $r'(a+r)$ is right quasi-regular in R_0. Therefore we now have

$$J(R) = 0 \oplus J(R_0) = \{(0, r) \in R \mid r \in J(R_0)\}.$$

Write $\bar{R}_0 = R_0/J(R_0)$ and $\bar{R} = R/J(R)$. Note that $\bar{R} = \frac{R}{J(R)} \cong F \oplus \frac{R_0}{J(R_0)} = F \oplus \bar{R}_0$, the Dorroh extension $\bar{R}_0$ by F.

Suppose that R_0 is feckly Armendariz. Let

$$f(x) = \sum_{i=0}^{m} (a(0)_i, a(1)_i)x^i, g(x) = \sum_{j=0}^{n} (b(0)_j, b(1)_j)x^j \in \bar{R}[x]$$

such that $f(x)g(x) = 0$. Here letting

$$f_0(x) = \sum_{i=0}^{m} a(0)_i x^i, g_0(x) = \sum_{j=0}^{n} b(0)_j x^j \in F[x] \text{ and}$$
$$f_1(x) = \sum_{i=0}^{m} a(1)_i x^i, g_1(x) = \sum_{j=0}^{n} b(1)_j x^j \in \bar{R}_0[x],$$

we obtain $f_0(x)g_0(x) = 0$ and $f_0(x)g_1(x) + f_1(x)g_0(x) + f_1(x)g_1(x) = 0$ from $f(x)g(x) = 0$. Since F is a field, $f_0(x) = 0$ or $g_0(x) = 0$. Assume $f_0(x) = 0$. Then $f_1(x)(g_0(x)+g_1(x)) = f_1(x)g_0(x)+f_1(x)g_1(x) = 0$. Here we can consider the polynomials $f_1(x), g_0(x), g_1(x)$ in $\bar{R}_0[x]$, and this yields

$$(\sum_{i=0}^{m} a(1)_i x^i)(\sum_{j=0}^{n}(b(0)_j + b(1)_j)x^j) = 0.$$

Since $\bar{R}_0$ is Armendariz, we get $a(1)_i(b(0)_j+b(1)_j) = 0$ for all i and j, entailing

$$(0, a(1)_i)(b(0)_j, b(1)_j) = 0 \text{ for all } i \text{ and } j.$$

This implies that $\alpha\beta = 0$ for all $\alpha \in C_{f(x)}$ and $\beta \in C_{g(x)}$. We can get the same result for the case of $g_0(x) = 0$ via a symmetric method. Therefore R is feckly Armendariz.

Conversely, assume that R is feckly Armendariz and let $f(x)g(x) = 0$ with

$$f(x) = \sum_{i=0}^{m} \bar{a}_i x^i \text{ and } g(x) = \sum_{j=0}^{n} \bar{b}_j x^j \in \bar{R}_0[x],$$

where $\bar{a}_i = a_i + J(R_0)$ and $\bar{b}_j = b_j + J(R_0)$ for each i, j. Let

$$F(x) = (0, f(x)) = \sum_{i=0}^{m}(0, \bar{a}_i)x^i \text{ and } G(x) = (0, g(x)) = \sum_{j=0}^{n}(0, \bar{b}_j)x^j \in \bar{R}[x].$$

Then we have $F(x)G(x) = 0$. Since $\bar{R}$ is Armendariz,

$$(0, a_i)(0, b_j) \in J(R) = 0 \oplus J(R_0),$$

and so $a_i b_j \in J(R_0)$ for all i, j. This implies that R_0 is feckly Armendariz. □

Clearly, R is feckly Armendariz if and only if whenever $f(x)g(x) \in J(R)[x]$ for any $f(x), g(x) \in R[x]$, then $ab \in J(R)$ for all $a \in C_{f(x)}$, $b \in C_{g(x)}$. We also consider the following conditions.

Let $U(R)$ denote the set of all units (i.e., invertible elements) in a ring R.

Theorem 2.8. *Let R be a ring. Consider the following conditions.*

(1) *R is an Armendariz ring in which 1 is the only unit* (*whence the characteristic of R is two*).

(2) *For any $f(x), g(x) \in R[x]$, if each coefficient of $f(x)g(x)$ is of the form $1 + u$ where $u \in U(R)$, then $ab = 0$ for all $a \in C_{f(x)}, b \in C_{g(x)}$.*

(3) *For any* $f(x), g(x) \in R[x]$, *if every coefficient of* $f(x)g(x)$ *is of the form* $1+u$ *where* $u \in U(R)$, *then* ab *is of the same form for each* $a \in C_{f(x)}, b \in C_{g(x)}$.

(4) R *is feckly Armendariz.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). *If* $J(R) = \{r \in R \mid 1-r \in U(R)\}$, *then* (4) $\Rightarrow$ (3).

Proof. (1)$\Rightarrow$(2). Assume the condition (1). Let $f(x), g(x) \in R[x]$ such that each coefficient of $f(x)g(x)$ is of the form $1+u$ with $u \in U(R)$. Then $f(x)g(x) = 0$ by the condition (1). Since R is Armendariz, $ab = 0$ for all $a \in C_{f(x)}, b \in C_{g(x)}$.

(2)$\Rightarrow$(1). If the condition (2) holds, then R is Armendariz obviously. On the other hand, if r is an invertible element in R, then $1-r = r(r^{-1}-1) = f(x)g(x)$ ($f(x) = r$ and $g(x) = r^{-1} - 1$) implies that $r = 1$.

(2) $\Rightarrow$(3). It is obvious since $0 = 1 + (-1)$.

(3)$\Rightarrow$(4). First note that $\overline{a}$ will mean $a + J(R)$ in the ring $R/J(R)$. Let $\overline{f}(x) = \overline{a}_0 + \overline{a}_1 x + \cdots + \overline{a}_n x^n$ and $\overline{g}(x) = \overline{b}_0 + \overline{b}_1 x + \cdots + \overline{b}_n x^n \in (R/J(R))[x]$ be such that $\overline{f}(x)\overline{g}(x) = \overline{0}$. We claim that $a_i b_j \in J(R)$ for each i, j. For that, it is enough to show that $1 - a_i b_j r$ is a unit in R for all $r \in R$. Now let $r \in R$. Then we have

$$\begin{aligned} \overline{a}_0\overline{b}_0\overline{r} &= \overline{0}, \\ \overline{a}_0\overline{b}_1\overline{r} + \overline{a}_1\overline{b}_0\overline{r} &= \overline{0}, \\ &\vdots \\ \overline{a}_n\overline{b}_n\overline{r} &= \overline{0}. \end{aligned}$$

since $\overline{f}(x)\overline{g}(x)\overline{r} = \overline{0}$. Note the fact that if $a \in J(R)$, then $1 - a$ is a unit. If we take $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $g'(x) = b_0 r + b_1 r x + \cdots + b_n r x^n$, then each coefficient of $f(x)g'(x)$ is of the form $1 + u$ where u is a unit in R. By hypothesis, for each i and j, $a_i b_j r$ is of the form $1 + u$ where $u \in U(R)$. Then for each i, j, $1 - a_i b_j r$ is a unit in R. Since r is arbitrary, we get the claim.

(4)$\Rightarrow$(3). Assume that $J(R) = \{r \in R \mid 1 - r \in U(R)\}$ and R is feckly Armendariz. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $g(x) = b_0 + b_1 x + \cdots + b_n x^n$ in $R[x]$ be such that each coefficient of $f(x)g(x)$ is of the form $1 + u$ where $u \in U(R)$. Then by hypothesis, each coefficient of $f(x)g(x)$ is in $J(R)$. This gives that $\overline{f}(x)\overline{g}(x) = \overline{0}$ where $\overline{f}(x) = \overline{a}_0 + \overline{a}_1 x + \cdots + \overline{a}_n x^n$ and $\overline{g}(x) = \overline{b}_0 + \overline{b}_1 x + \cdots + \overline{b}_n x^n$. Since R is feckly Armendariz, each $a_i b_j$ is of the form $1 + u$ where $u \in U(R)$. $\square$

Considering Theorem 2.8, there exist many feckly Armendariz rings that do not satisfy the condition (2). Let $R = \mathbb{Z}[[x]]$. Then $R/J(R) = \mathbb{Z}[[x]]/x\mathbb{Z}[[x]] \cong \mathbb{Z}$, so R is feckly Armendariz. But $0 \neq (1+x)(x^2) = x^2 + x^3 = 1 + (-1 + x^2 + x^3)$ with $-1 + x^2 + x^3 \in U(R)$.

There exist many (Armendariz) rings which satisfies the condition (2) in Theorem 2.8. For example, consider $\mathbb{Z}_2$ and $\mathbb{Z}_2[X]$ (X is any set of commuting indeterminates, possibly infinite), etc.

Remark. (1) When a given ring R is feckly Armendariz, $J(R)$ need not equal to $\{r \in R \mid 1-r \in U(R)\}$. For example, let $R = \mathbb{Z}$. Then R is (feckly) Armendariz with $J(R) = 0$. But $-1 = 1 - 2 \in U(R)$ and $2 \notin J(R)$.

As another example, let $R = D[[x]]$ over a division ring D of order ≥ 3. Take $0 \neq d \in D$ such that $d \neq 1$. Then $J(R) = xD[[x]]$, but $J(R) \subsetneq \{r \in R \mid 1 - r \in U(R)\}$ since $1 - d \in U(R)$. Note that R is (feckly) Armendariz.

(2) It is well-known that $J(R) \subseteq \{r \in R \mid 1 - r \in U(R)\}$ for any ring R. Equality holds in the following cases:

(i) If R is a local ring (hence R is feckly Armendariz). For, in this case $J(R) = \{r \in R \mid r \notin U(R)\}$. For example, consider $R = \mathbb{Z}_2[[x]]$. Note that $J(R) = x\mathbb{Z}_2[[x]]$.

(ii) If $R/J(R)$ is a Boolean ring (hence R is (feckly) Armendariz). For, let $r \in R$ be such that $1 - r \in U(R)$. Assume that $r \notin J(R)$. Since $\overline{r}(\overline{r} - \overline{1}) = \overline{0}$ in $R/J(R)$ and $1 - r$ is invertible, we have that $\overline{r} = \overline{0}$, which is a contradiction.

Finally, we will consider some special feckly Armendariz rings.

A ring R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo the Jacobson radical $J(R)$ of R. Local rings are Abelian and semilocal. However, the classes of Abelian rings and feckly Armendariz rings do not imply each other by Example 1.2 and by help of Proposition 2.5(2), considering $R = U_2(D)$ over a division ring D.

Proposition 2.9. (1) *Let R be an Abelian ring. Then R is a semiperfect feckly Armendariz ring if and only if R is a finite direct sum of local feckly Armendariz rings.*

(2) *If R is a minimal noncommutative feckly Armendariz ring, then R is of order* 8 *and is isomorphic to $U_2(\mathbb{Z}_2)$. Here by minimal we mean having smallest cardinality.*

Proof. (1) Assume that R is a semiperfect feckly Armendariz ring. Since R is semiperfect, R contains a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by [21, Corollary 3.7.2], say $R = \sum_{i=1}^{n} e_i R$ such that each $e_i R e_i$ is a local ring. Since R is Abelian and feckly Armendariz, $e_i R = e_i R e_i$ is a feckly Armendariz ring by Proposition 2.1(1).

Conversely, let R is a finite direct sum of local feckly Armendariz rings. Then R is semiperfect since local rings are semiperfect by [21, Corollary 3.7.1]. Note that R is also feckly Armendariz by Proposition 2.1(2).

(2) Suppose that R is a minimal noncommutative feckly Armendariz ring. Then $|R| \geq 2^3$ by [9, Theorem]. If $|R| = 2^3$, then R is isomorphic to $U_2(\mathbb{Z}_2)$ by [9, Proposition]. But $U_2(\mathbb{Z}_2)$ is a feckly Armendariz ring by Proposition 2.5(2). This implies that R is of order 8 and is isomorphic to $U_2(\mathbb{Z}_2)$. $\square$

Nicholson [26] called a ring R *clean* if every element of R is a sum of an idempotent and a unit in R, also gave a useful characterization of a exchange ring which is introduced by Warfield [31] as a generalization of the class of clean rings. It is proved that a ring R is *exchange* if and only if for any $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$ and that clean rings are exchange rings and that the two concepts are equivalent for Abelian rings by [26, Proposition 1.1 and Proposition 1.8].

Proposition 2.10. *Let R be a feckly Armendariz ring. If R is an exchange ring, then R is clean.*

Proof. Suppose that a ring R is both feckly Armendariz and exchange. Then $R/J(R)$ is exchange and idempotents can be lifted modulo $J(R)$ by [26, Proposition 1.4] and Abelian, and so $R/J(R)$ is clean and idempotents lift modulo $J(R)$. Thus R is clean by [13, Proposition 6]. □

A ring R is called (*von Neumann*) *regular* if for every $x \in R$ there exists $y \in R$ such that $xyx = x$ in [10]. It is proved that R is regular if and only if every principal right (left) ideal of R is generated by an idempotent in [10, Theorem 1.1]. Regular rings are clearly semiprimitive. For a semiprimitive ring R, it is obvious that R is Armendariz if and only if R is feckly Armendariz. Note that the reduced, Armendariz and Abelian ring properties are coincided for a regular ring in [3, Theorem 6]. Hence, regular feckly Armendariz rings are feckly reduced.

Following the literature, a ring R is called *semipotent* if every right (equivalently left) ideal not contained in $J(R)$ contains a nonzero idempotent. Clearly, if R is semipotent, then so is $R/J(R)$. Any regular ring or exchange ring is known to be semipotent.

Proposition 2.11. *Let R be a semipotent ring. Then the following are equivalent*:

(1) *R is feckly Armendariz.*
(2) *$R/J(R)$ is Abelian.*
(3) *R is feckly reduced.*

Proof. It is enough to show that (2)⇒(3). Let R be a semipotent ring. Assume that $R/J(R)$ is Abelian. Then $R/J(R)$ is semipotent with $J(R/J(R)) = 0$, and so $R/J(R)$ is reduced by the proof of [29, Lemma 4.10], i.e., R is feckly reduced. □

Recall the ring $S = R[x]$ with $J(S) = 0$ in Example 1.9. Let I be the ideal of S generated by x, i.e., $I = xR[x]$. Then I cannot contain nonzero idempotent by help of [20, Lemma 8]. Thus S cannot be semipotent, noting that S is not feckly reduced.

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