ON μ–ESSENTIAL AND μ–M–SINGULAR MODULES

A. Çiğdem ÖZCAN

Hacettepe University
Department of Mathematics
06800 Beytepe Ankara, Turkey
E-mail: ozcan@hacettepe.edu.tr

As a generalization of essential submodules Zhou defines a μ-essential submodule provided it has a non-zero intersection with any non-zero submodule in μ for any class μ. Let M be a module. In this article we study δ–essential submodules as a dual of δ-small submodules of Zhou where $\delta = \{N \in \sigma[M] : \mathrm{Rej}(N, \mathcal{M}) = 0\}$ and $\mathcal{M} = \{N \in \sigma[M] : N \ll \widehat{N}\}$, and also define μ-M–singular modules as modules $N \in \sigma[M]$ such that $N \cong K/L$ for some $K \in \sigma[M]$ and L is μ–essential in K. By $\mathcal{M}$–M–singular modules and δ–M–singular modules a characterization of GCO–modules, and by $\mathcal{FC}$–M–singular modules where $\mathcal{FC}$ is the class of finitely cogenerated modules, a characterization of semisimple Artinian rings are given.

Keywords: essential submodule, singular module

1. Preliminaries

Let M be a module, $N \in \sigma[M]$ and μ a class of modules in $\sigma[M]$ which is closed under isomorphisms and submodules. Following Zhou[16] we call a submodule N a μ–*essential submodule* of $K \in \sigma[M]$ if for any nonzero μ-submodule X in K, $N \cap X \neq 0$, denoted by $N \leq_{\mu e} K$. In this article after studying some properties of μ–essential submodules we consider δ–essential submodules as a dual of δ-small submodules of Zhou by denoting the class $\delta = \{N \in \sigma[M] : \mathrm{Rej}(N, \mathcal{M}) := \overline{Z}_M(N) = 0\}$ where $\mathcal{M} = \{N \in \sigma[M] : N \ll \widehat{N}\}$ and $\overline{Z}_M(.)$ is defined by Talebi and Vanaja as a dual of the singular submodule $Z_M(.)$. If $\mathcal{F} = \{F \in \sigma[M] : \forall\, 0 \neq K \subseteq F,\ \overline{Z}_M(K) \neq K\}$, then it is known that $\mathcal{M} \subseteq \delta \subseteq \mathcal{F}$. We prove a result on when an $\mathcal{F}$–essential submodule is δ–essential and a δ–essential submodule is $\mathcal{M}$–essential. Also we prove that $Tr(\mathcal{S} \cap \mathcal{M}, N) = Tr(\mathcal{S} \cap \delta, N) = Tr(\mathcal{S} \cap \mathcal{F}, N)$ where $\mathcal{S}$ is the class of simple modules in $\sigma[M]$ and Tr is used for the trace.

In the last section we define μ-M–singular modules $N \in \sigma[M]$ for a module M. N is called μ-M*–singular module* if $N \cong K/L$ for some $K \in \sigma[M]$ and $L \leq_{\mu e} K$. It is proved that M is a GCO–module (i.e. every simple M–singular module is injective in $\sigma[M]$) if and only if for every $\mathcal{M}$–M–singular module N in $\sigma[M]$, $\overline{Z}_M(N) = N$ if and only if for every δ–M–singular module N in $\sigma[M]$, $\overline{Z}_M(N) = N$. When we consider the class of all finitely cogenerated modules $\mathcal{FC}$ we prove that every finitely cogenerated R–module is projective if and only if for every $\mathcal{FC}$–R–singular

R–module N, $\mathrm{Rej}(N, \mathcal{F}C) = N$ if and only if R is semisimple Artinian.

Let R be a ring with identity. All modules we consider are unitary right R-modules and we denote the category of all such modules by Mod–R. Let M be an R-module. The R–injective hull of M is denoted by $\mathrm{E}(M)$, and the *M–injective hull* of N in the category of $\sigma[M]$ is denoted by $\widehat{N}$. For the definition of $\sigma[M]$ and $\widehat{N}$ see.[15]

Let μ be a class of modules. For any module N, the *trace* of μ in N is denoted by $\mathrm{Tr}(\mu, N) = \Sigma\{Imf : f \in Hom(C, N), C \in \mu\}$. Dually the *reject* of μ in N is denoted by $\mathrm{Rej}(N, \mu) = \bigcap\{kerg : g \in Hom(N, C), C \in \mu\}$.

Let N be a submodule of M ($N \leq M$). The notations $N \ll M$, $N \leq_e M$ and $N \leq_d M$ is used for a small submodule, an essential submodule and a direct summand of M, respectively. $Soc(M)$ will denote the socle of M. A module $N \in \sigma[M]$ is said to be *M–small* (or small in $\sigma[M]$) if $N \cong K \ll L$ for $K, L \in \sigma[M]$. Then an R–module $N \in \sigma[M]$ is M–small if and only if $N \ll \widehat{N}$.

Dually, a module $N \in \sigma[M]$ is called *M–singular* (or singular in $\sigma[M]$) if $N \cong L/K$ for an $L \in \sigma[M]$ and $K \leq_e L$. Every module $N \in \sigma[M]$ contains a largest M–singular submodule which is denoted by $\mathrm{Z}_M(N)$. Then $\mathrm{Z}_M(N) = \mathrm{Tr}(\mathcal{U}, N)$ where $\mathcal{U}$ denotes the class of all M–singular modules (see[15]).

Simple modules in $\sigma[M]$ split into four disjoint classes by combining the exclusive choices [M–injective or M–small] and [M–projective or M–singular]. Also note that if a module N in $\sigma[M]$ is M–singular and projective in $\sigma[M]$, then it is zero.

Let $N \subseteq K \in \sigma[M]$. N is called *δ–M–small* in K if, whenever $N + X = K$ with K/X is M–singular, we have $X = K$ (see[7]). Zhou[17] studies δ–R–small submodules in Mod–R. By [17, Lemma 1.2], in the definition of δ–R–small submodule, K/X can be taken Goldie torsion, i.e. K/X can be a member of the torsion class of the Goldie torsion theory in Mod–R.

In this paper μ **will be a class in** $\sigma[M]$ **which is closed under isomorphisms and submodules**, unless otherwise stated. Any member of μ we shall call a *μ–module*. In this article we denote the following classes:

$\mathcal{S} = \{N \in \sigma[M] : N \text{ is simple}\}$,
$\mathcal{M} = \{N \in \sigma[M] : N \text{ is } M\text{-small}\}$
$\delta = \{N \in \sigma[M] : \overline{Z}_M(N) = 0\}$,
$\mathcal{F} = \{F \in \sigma[M] : \forall\, 0 \neq K \subseteq F,\ \overline{Z}_M(K) \neq K\}$
$\mu M - Sing = \{N \in \sigma[M] : N \text{ is } \mu\text{-}M\text{-singular}\}$
$\mathcal{FC} = \{N \in \sigma[M] : N \text{ is finitely cogenerated}\}$

Definition 1.1. Let $N \in \sigma[M]$. Following Zhou[16] N is called a *μ–essential submodule* of $K \in \sigma[M]$ if for any nonzero μ–module X in K, $N \cap X \neq 0$. It is denoted by $N \leq_{\mu e} K$.

Clearly every essential submodule is μ–essential. But the converse is not true in general.

Example 1.1. Let μ be the class of simple modules and zero modules in Mod–R. Then a submodule N of a module M is μ–essential if and only if N contained the socle of M but this is not enough to make N essential. For example in the $\mathbb{Z}$–module $\mathbb{Z} \oplus \mathbb{Z}_p$, where p is a prime, $0 \oplus \mathbb{Z}_p$ is μ–essential but not essential.

Example 1.2. Consider the class of M–small modules $\mathcal{M}$ in $\sigma[M]$. Let N be an injective module in $\sigma[M]$ with $0 \neq Rad(N) \not\leq_e N$ (for example let $N = U \oplus V$ where U is injective with essential radical and V is injective simple module.) Let X be a non-zero M–small submodule of N. Then $X \ll N$ so $X = X \cap Rad(N) \neq 0$. Thus $Rad(N)$ is $\mathcal{M}$–essential but not essential in N.

The following lemma is clear from definitions.

Lemma 1.1. *Let $K \in \sigma[M]$. If every nonzero submodule of K contains a nonzero μ–module, then for any submodule N of K, $N \leq_e K$ if and only if $N \leq_{\mu e} K$.*

Corollary 1.1. *Let $N \leq K \in \sigma[M]$. If $N \leq_{\mu e} K$ and K is a μ–module, then $N \leq_e K$.*

Now we list the properties of μ–essential submodules. We omit the proofs because they are similar to those for essential submodules (see, for example[2]).

Lemma 1.2. *Let M be a module.*

a) Let $N \leq L \leq K \in \sigma[M]$. Then $N \leq_{\mu e} K$ if and only if $N \leq_{\mu e} L \leq_{\mu e} K$.

b) If $K_1 \leq_{\mu e} L_1$, $K_2 \leq_{\mu e} L_2$, then $K_1 \cap K_2 \leq_{\mu e} L_1 \cap L_2$ for $L_1, L_2 \in \sigma[M]$.

c) Let $N, L \in \sigma[M]$. If $f : N \to L$ is a homomorphism and $K \leq_{\mu e} L$, then $f^{-1}(K) \leq_{\mu e} N$.

d) If $N/L \leq_{\mu e} K/L$, then $N \leq_{\mu e} K$.

e) Let $N \in \sigma[M]$, $\{K_i\}$ an independent family of submodules of N and if $K_i \leq_{\mu e} L_i \leq N$ for all $i \in I$, then $\oplus_{i\in I} K_i \leq_{\mu e} \oplus_{i\in I} L_i$.

Example 1.3. In Lemma 1.2(e), $\{L_i\}$ need not be an independent family. For example, let μ be the class of simple modules and zero modules and put $K_1 = 0 \oplus \mathbb{Z}_p \leq \mathbb{Z} \oplus \mathbb{Z}_p = L_1$ and $K_2 = L_2 = \mathbb{Z} \oplus \overline{0} \leq L_1$. Then $K_1 \leq_{\mu e} L_1$, $K_2 \leq_{\mu e} L_2$ and $K_1 \cap K_2 = 0$ but $L_1 \cap L_2 \neq 0$.

2. δ–essential Submodules Where $\delta = \{N \in \sigma[M] \mid \overline{Z}_M(N) = 0\}$

Talebi and Vanaja[14] define $\overline{Z}_M(N)$ as a dual of $Z_M(N)$ as follows:

$$\overline{Z}_M(N) = \mathrm{Rej}(N, \mathcal{M}) = \cap\{\ker g \mid g \in \mathrm{Hom}(N, L), L \in \mathcal{M}\}$$

where $N \in \sigma[M]$. They call N an *M–cosingular (non–M–cosingular) module* if $\overline{Z}_M(N) = 0$ ($\overline{Z}_M(N) = N$). If N is M–small, then N is M–cosingular. The class of all M–cosingular modules is closed under submodules, direct sums and direct products [14, Corollary 2.2]. Note that $\overline{Z}^2_M(N) = \overline{Z}_M(\overline{Z}_M(N))$. Talebi and Vanaja study the torsion theory cogenerated by M–small modules, $\tau = (\mathcal{T}, \mathcal{F})$ where

$$\mathcal{T} = \{T \in \sigma[M] \mid \overline{Z}_M(T) = T\},$$
$$\mathcal{F} = \{F \in \sigma[M] \mid \forall 0 \neq K \leq F, \overline{Z}_M(K) \neq K\}.$$

This torsion theory is also studied by Özcan and Harmancı.[9] This is a dual of the Goldie torsion theory and not necessarily hereditary. Also $\mathcal{M} \subseteq \delta \subseteq \mathcal{F}$. Now we investigate the relationship between $\mathcal{M}$–essential, δ–essential and $\mathcal{F}$–essential submodules by inspired [17, Lemma 1.2]. First we note that the following two theorems which are characterize the torsion free class $\mathcal{F}$.

Theorem 2.1. *[9, Theorem 15] Let M be a module and assume that M has a projective cover in $\sigma[M]$. If $\overline{Z}_M(M) = M$, then $\mathcal{M} = \delta = \mathcal{F}$.*

Let N and L be submodules of a module M. N is called a *supplement* of L (in M) if $N + L = M$ and $N \cap L \ll N$. M is called *amply supplemented* if, for all submodules N and L of M with $N + L = M$, N contains a supplement of L in M.

Theorem 2.2. *[14, Theorem 3.6] Let M be a module such that every injective module in $\sigma[M]$ is amply supplemented. Then $\mathcal{F}$ is closed under factor modules and $\mathcal{F} = \{N \in \sigma[M] \mid \overline{Z}^2_M(N) = 0\}$.*

For shortness we denote

(A) M has a projective cover and $\overline{Z}_M(M) = M$.

(B) Every injective module in $\sigma[M]$ is amply supplemented.

Proposition 2.1. *Consider the following conditions for $K \leq N \in \sigma[M]$.*

a) $K \leq_{\mathcal{F}e} N$.

b) $K \leq_{\delta e} N$.

c) $K \leq_{\mathcal{M}e} N$.

Then $(a) \Rightarrow (b) \Rightarrow (c)$. If M has (B), then $(b) \Rightarrow (a)$. If M has (A), then $(c) \Rightarrow (a)$.

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ They are clear.
$(b) \Rightarrow (a)$ Let $X \leq N$ with $X \cap K = 0$ and $X \in \mathcal{F}$. Then $\overline{Z}^2_M(X) = \overline{Z}_M(\overline{Z}_M(X)) = 0$ by Theorem 2.2. Since $\overline{Z}_M(X) \cap K = 0$, $\overline{Z}_M(X) = 0$ by (b). Again by (b), $X = 0$.
$(c) \Rightarrow (a)$ It is clear by Theorem 2.1. $\square$

Let M be a module. Define

$$Soc_M(N) = Tr(\mathcal{S} \cap \mathcal{M}, N)$$

for any module $N \in \sigma[M]$. Then $Soc_M(N) \leq Soc(N)$. Clearly if $Soc_M(N) \leq_e N$, then $Soc_M(N) = Soc(N)$. The following lemma shows that in the definition of $Soc_M(N)$ we can take $\mathcal{F}$–modules or δ–modules instead of $\mathcal{M}$–modules. That is

$$Soc_M(N) = Tr(\mathcal{S} \cap \delta, N) = Tr(\mathcal{S} \cap \mathcal{F}, N).$$

Lemma 2.1. *Let M be a module. Any simple $\mathcal{F}$–module in $\sigma[M]$ is M–small.*

Proof. Let X be a simple $\mathcal{F}$–module in $\sigma[M]$. If $\overline{Z}_M(X) = X$, then $X \in \mathcal{T} \cap \mathcal{F} = 0$, a contradiction.Then $\overline{Z}_M(X) = 0$. If X is M–injective, then $\overline{Z}_M(X) = X$. For, let L be a submodule of X such that X/L is M–small. If $L = 0$, then X is M–small, a contradiction. Hence $L = X$, that is $\overline{Z}_M(X) = X$. This contradiction implies that X is M–small. □

The following proposition can be seen by [16, Proposition 3], but we give the proof for completeness.

Proposition 2.2. *Let $N \in \sigma[M]$. $Soc_M(N)$ is the intersection of all its $\mathcal{F}$–essential submodules of N.*

Proof. Let S be a simple M–small submodule of N and K be an $\mathcal{F}$–essential submodule of N, then $S \cap K \neq 0$. Therefore $S \leq K$. It follows that the intersection of all $\mathcal{F}$–essential submodules contains all simple M–small submodules and hence it contains their sum. Thus $Soc_M(N)$ is contained in the intersection of all $\mathcal{F}$–essential submodules of N.

If $N = Soc_M(N)$, then the proof is completed. Suppose that $N \neq Soc_M(N)$. Let $n \in N - Soc_M(N)$. Then there exists a submodule K maximal with respect to $K \supseteq Soc_M(N)$ and $n \notin K$. If we can show that $K \leq_{\mathcal{F}e} N$, then n lies outside an $\mathcal{F}$–essential submodule, and so $Soc_M(N)$ is the intersection of all $\mathcal{F}$–essential submodules of N.

Suppose that $L \cap K = 0$ for some nonzero submodule L of N with $L \in \mathcal{F}$. Consider the natural epimorphism $\pi : N \to N/K$. Then $L \cong \pi(L) \leq N/K$. Since K is maximal with respect to $K \supseteq Soc_M(N)$ and $n \notin K$, N/K has a minimal submodule contained in every nonzero submodule. Also since $L \in \mathcal{F}$, then $L \cap Soc_M(N) \neq 0$ by Lemma 2.1. But $L \cap Soc_M(N) \leq L \cap K = 0$, a contradiction. □

Hence intersections of $\mathcal{M}$–essential, δ–essential and $\mathcal{F}$–essential submodules are equal.

$\mathrm{Tr}(\mathcal{M}, N)$ is investigated in[7] and denoted by $Z^*_M(N)$. Then it can be seen that

$$Soc(Z^*_M(N)) = Soc_M(N).$$

There are some examples of modules M such that $Soc_M(N) \neq 0$, $Soc_M(N) \neq Soc(N)$ and $Soc_M(N) = Soc(N)$.

Example 2.1. 1) If M is a cosemisimple module (i.e. every simple module is M–injective) and $N \in \sigma[M]$, then $Soc_M(N) = 0$, because $Soc(N) = Soc_M(N) \oplus T$ where T is a direct sum of simple M–injective submodules of N.

2) If R is a small ring (for example a commutative integral domain) then every finitely generated R–module is small.[12] This implies that $Soc_R(N) = Soc(N)$ for

every R–module N.

3) Let R be the ring $\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where F is a field. Then $Z^*(R_R) = Soc(R_R)$ by [6, Example 11]. This imples that $Soc(Z^*(R_R)) = Soc_R(R_R) = Soc(R_R)$. Since $Soc(R_R) \leq_e R_R$, every δ–essential right ideal is essential.

4)[4] Let Q be a local quasi-Frobenius ring and $J = J(Q)$ (the Jacobson radical of Q), $S = Soc(Q_Q) = Soc({}_QQ)$. Then $W = \begin{bmatrix} Q & Q/S \\ J & Q/S \end{bmatrix}$ is a well–defined ring by the usual matrix addition, equality and the following multiplication

$$\begin{bmatrix} u & v+S \\ j & w+S \end{bmatrix} \begin{bmatrix} x & y+S \\ k & z+S \end{bmatrix} = \begin{bmatrix} ux+vk & uy+vz+S \\ jx+wk & jy+wz+S \end{bmatrix}$$

where $u, v, w, x, y, z \in Q$ and $j, k \in J$. W is a right and left Artinian ring. $J(W) = \begin{bmatrix} J & Q/S \\ J & J/S \end{bmatrix}$ and $Soc({}_WW) = \begin{bmatrix} S & Soc(Q/S) \\ 0 & 0 \end{bmatrix}$, $Soc(W_W) = \begin{bmatrix} S & 0 \\ S & 0 \end{bmatrix}$. By [12, Theorem 3] or [8, Proposition 2.8], it can be shown that $Z^*(W_W) = l_W(Soc({}_WW)) = \begin{bmatrix} J & Q/S \\ J & Q/S \end{bmatrix}$ where $l_W(.)$ is the left annihilator over W. Since $S \leq J$, then $Soc(W_W) \leq Z^*(W_W)$. This implies that $Soc_W(W_W) = Soc(W_W)$.

5)[5] Let $R = F[x; \sigma]$ be the twisted polynomial rings where F is a field of characteristic $p > 0$ and $\sigma : F \to F$ is the endomorphism given by $\sigma(a) = a^p (a \in F)$. The ring R consists of all polynomials $a_0 + xa_1 + x^2a_2 + \ldots + x^na_n$ where n is a non–negative integer, $a_i \in F$ $(0 \leq i \leq n)$, multiplication is given by the relation $ax = x\sigma(a)(a \in F)$ Note that R is a principal right ideal domain [5, p.597]. Let A denote the ideal xR of R. Clearly A is a maximal right ideal of R and the R–module R/A is not injective because $R/A \neq (R/A)x$ (see [13, Proposition 2.6]). In [5, Proposition 9], it is given an example of a field F such that the R–module R/sR is injective for all $s \in R - xR$. Thus some simple R–modules are injective and some are not. In particular, for the principal right ideal domain R, $Z^*(M_1) = M_1$ and $Z^*(M_2) = 0$ for some simple R–modules M_1 and M_2. In this case, $Z^*(M_1 \oplus M_2) = M_1 \oplus 0 \neq 0, M_1 \oplus M_2$ (see [8, p.4918]). Hence $Soc_R(M_1 \oplus M_2) = Soc(Z^*(M_1 \oplus M_2)) = M_1 \oplus 0 \neq Soc(M_1 \oplus M_2)$.

3. μ–M–Singular Modules

Definition 3.1. Let M be a module and $N \in \sigma[M]$. N is called *μ–M–singular* if $N \cong K/L$ for some $K \in \sigma[M]$ and $L \leq_{\mu e} K$. In case $M = R$, we use μ–singular.

The class of μ–M–singular modules is closed under submodules, homomorphic images, direct sums and isomorphisms.

Hence every module $N \in \sigma[M]$ contains a largest μ–M–singular submodule which we denote by $\mathrm{Z}_{\mu M}(N) = \mathrm{Tr}(\mu M\text{-}Sing, N)$ where μM-*Sing* is the class of all μ–M–singular modules. Then $\mathrm{Z}_M(N) \leq \mathrm{Z}_{\mu M}(N)$. If $N \in \sigma[M]$ is μ–M–singular (i.e. $\mathrm{Z}_{\mu M}(N) = N$) and a μ–module, then N is M–singular. For, let $N \in \mu$ and $N \cong K/L$ where $K \in \sigma[M]$, $L \leq_{\mu e} K$. We claim that $L \leq_e K$. Let $0 \neq X \leq K$ and assume that $L \cap X = 0$. Then $X \cong (L \oplus X)/L \leq K/L$ and so $X \in \mu$. Since $L \leq_{\mu e} K$ we have a contradiction. This proves that N is M-singular.

If $\mathrm{Z}_{\mu M}(N) = 0$, then N is called *non–μ–singular* in $\sigma[M]$ or *non–μ–M–singular*.

Proposition 3.1. *Let N be a μ–M–singular module and $f \in Hom_R(M, N)$.*

(1) If M is quasi–projective and $f(M)$ is finitely generated, then $kerf \leq_{\mu e} M$.

(2) If M is projective in $\sigma[M]$, then $kerf \leq_{\mu e} M$.

Proof. (1) We may assume $f(M) \cong L/K$ where $L \in \sigma[M]$ is finitely generated and $K \leq_{\mu e} L$. Since $L \in \sigma[M]$ and L is finitely generated, then M is L–projective. Hence there exists a homomorphism $g : M \to L$ such that $\pi g = f$ where π is the natural epimorphism $L \to L/K$. Then $kerf = g^{-1}(K) \leq_{\mu e} M$ by Lemma 1.2.
(2) By the proof of (1). □

Proposition 3.2. *Let P be a projective R–module and $X \leq P$. Then P/X is μ–singular if and only if $X \leq_{\mu e} P$.*

Proof. If $I \leq R_R$ and R/I is μ–singular, then $I \leq_{\mu e} R$ by Proposition 3.1. Now let P/X be μ–singular and assume $X \not\leq_{\mu e} P$. Let F be a free module such that $F = P \oplus P'$, $P' \leq F$. Then $F/(X \oplus P') \cong P/X$ is μ–singular and $X \oplus P' \not\leq_{\mu e} F$. So we may assume without loss of generality P is free, i.e. $P = \oplus R_\lambda$, each R_λ is a copy of R. Take R_λ. Then $R_\lambda/(R_\lambda \cap X) \cong (R_\lambda + X)/X \hookrightarrow P/X$ is μ–singular. So $R_\lambda \cap X \leq_{\mu e} R_\lambda$. This implies that $\oplus R_\lambda \cap X \leq_{\mu e} \oplus R_\lambda = P$, i.e. $X \leq_{\mu e} P$. □

From the properties of μ–singular modules and the above propositions the following can be seen easily.

Proposition 3.3. *For an R–module N the following are equivalent.*

a) N is μ–singular (in Mod–R).

b) $N \cong F/K$ with F a projective (free) R–module and $K \leq_{\mu e} F$.

c) For every $n \in N$, the right annihilator $r(n)$ is μ–essential in R.

Recall that a submodule N of a module M is said to be *closed* in M if N has no proper essential extension in M, denote $N \leq_c M$.

Lemma 3.1. *Let M be a module and $N \in \sigma[M]$. If $Z_{\mu M}(N) = 0$ and $K \leq_c N$, then $Z_{\mu M}(N/K)$=0.*

Proof. Clear by definitions. □

From now on we consider the condition that for every M–singular module N, $\mathrm{Rej}(N,\mu) = N$ and give a characterization of GCO-modules and semisimple Artinian rings by considering the classes $\mathcal{M}, \delta$ and $\mathcal{FC}$.

Theorem 3.1. *Let M be a module. Consider the following conditions.*

a) Every μ–module is projective in $\sigma[M]$.

b) For every M–singular module N, $Rej(N,\mu) = N$.

c) For every μ–M–singular module N, $Rej(N,\mu) = N$.

d) For every simple M–singular module N, $Rej(N,\mu) = N$.

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d) . If μ is closed under factor modules, then (a)-(d) are equivalent.

Proof. (a)$\Rightarrow$ (b) Let N be an M–singular module. Let $g : N \to L$ where $L \in \mu$. Then $N/kerg \in \mu$. By (a), $N/kerg$ is projective in $\sigma[M]$. Since N is M–singular, we have that $N = kerg$. Hence $\mathrm{Rej}(N,\mu) = N$.
(b)$\Rightarrow$ (c) Let N be a μ–M–singular module and $g : N \to L$ a homomorphism where $L \in \mu$. Then $N/kerg \in \mu$. This implies that $\mathrm{Rej}(N/kerg,\mu) = 0$. Since $N/kerg$ is μ–M–singular and a μ–module, it is M–singular. Then by (b), $N = kerg$. Hence $\mathrm{Rej}(N,\mu) = N$.
(c)$\Rightarrow$ (b) and (b)$\Rightarrow$ (d) are clear.
(d)$\Rightarrow$ (a) Assume that μ is closed under factor modules. Let $N \in \sigma[M]$ be a μ–module. We claim that N is semisimple. Let $x \in N$ and K be a maximal submodule of xR. Then xR/K is a simple μ–module. By (d) it cannot be M–singular. Hence xR/K is projective in $\sigma[M]$. This implies that K is a direct summand of xR. Hence N is semisimple. Because of the above process, any simple submodule of N is projective in $\sigma[M]$. It follows that N is projective in $\sigma[M]$. $\square$

If we consider the class $\mathcal{M}$ of all M–small modules we have a characterization of GCO–modules: A module M is called a *GCO–module* if every simple M–singular module is injective in $\sigma[M]$. (see[1]).

Corollary 3.1. *Let M be a module. Then the following are equivalent.*

a) Every M–small module is projective in $\sigma[M]$.

b) Every M–singular module is non–M–cosingular.

c) Every $\mathcal{M}$–M–singular module is non–M–cosingular.

d) M is a GCO–module.

e) Every δ–M–singular module is non–M–cosingular.

Proof. (d) $\Leftrightarrow$ (a) is by[7] and (b) $\Leftrightarrow$ (d) is by.[10]

Simple modules are either M–injective or M–small. Hence (a)-(d) are equivalent by Theorem 3.1. (e) $\Rightarrow$ (b) is clear. Since $\mathcal{M} \subseteq \delta$, every δ–M–singular module is $\mathcal{M}$–M–singular. Hence (c) $\Rightarrow$ (e) is clear. $\square$

For the class δ of all M–cosingular modules, we immediately have the following corollary. The equivalencies of (a), (b) and (d) are given in.[10]

Corollary 3.2. *Let M be a module. Consider the following conditions.*

a) Every M–cosingular module is projective in $\sigma[M]$.

b) For every M–singular module N, Rej$(N, \delta) = N$.

c) For every δ–M–singular module N, Rej$(N, \delta) = N$.

d) M is a GCO–module.

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d). If δ is closed under factor modules (see Theorem 2.1), then (a)-(d) are all equivalent.

Talebi and Vanaja[14] are also studied the modules M such that every M–cosingular module is projective in $\sigma[M]$.

A module M is called *finitely cogenerated* if Soc(M) is finitely generated and essential submodule of M. Let $\mathcal{FC}$ be the class of all finitely cogenerated R–modules. Note that $\mathcal{FC}$ is closed under submodules.

Corollary 3.3. *The following are equivalent for a ring R.*

a) Every finitely cogenerated R–module is projective.

b) For every singular R–module N, Rej$(N, \mathcal{FC}) = N$.

c) For every $\mathcal{FC}$–singular R–module N, Rej$(N, \mathcal{FC}) = N$.

d) R is semisimple Artinian.

Proof. (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) By Theorem 3.1.
(d) $\Rightarrow$ (a) is clear.
(b) $\Rightarrow$ (d) Let E be an essential right ideal of R. Suppose that a is an element of R but a does not belong to E. Let F be a right ideal of R maximal with respect to the properties that E is contained in F and a does not belong to F. Then $(aR + F)/F$ is simple singular. By (b), we have a contradiction. Hence R is semisimple Artinian. $\square$

A ring R is a quasi-Frobenius ring (briefly QF–ring) if and only if every right R–module is a direct sum of an injective module and a singular module.[11] In this result we may take μ–singular modules instead of singular as the following result shows.

Theorem 3.2. *The following are equivalent for a ring R.*

a) R is a QF–ring.

b) Every right R–module is a direct sum of an injective module and a μ–singular module.

Proof. (a)$\Rightarrow$ (b) It is clear.
(b)$\Rightarrow$ (a) Let M be a projective R–module. Then M is a direct sum of an injective module and a μ–singular module. Since projective μ–singular modules are zero, M is injective. Then R is a QF–ring (see for example[1]). $\square$

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