Primitive Submodules, Co-semisimple and Regular Modules

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Abstract

In this paper, primitive submodules are defined and various properties of them are investigated. Some characterizations of co-semisimple modules are given and several conditions under which co-semisimple and regular modules coincide are discussed.

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1 Introduction

Let M be a module. Any module that is isomorphic to a submodule of some homomorphic image of a direct sum of copies of M is called an *M-subgenerated module*. The full subcategory of the category of all modules whose objects are all M-subgenerated modules is denoted by $\sigma[M]$. For a ring R, $\sigma[M]$ consists of all R-modules if and only if $R \in \sigma[M]$. Let M and N be modules. M is called *N-projective* if for every epimorphism $g : N \to X$ and homomorphism $f : M \to X$, there exists a homomorphism $h : M \to N$ such that $g \circ h = f$. A module M is called *projective in $\sigma[M]$* if M is N-projective for every $N \in \sigma[M]$. A module M is called *quasi-projective* if M is M-projective. In [29, 18.3] it is proved that a finitely generated quasi-projective module is projective in $\sigma[M]$.

A module $N \in \sigma[M]$ is called *M-singular* [28] if there exists a short exact sequence $0 \to K \to L \to N \to 0$ in $\sigma[M]$ such that K is essential in L. The largest M-singular submodule of N is denoted by $\mathcal{Z}(N)$. If $\mathcal{Z}(N) = 0$, then N is called *non-M-singular*.

Let M be a module and N and K be submodules of M. The *product of N with K in M* is defined as follows [5]:

$$N_M K = \sum\{f(N) \mid f \in Hom_R(M, K)\}.$$

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The product $N_M N$ will be denoted by N^2 and N will be called an *idempotent* in M if $N^2 = N$. If every submodule of M is an idempotent, then M is called *fully idempotent*. It is obvious that, for any left ideal I of a ring R, ${}_R I$ is an idempotent in ${}_R R$ iff I is an idempotent left ideal. If every left ideal (resp., two-sided ideal) of the ring R is an idempotent, then R is called a *fully left idempotent* (resp., *fully idempotent*) *ring*.

Beachy proved in [4, Proposition 5.6] that if M is projective in $\sigma[M]$, then the product of submodules is associative, i.e. $(N_M K)_M L = N_M(K_M L)$ for any submodules N, K and L of M.

Definition 1.1. Let M be a nonzero module.

1) A proper fully invariant submodule N of M is called *prime* in M [19] if $K_M L \subseteq N$, then $K \subseteq N$ or $L \subseteq N$ for any fully invariant submodules K, L of M. The module M is called a *prime module* if 0 is a prime submodule in M. Note that if M has no nonzero proper fully invariant submodules, then M is prime [19, Remark 20].

2) A proper fully invariant submodule N of M is called *semiprime* in M [20] if $K_M K \subseteq N$, then $K \subseteq N$ for any fully invariant submodule K of M. The module M is called a *semiprime module* if 0 is a semiprime submodule in M. More information on semiprime submodules can be found in [9].

By a *fully invariant (resp., prime, semiprime) factor module of* M, we mean a factor module M/N for a fully invariant (resp., prime, semiprime) submodule N of M.

A module M is called *regular* if every cyclic submodule of M is a direct summand of M (see [25] for more information). We should note that, Zelmanowitz in [32] defined a regular module provided that for any $m \in M$ there exists $f \in Hom_R(M, R)$ such that $m = f(m)m$, and he proved that every cyclic submodule of such a module is a direct summand [32, Proposition 1.6]. But the converse is not true in general, e.g. consider the abelian group $\mathbb{Z}_p$. In the ring case, they are the same notions (see [13, Theorem 1.1]) and called a von Neumann regular ring. In this paper, we use the aforementioned definition of regular modules.

The paper is organized as follows. In Section 2, we characterize regular modules in terms of semiprime modules (Theorem 2.3). In Section 3, using the annihilator of a module defined by Beachy in [4], we introduce primitive submodules inspired by left primitive ideals. Various basic properties of primitive submodules are investigated. If M is a projective module in $\sigma[M]$, then any proper primitive submodule of M is prime; and maximal and primitive submodules coincide if, in addition, M is quasi-duo (Proposition 3.9). Section 4 is devoted to modules whose primitive factors are artinian. We prove that every primitive factor module of a projective fully-bounded Noetherian module is artinian and FI-simple, i.e. it has no fully invariant submodules except 0 and M (Theorem 4.13).

In the final section, Section 5, we consider co-semisimple and regular modules and determine some relations between them. A well-known theorem of Kaplansky states that the concepts of von Neumann regular rings and V-rings coincide for commutative rings. As a generalization of this result, Baccella proved in [3, Theorem] that if R is a ring whose right primitive factor rings are artinian, then R is von Neumann regular iff R is a right V-ring (i.e. R_R is co-semisimple). But his proof is not correct, because in the proof he used the fact that *"a prime fully idempotent ring is right and left nonsingular"* (see [2, Lemma 4.3]). We see that this fact is not true making use of an example due to Bergman, see Remark 5.17. In this respect, we investigate some conditions under which any co-semisimple module with every primitive factor module artinian is regular. We prove that if M is finitely generated, quasi-projective, co-semisimple, fully bounded, and every primitive factor module of M is artinian, then M is regular (Corollary 5.12). Also, if M is finitely generated, quasi-projective, co-semisimple, and every essential submodule of M is a finite intersection of maximal submodules and every primitive factor module of M is artinian, then M is regular (Theorem 5.15). On the other hand, if $Hom_R(M,S) \neq 0$ for every simple module in $\sigma[M]$, every primitive factor module of M is co-semisimple, and M is regular, then M is cosemisimple (Theorem 5.7). Furthermore, Kaplansky's result was also extended to left quasi-duo rings by Yu [31]. In this section, we also provide the module-theoretic version of Yu's theorem (Proposition 5.13).

Throughout this paper, rings are associative with identity, and modules are left modules. Let R be a ring. We write ${}_RM$ for a left R-module M. The notation $N \leq M$ ($N \leq_e M$) means that N is an (essential) submodule of a module M. The Jacobson radical and the socle of M are denoted by $Rad(M)$ and $Soc(M)$, respectively. We denote by $Hom_R(M,K)$ the R-homomorphisms from the module M to the module K, and by $End_R(M)$ the endomorphism ring of a module M over a ring R. We refer to [1,29] for all undefined terminology in this paper.

2 Regular modules

Recall that a module M is called *regular* if every cyclic submodule of M is a direct summand of M (see [25] for more information). In this section, we characterize regular modules in terms of semiprime modules.

Note that any regular module is semiprime. For, let M be a regular module, N a submodule of M and $n \in N$. Since Rn is a direct summand of M, $n = \pi(n) \in N_MN$ where $\pi : M \to Rn$ is the canonical projection. Hence $N_MN = N$. But semiprime modules need not be regular, for example, consider the $\mathbb{Z}$-module $\mathbb{Z}$. We need the following two results.

Lemma 2.1. *Let M be an R-module and $x \in M$. If there exists a morphism $f : M \to Rx$ such that $f(x) = x$, then Rx is a direct summand of M.*

Proof. Let $x \in M$ and $f : M \to Rx$ be a morphism such that $f(x) = x$. Notice that $Rx \cap Ker(f) = 0$. If $m \in M$, then $f(m) = rx = f(rx)$, so $m - rx \in Ker(f)$. Thus $m = rx + (m - rx) \in Rx \oplus Ker(f)$. □

Proposition 2.2. *An R-module M is regular if and only if for all $x \in M$, there exists a morphism $f : M \to Rx$ such that $f(x) = x$.*

Proof. $\Rightarrow$. Let $x \in M$. Since M is regular, $M = Rx \oplus K$ for some submodule K. Then $\pi(x) = x$ where $\pi : M \to Rx$ is the canonical projection of M.

$\Leftarrow$. It is by Lemma 2.1. □

According to Proposition 2.2, every submodule and every fully invariant factor module of a regular module is regular. Also any regular module is fully idempotent.

Theorem 2.3. *Let M be a nonzero projective module in $\sigma[M]$. Consider the following:*

1) M is semiprime, the union of every chain of semiprime submodules in M is semiprime, and every prime factor module of M is regular.

2) M is regular.

Then (1) $\Rightarrow$ (2). If M is finitely generated, then (2) $\Rightarrow$ (1).

Proof. (2) $\Rightarrow$ (1) We just note that the union of semiprime submodules is a proper submodule of M since M is finitely generated.

(1) $\Rightarrow$ (2). Let $0 \neq x \in M$. Suppose Rx has no direct complements in M. Consider

$$\Gamma = \{A \leq M \mid A \text{ is semiprime in } M \text{ and } (Rx + A)/A \text{ has no direct complements in } M/A\}$$

$\Gamma \neq \emptyset$ because $0 \in \Gamma$. Let $\{A_i\}_I$ be a chain in Γ. By hypothesis $A := \bigcup_I A_i$ is semiprime in M. Now, suppose that

$$\frac{M}{A} = \frac{Rx + A}{A} \oplus \frac{N}{A}.$$

Consider the canonical projections $\rho : M/A \to (Rx + A)/A$ and $\pi : M \to M/A$. Since M is projective in $\sigma[M]$, there exists $f : M \to Rx$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi"'] & Rx \arrow[d, "\pi|_{Rx}"] \\ M/A \arrow[r, "\rho"] & Rx + A/A. \end{tikzcd}$$

So, $f(x)+A = \pi(f(x)) = \rho(\pi(x)) = x+A$, hence $f(x)-x \in A$. Let $j \in I$ such that $f(x)-x \in A_j$. Since A_j is semiprime, then it is fully invariant, so f defines $\overline{f} : M/A_j \to (Rx + A_j)/A_j$. Notice that $\overline{f}(x + A_j) = f(x) + A_j = x + A_j$. Therefore, by Lemma 2.1, $(Rx + A_j)/A_j$ has a direct complement in M/A_j, which is a contradiction. Thus $A \in \Gamma$.

Then, by Zorn's Lemma, there exists a semiprime submodule A of M maximal with respect to the property that $Rx + A/A$ has no direct complements in M/A. By hypothesis, A cannot be prime in M. So, we can assume that $A = 0$ and M is semiprime but not prime. Hence, there exist nonzero fully invariant submodules B and C of M such that $B_M C = 0$. Let $N = Ann_M(C)$ (see Definition 3.1). Then $0 \neq B \leq N$. Now, denote $K = Ann_M(N)$. Since $N_M C = 0$, $C_M N = 0$ by [9, Lemma 1.19], so $0 \neq C \leq K$, and by construction $K_M N = 0$. By [9, Lemma 1.19] $N \cap K = 0$, and N and K are semiprime by [9, Lemma 1.23]. Hence, by the choice of A, $(Rx + N)/N$ has a direct complement in M/N and $(Rx + K)/K$ has a direct complement in M/K.

Let $\pi : M \to M/N$, $\rho : M \to M/K$, $\overline{f} : M/N \to (Rx + N)/N$ and $\overline{g} : M/K \to (Rx + K)/K$ be the canonical projections. Since M is projective in $\sigma[M]$, there exist $f : M \to Rx$ and

$g : M \to Rx$ such that $\pi f = \overline{f}\pi$ and $\rho g = \overline{g}\rho$.

$$\begin{tikzcd} & M \arrow[ddl, "f"'] \arrow[d, "\pi"] & & & M \arrow[ddl, "g"'] \arrow[d, "\rho"] \\ & M/N \arrow[d, "\overline{f}"] & & & M/K \arrow[d, "\overline{g}"] \\ Rx \arrow[r, "\pi"] & (Rx+N)/N & & Rx \arrow[r, "\rho"] & (Rx+K)/K \end{tikzcd}$$

Then $x - f(x) \in N$ and $x - g(x) \in K$. On the other hand, $\rho g f(x) = \overline{g}\rho f(x) = f(x) + K$ because $f(x) \in Rx$. So $gf(x) - f(x) \in K$. Therefore,

$$x + (gf - f - g)(x) = x - f(x) + g(f(x) - x) \in N$$

and

$$x + (gf - f - g)(x) = x - g(x) + gf(x) - f(x) \in K.$$

Hence, $x + (gf - f - g)(x) = 0$. This implies that $(gf - f - g)(x) = -x$, and so $(g + f - gf)(x) = x$. Thus M is regular by Proposition 2.2, a contradiction. $\square$

Corollary 2.4. *Let M be a nonzero finitely generated quasi-projective module. Then M is regular if and only if every nonzero fully invariant factor module of M is semiprime and every prime factor module of M is regular.*

Proof. $\Rightarrow$. Since M is regular, $N_M N = N$ for every submodule N of M. Then every proper fully invariant submodule of M is semiprime. Hence every nonzero fully invariant factor module of M is semiprime.

$\Leftarrow$. First note that M is projective in $\sigma[M]$ by [29, 18.3]. Let $\{A_i\}_I$ be a chain of semiprime submodules of M. Since each A_i is proper and M is finitely generated, $\bigcup_I A_i$ is a proper fully invariant submodule of M. By hypothesis $M/\bigcup_I A_i$ is a semiprime module. Thus $\bigcup_I A_i$ is semiprime in M. $\square$

3 Primitive submodules

In this section, we define primitive submodules and consider some of their basic properties. First, recall the annihilator of a module.

Definition 3.1. [4] Let M and X be R-modules. *The annihilator of X in M* is defined as

$$Ann_M(X) = \bigcap \{Ker(f) \mid f \in Hom_R(M, X)\}.$$

This is also defined as $Rej_M(X)$ in the literature (see [1]). Note that $Ann_R(X) = l_R(X)$, the usual left annihilator of X in R by [1, Proposition 8.22].

According to [4, Proposition 1.6], $Ann_M(X)$ is a fully invariant submodule of M and is the greatest submodule of M such that $Ann_M(X)_M X = 0$. Also, notice that $Ann_M(X) = M$ if and only if $Hom_R(M, X) = 0$.

Definition 3.2. Let M be a module and P a submodule of M. The module P is called *a primitive submodule* of M if there exists a simple module $S \in \sigma[M]$ such that $P = Ann_M(S)$. The module M is called *primitive* if 0 is a primitive submodule of M.

Remark 3.3.

1. $Rad(M) \subseteq Ann_M(S)$ for any module M and any simple module $S \in \sigma[M]$.

2. The following are equivalent for an R-module M.

   i) $Rad(M) = M$.

   ii) $Ann_M(S) = M$ for every simple module $S \in \sigma[M]$.

   iii) $Hom_R(M, S) = 0$ for every simple module $S \in \sigma[M]$.

   Indeed, if M has a maximal submodule $\mathcal{M}$, then M has a proper primitive submodule, namely $P = Ann_M(M/\mathcal{M})$. Hence every nonzero module has a primitive submodule. Here, we should note that $\sigma[M]$ always has a simple module for any nonzero module M (see [29]).

3. Clearly, if M is a generator in $\sigma[M]$, then $Hom_R(M, S) \neq 0$ for every simple module $S \in \sigma[M]$. The converse is true if M is quasi-projective by [29, 18.5].

Proposition 3.4. *Let M be projective in $\sigma[M]$. Then any proper primitive submodule of M is prime in M.*

Proof. Let $P = Ann_M(S)$ be a proper primitive submodule of M where $S \in \sigma[M]$ is simple. Let N and L be fully invariant submodules of M such that $N_M L \leq P$. Then $(N_M L)_M S = 0$. Since M is projective in $\sigma[M]$, $N_M(L_M S) = 0$ by [4, Proposition 5.6]. On the other hand, $L_M S \leq S$ gives that $L_M S = 0$ or $L_M S = S$. If $L_M S = 0$, then we have $L \leq P$. If $L_M S = S$, then $0 = N_M(L_M S) = N_M S$. It follows that $N \leq P$. Hence P is a prime submodule of M. □

Example 3.5. Let $p \in \mathbb{Z}$ be a prime number and let k be any positive integer. Then the abelian group $\mathbb{Z}_{p^k}$ is self-projective. Since $\mathbb{Z}_{p^k}$ is finitely generated, it is projective in $\sigma[\mathbb{Z}_{p^k}]$ by [29, 18.3]. Note that $\mathbb{Z}_{p^k}/p\mathbb{Z}_{p^k} \cong \mathbb{Z}_p$. Since $p\mathbb{Z}_{p^k}$ is fully invariant in $\mathbb{Z}_{p^k}$, $Ann_{\mathbb{Z}_{p^k}}(\mathbb{Z}_p) = p\mathbb{Z}_{p^k}$ by [6, Proposition 1.8]. Hence the primitive submodule $Ann_{\mathbb{Z}_{p^k}}(\mathbb{Z}_p) = p\mathbb{Z}_{p^k}$ is prime in $\mathbb{Z}_{p^k}$.

Proposition 3.6. *For any module M, $Rad(M) = \bigcap\{P \leq M \mid P$ is primitive$\}$.*

Proof. If $M = 0$, then there is nothing to prove. Assume that $M \neq 0$. Denote $\mathbf{P} = \bigcap\{P \leq M \mid P$ is primitive$\}$. Since $Rad(M) \subseteq Ann_M(S)$ for every simple module $S \in \sigma[M]$, we have that $Rad(M) \subseteq \mathbf{P}$. On the other hand, if $Rad(M) = M$, then the only primitive submodule of M is M by Remark 3.3. Hence $\mathbf{P} \subseteq Rad(M)$. Assume that $Rad(M) \neq M$, and let $\mathcal{M}$ be a maximal submodule of M. Then $Ann_M(M/\mathcal{M}) \subseteq \mathcal{M}$ because $\mathcal{M}$ is the kernel of the natural epimorphism $M \to M/\mathcal{M}$. This implies that $\mathbf{P} \subseteq Ann_M(M/\mathcal{M}) \subseteq \mathcal{M}$. Hence again $\mathbf{P} \subseteq Rad(M)$. □

Lemma 3.7. *Let M be a module and $P = Ann_M(S)$ a proper primitive submodule of M for some simple $S \in \sigma[M]$. Then $S \in \sigma[M/P]$ and $Ann_{M/P}(S) = 0$.*

Proof. Since $P = Ann_M(S)$ is a proper submodule of M, there exists a nonzero homomorphism $f : M \to S$. Then $P \subseteq Ker(f)$ and so we have an epimorphism $M/P \to M/Ker(f) \cong S$. It follows that $S \in \sigma[M/P]$.

Let $x + P \in M/P$ be a nonzero element in $Ann_{M/P}(S)$. Since $x \notin P$, there exists a homomorphism $g : M \to S$ such that $g(x) \neq 0$. So $P \subseteq Ker(g)$. This implies that there exists a homomorphism $\overline{g} : M/P \to S$ such that $g = \overline{g}\pi$ where $\pi : M \to M/P$ is the canonical epimorphism. Thus $0 \neq g(x) = \overline{g}\pi(x) = \overline{g}(x + P) = 0$, a contradiction. Hence $Ann_{M/P}(S) = 0$. □

Definition 3.8. Let M be a module. M is a *quasi-duo module* if all maximal submodules of M are fully invariant.

In [23, 3.25], quasi-duo modules were presented as *quasi-invariant modules.*

Proposition 3.9. *Let M be projective in $\sigma[M]$. Then M is quasi-duo if and only if the maximal and proper primitive submodules of M are the same.*

Proof. It is clear that if the maximal and primitive submodules coincide, then M is quasi-duo.

Let $\mathcal{M}$ be a maximal submodule of M. Since M is quasi-duo, we have $\mathcal{M}_M(M/\mathcal{M}) = 0$ by [6, Proposition 1.8], so $\mathcal{M} \subseteq Ann_M(M/\mathcal{M})$. Thus $\mathcal{M} = Ann_M(M/\mathcal{M})$. Now, let $P = Ann_M(S)$ be a proper primitive submodule for some simple module S in $\sigma[M]$. Let $0 \neq f : M \to S$. Since $Ker(f)$ is maximal, it is fully invariant in M. So $0 = Ker(f)_M(M/Ker(f)) \cong Ker(f)_M S$. Thus $Ker(f) = P$. □

4 Modules whose primitive factors are artinian

In this section, we prove that every primitive factor module of a projective fully-bounded Noetherian module is artinian FI-simple.

Let us first consider the modules whose primitive factors are artinian. Obviously, if $M/Rad(M)$ is artinian, then M/P is artinian for any primitive submodule P of M. On the other hand, we will show that if M is a projective module in $\sigma[M]$ and M/P is artinian for any primitive submodule P of M, then M/P is semisimple. To prove it, we need the following.

Definition 4.1. A module M is called *retractable* [16] if $Hom_R(M, N) \neq 0$ for all $0 \neq N \leq M$.

Lemma 4.2. *Let M be projective in $\sigma[M]$. If M is a semiprime module, then it is retractable.*

Proof. Let M be semiprime and $N \leq M$. If $Hom(M, N) = 0$, then $M_M N = 0$. But $N_M N \subseteq M_M N = 0$, and so $N = 0$. □

Proposition 4.3. *Let M be a projective module in $\sigma[M]$ and let P be a primitive submodule of M. If M/P is artinian, then M/P is semisimple.*

Proof. Let M be a projective module in $\sigma[M]$ and P a primitive submodule of M such that M/P is artinian. If $P = M$, then there is nothing to prove. Assume that $P \neq M$. Since P is fully invariant, M/P is projective in $\sigma[M/P]$ (see [26, Lemma 9]) and it is a prime module by Proposition 3.4 and [19, Proposition 18]. So M/P is retractable by Lemma 4.2. Any retractable semiprime artinian module is semisimple by [9, Theorem 1.17]. Hence M/P is semisimple. $\square$

Now, we give the following definitions generalizing the concept of left bounded (resp., fully bounded) rings given by Chatters and Hajarnavis in [11] to the module theory.

Definition 4.4. 1) A module M is *bounded* if any essential submodule of M contains a fully invariant submodule of M which is an essential submodule.

2) M is *fully bounded* if every prime factor module of M is bounded.

3) M is an *FBN-module* if M is fully bounded and noetherian.

Example 4.5. If M is an artinian uniserial module (i.e. a module whose submodules are linearly ordered), then $Soc(M)$ is simple, fully invariant and essential in M. Since $Soc(M)$ is contained in all nonzero submodules of M, we have that M is bounded. Moreover, since every factor module of an artinian uniserial module is artinian uniserial, M is also fully bounded.

In the literature, there are many other generalizations of bounded (resp., fully bounded) rings to modules, for example see [7], [15], [17], and [22]. The definitions of bounded and fully bounded modules given in [7, Definition 2.1] are very close to ours. For convenience of the reader, we will give the definitions here and compare them with ours.

Definition 4.6. [7, Definition 2.1] Let M be a module and τ a hereditary torsion theory in $\sigma[M]$. A submodule N of M is *τ-pure* if M/N is τ-torsion-free. The module M is *τ-bounded* if every τ-pure essential submodule of M contains a nonzero fully invariant submodule of M. The module M is *fully τ-bounded* if for every prime submodule P in M, the module M/P is τ-bounded.

It is clear that if M is a bounded module, then M is ξ-bounded, where ξ is the hereditary torsion theory $(0, \sigma[M])$. In general, the converse is not true, as the following example shows.

Example 4.7. Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}_p$ where $p \in \mathbb{Z}$ is a prime number. Note that the submodule $\mathbb{Z} \oplus \mathbb{Z}_p$ is essential in M. Now, we claim that M has no nontrivial essential fully invariant submodules. In fact, if $N \leq_e M$ and fully invariant in M, then $0 \neq N \cap (\mathbb{Q} \oplus 0)$, but $\mathbb{Q}$ has no nontrivial fully invariant submodules, hence $N \cap (\mathbb{Q} \oplus 0) = \mathbb{Q} \oplus 0$. On the other hand, since $0 \neq N \cap (0 \oplus \mathbb{Z}_p)$, then $0 \oplus \mathbb{Z}_p \subseteq N$. Thus, $N = M$. Now, if $K \leq_e M$, then $0 \neq K \cap (0 \oplus \mathbb{Z}_p)$. Hence $0 \oplus \mathbb{Z}_p \subseteq K$ and $0 \oplus \mathbb{Z}_p$ is fully invariant in M. Thus, M is ξ-bounded but not bounded.

However, for the trivial hereditary torsion theory $\xi = (0, \sigma[M])$ in $\sigma[M]$, fully ξ-bounded and fully bounded modules are equivalent provided that M is projective in $\sigma[M]$:

Proposition 4.8. *Let M be projective in $\sigma[M]$. Then M is a fully bounded module if and only if M is a fully ξ-bounded module.*

Proof. $\Rightarrow$. It is obvious.

$\Leftarrow$. Let P be a prime submodule of M and consider the factor module M/P. Let $N \leq_e M/P$. Since M is fully ξ-bounded, then there exists a nonzero fully invariant submodule K of M/P such that $K \subseteq N$. Since M/P is a prime module, by [10, Lemma 4.5] (for a proof of this lemma see [9, Proposition 1.3]), every nonzero fully invariant submodule is essential, hence $K \leq_e M/P$. Thus, M/P is fully bounded. $\square$

Definition 4.9. A module M is called *FI-simple* [10], if it has no fully invariant submodules except 0 and M.

Lemma 4.10. *A module M is semisimple Artinian FI-simple if and only if $M \cong S^{(n)}$ where n is a natural number and S is a simple module.*

Proof. It follows from the facts that, if M is a semisimple module, then each homogeneous component (i.e. the direct sum of all isomorphic simple submodules) of M is fully invariant in M, and each fully invariant submodule F of M is the direct sum of the homogeneous components of M such that each of which has nonzero intersection with F. $\square$

Definition 4.11. [8] Let M be an R-module. A *left annihilator* in M is a submodule of the form

$$\mathcal{A}_X = \bigcap \{Ker(f) \mid f \in X\}$$

for some $X \subseteq End_R(M)$.

The following result will be used to prove the main theorem.

Proposition 4.12. [10, Proposition 2.11] *Let M be projective in $\sigma[M]$ with nonzero socle. If M is a prime module satisfying ACC (ascending chain condition) on left annihilators, then M is semisimple artinian and FI-simple.*

We are now going to prove the main theorem of this section. Note that the ring version was proved in [14, Proposition 8.4].

Theorem 4.13. *Let M be projective in $\sigma[M]$. If M is an FBN-module, then M/P is artinian and FI-simple for every primitive submodule P of M.*

Proof. Assume that $M \neq 0$. For any proper primitive submodule P of M, M/P is projective in $\sigma[M/P]$ (see [26, Lemma 9]), FBN and prime. Hence without loss of generality, we can assume $P = 0$. According to Proposition 4.12, the proof will be completed if $Soc(M) \neq 0$. Let S be simple in $\sigma[M]$ such that $P = Ann_M(S) = 0$. Then $M_M S \neq 0$ and so there exists a nonzero morphism $f : M \to S$. Now we claim that $Ker(f)$ is not essential in M. Since M is bounded, it is enough to prove that $Ker(f)$ has no nonzero fully invariant submodule of M. Assume on the contrary that N is a nonzero fully invariant submodule of M such that $N \subseteq Ker(f)$. Then there is an epimorphism $M/N \to M/Ker(f) \cong S$. By [4, Lemma 5.4], there exists an epimorphism $N_M(M/N) \to N_M S$. Since N is fully invariant in M, $N_M(M/N) = 0$ by [6, Proposition 1.8].

Hence $N_M S = 0$. Since $Ann_M(S) = 0$, $N = 0$. This proves the claim. Then there exists $0 \neq K \leq M$ such that $Ker(f) \cap K = 0$. Since $Ker(f)$ is a maximal submodule of M, we conclude that $M = Ker(f) \oplus K$. Thus $K \cong S$, and so $Soc(M) \neq 0$. $\square$

Recall that a module M is called *Kasch* if every simple module in $\sigma[M]$ can be embedded in M.

Proposition 4.14. *Let M be projective in $\sigma[M]$. If M is a Kasch and an FBN-module, then $M/Rad(M)$ is semisimple artinian.*

Proof. First notice that since M is noetherian, $Rad(M) \neq M$. Since M is Kasch, every simple module in $\sigma[M]$ can be embedded in $Soc(M)$, and since M is noetherian, $Soc(M)$ is finitely generated. Then the set of simple modules in $\sigma[M]$ is finite up to isomorphism. Hence the set of primitive submodules of M is finite, say $P_1, ..., P_n$. This implies that $Rad(M) = P_1 \cap \cdots \cap P_n$ by Proposition 3.6. Then by the monomorphism $M/Rad(M) \to M/P_1 \oplus \cdots \oplus M/P_n$ defined by $m + Rad(M) \mapsto (m + P_1, \ldots, m + P_n)$ and Theorem 4.13, we obtain that $M/Rad(M)$ is semisimple artinian. $\square$

5 Co-semisimple and regular modules

Dual to semisimple modules are the co-semisimple modules which were introduced by Fuller [12] and also called V-modules by Ramamurthi [21].

Definition 5.1. A module M is *co-semisimple* if each simple module (in $\sigma[M]$) is M-injective. If $_RR$ is co-semisimple, we call R *a left V-ring.*

Notice that any simple module not belonging to $\sigma[M]$ is M-injective, and semisimple modules are co-semisimple (see [29, p. 190]). It is well known that co-semisimple and regular modules are independent notions (see [29, Example 23.6]). In this section, we investigate some conditions under which regular modules and co-semisimple modules coincide.

The following characterization of co-semisimple modules was given in [12, Proposition 3.1], for further characterizations see [29, 23.1].

Proposition 5.2. *The following are equivalent for a module M:*

1. *M is co-semisimple.*

2. *Any proper submodule of M is an intersection of maximal submodules.*

Furthermore, we recall the following lemma.

Lemma 5.3. [10, Lemma 1.1] *Let M be projective in $\sigma[M]$, $K \leq M$, and $\{X_i\}_I$ a family of modules in $\sigma[M]$. Then*

$$K_M(\sum_I X_i) = \sum_I (K_M X_i).$$

Proposition 5.4. [10, Proposition 4.4] *Let M be a projective module in $\sigma[M]$. If M is co-semisimple, then it is fully idempotent.*

Proof. Let N be a submodule of M and suppose that $N_M N \subset N$. According to Proposition 5.2, there exists a maximal submodule $\mathcal{M}$ of M such that $N_M N \subseteq \mathcal{M}$ but $N \nsubseteq \mathcal{M}$. Hence $M = N + \mathcal{M}$. Therefore, by Lemma 5.3,

$$N \subseteq N_M M = N_M(\mathcal{M} + N) = N_M \mathcal{M} + N_M N \subseteq \mathcal{M},$$

a contradiction. Thus $N_M N = N^2 = N$. $\square$

Proposition 5.5. *Let M be a nonzero co-semisimple module. Then M has a proper primitive submodule.*

Proof. Let L be a nonzero finitely generated submodule of M. Since L has a maximal submodule, there exists an epimorphism $L \to S$ for some simple module $S \in \sigma[M]$. Since M is co-semisimple, S is M-injective. So there exists a nonzero morphism $M \to S$. Then $Hom_R(M, S) \neq 0$, and hence $Ann_M(S)$ is the proper primitive submodule of M. $\square$

Theorem 5.6. *Let M be projective in $\sigma[M]$. Then the following are equivalent:*

1) M is co-semisimple.

2) M is a generator in $\sigma[M]$, fully idempotent, and every primitive factor module of M is co-semisimple.

Proof. (1) $\Rightarrow$ (2). Since M is co-semisimple, it is fully idempotent by Proposition 5.4. The result 23.8 in [29] states that if M is M-projective and co-semisimple, then M is a generator in $\sigma[M]$. So every primitive submodule of M is proper by Remark 3.3(3). Now, let P be a primitive submodule of M and $S \in \sigma[M/P]$ be simple. Since $\sigma[M/P] \subseteq \sigma[M]$, S is simple in $\sigma[M]$. Then S is M-injective, and so it is M/P-injective. Hence M/P is co-semisimple.

(2) $\Rightarrow$ (1). Let S be a simple module in $\sigma[M]$ and $\widehat{S} = E_M(S)$ the M-injective hull of S in $\sigma[M]$. Denote $P := Ann_M(S)$. Since M is a generator in $\sigma[M]$, $P \neq M$ (see Remark 3.3(3)). Then M/P is co-semisimple by hypothesis. Now, assume that $P = Ann_M(\widehat{S})$. By [7, Proposition 1.5], $\widehat{S} \in \sigma[M/P]$. Since M/P is co-semisimple, S is injective in $\sigma[M/P]$ by Lemma 3.7. So S is a direct summand of $\widehat{S}$, and hence $S = \widehat{S}$ is injective in $\sigma[M]$. In this case M is co-semisimple.

Assume that $P \neq Ann_M(\widehat{S})$. Then there exists a morphism $f : M \to \widehat{S}$ such that $f(P) \neq 0$. Since $S \leq_e \widehat{S}$, $S \subseteq f(P)$. Let $p \in P$ be such that $S = Rf(p)$. Then $S = f(Rp) = f(Rp\,_M Rp) \subseteq Rp\,_M f(Rp) \subseteq P_M S = 0$. Hence $S = 0$, a contradiction. $\square$

Theorem 5.7. *Let M be a module. Assume that $Hom_R(M, S) \neq 0$ for every simple module in $\sigma[M]$, and every primitive factor module of M is co-semisimple. If M is regular, then M is cosemisimple.*

Proof. Let S be a simple module in $\sigma[M]$ and consider the proper primitive submodule $P = Ann_M(S)$ of M. By hypothesis, M/P is co-semisimple. By Lemma 3.7, $S \in \sigma[M/P]$, and then S is M/P-injective. Now, we claim that S is M-injective. Let $N \leq M$ and $f : N \to S$ be a morphism. Since M is fully idempotent, $N \cap P = N \cap (P_M P)$, so let $n = f_1(p_1) + \cdots + f_k(p_k) \in N \cap P$ where each $f_i \in Hom(M, P)$. Since M is regular, Rn is a direct summand of M, and then we have the canonical projection $\pi : M \to Rn$. Therefore, $n = \pi(n) = \pi(f_1(p_1) + \cdots + f_k(p_k)) = \pi f_1(p_1) + \cdots + \pi f_k(p_k)$. Thus

$$f(n) = f(\pi f_1(p_1) + \cdots + \pi f_k(p_k)) = f\pi f_1(p_1) + \cdots + f\pi f_k(p_k) = 0$$

because $f\pi f_i : M \to S$ for all $1 \leq i \leq k$. Then, we have a well-defined morphism $\overline{f} : (N + P)/P \to S$. Since S is M/P-injective, there exists $f' : M/P \to S$ such that $f'|_{(N+P)/P} = \overline{f}$. Thus, if $\rho : M \to M/P$ is the canonical epimorphism, $f'(\rho(N)) = f'((N + P)/P) = \overline{f}(\rho(N)) = f(N)$. Hence S is M-injective. Thus M is co-semisimple. □

Corollary 5.8. *Consider the following conditions for a ring R:*

1) R is von Neumann regular, and every left primitive factor ring of R is a left V-ring.

2) R is fully left idempotent, and every left primitive factor ring of R is a left V-ring.

3) R is a left V-ring.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

Proof. This follows from [25, Proposition 22.2] and Theorems 5.6 and 5.7. □

Lemma 5.9. 1) *If M is projective in $\sigma[M]$ and a prime module, then $E = End_R(M)$ is a prime ring.*

2) [9, Corollary 1.10] *If M is retractable and $E = End_R(M)$ is a prime ring, then M is a prime module.*

Proof. 1) Let I and J be ideals of E such that $IJ = 0$. We can assume that I and J are cyclic. On the other hand, since M is M-projective, $J = Hom_R(M, JM)$ by [29, 18.4]. Now, assume that $J \neq 0$. Then there exists a nonzero morphism $g : M \to JM$ such that $fg = 0$ for all $f \in I$. Also since $I = Hom_R(M, IM)$, we have $g(M)_M IM = 0$. By hypothesis, M is prime and projective in $\sigma[M]$, and so $IM = 0$ by [6, Proposition 1.11]. Thus $I = 0$. □

Lemma 5.10. *If M is projective in $\sigma[M]$, semiprime, and bounded, then it is non-M-singular.*

Proof. Suppose that $\mathcal{Z}(M) \neq 0$. Then there exists $0 \neq f \in Hom(M, \mathcal{Z}(M))$ by Lemma 4.2. Since $\mathcal{Z}(M)$ is M-singular and M is projective in $\sigma[M]$, $Ker(f) \leq_e M$ by [7, Proposition 1.2]. Since M is bounded, there exists a fully invariant submodule N of M such that $N \subseteq Ker(f)$ and $N \leq_e M$. By [4, Lemma 5.4], there is an epimorphism

$$N_M(M/N) \to N_M(M/Ker(f)).$$

Since N is fully invariant, $N_M(M/N) = 0$ by [6, Proposition 1.8]. It follows that $N_M f(M) \cong N_M(M/Ker(f)) = 0$. Since $(N \cap f(M))_M(N \cap f(M)) \subseteq N_M f(M) = 0$ and M is semiprime, we have that $N \cap f(M) = 0$. But $N \leq_e M$, this is a contradiction. Thus M is non-M-singular. $\square$

A module M is called *Goldie* [27] if it satisfies the ACC on left annihilators and has finite uniform dimension.

Theorem 5.11. *Let M be finitely generated, quasi-projective, co-semisimple, and non-M-singular with $E := End_R(M)$ prime. If every primitive factor module of M is artinian, then M is semisimple artinian.*

Proof. First we claim that M is a prime Goldie module.

Assume that $M \neq 0$. Since M is co-semisimple, it is semiprime and then it is retractable by Lemma 4.2. So M is prime by Lemma 5.9.

On the other hand, $T := End_R(\widehat{M})$ is the maximal ring of quotients of E by [30, 11.1 and 11.5] where $\widehat{M}$ is the M-injective hull of M. Let $\{N_n : n \in \mathbb{N}\}$ be an independent family of submodules of $\widehat{M}$. Then $\{Hom_R(\widehat{M}, N_n) : n \in \mathbb{N}\}$ is an independent family of right ideals of T. Since E_E is essential in T by [30, 11.5], we have an independent family of cyclic right ideals of E, say g_nE $(n \in \mathbb{N})$, such that all g_nT $(n \in \mathbb{N})$ is independent in T. Since E is a prime ring, there exists $h_n \in E$ $(n \in \mathbb{N})$ such that $z_n := g_nh_{n-1}g_{n-1}\dots h_1g_1 \neq 0$. So, we have a descending chain of left ideals $Tz_1 \supseteq \cdots \supseteq Tz_n \supseteq \cdots$. Since T is a regular ring, there exist nonzero idempotents $f_n \in T$ such that $Tz_n = Tf_n$ for all $n \in \mathbb{N}$. Then there is an ascending chain $(1 - f_1)T \subseteq \cdots \subseteq (1 - f_n)T \subseteq \cdots$. Consider the R-submodule

$$K := \Big(\bigcup_{n>0}(1 - f_n)T\Big)\widehat{M}$$

of $\widehat{M}$.

If $M \subseteq K$, then $M \subseteq (1 - f_n)T\widehat{M} \subseteq K \subseteq \widehat{M}$ for some n because M is finitely generated. Then $(1 - f_n)T\widehat{M} \leq_e \widehat{M}$, but $(1 - f_n)\widehat{M} \cap f_n\widehat{M} = 0$ gives a contradiction. Thus $K \subset \widehat{M}$ and $M \nsubseteq K$. Since M is co-semisimple, so is $\widehat{M}$. Then there exists a maximal submodule $\mathcal{M}$ of $\widehat{M}$ such that $K \subseteq \mathcal{M}$ and $M \nsubseteq \mathcal{M}$.

Assume that $j : M \to \widehat{M}$ and $\pi : \widehat{M} \to \widehat{M}/\mathcal{M}$ are the canonical inclusion and projection respectively. Since $M \nsubseteq \mathcal{M}$, $\pi j \neq 0$. This implies that $P = Ann_M(\widehat{M}/\mathcal{M})$ is a proper primitive submodule of M. Notice that $TP \subseteq \mathcal{M}$, in fact, if $\alpha \in T$, then $\pi\alpha(P) = \pi\alpha j(P) = 0$ because $\pi\alpha j : M \to \widehat{M}/\mathcal{M}$. Thus $TP \subseteq \mathcal{M}$.

By hypothesis M/P is artinian, hence $Soc(M/P) \neq 0$, therefore by Propositions 4.3 and 4.12, M/P is FI-simple. Hence P is a maximal fully invariant submodule of M. On the other hand, $TP \cap M$ is a fully invariant submodule of M containing P, so $P = TP \cap M$ or $TP \cap M = M$. But, since $M \nsubseteq \mathcal{M}$, we have $P = TP \cap M$.

We now claim that $z_n(M) \nsubseteq P$ for all n. Suppose that $z_n(M) \subseteq P$ for some n. By construction, write $f_n = \alpha z_n$ for some $\alpha \in T$. Then $f_n(M) = \alpha z_n(M) \subseteq \alpha(P) \subseteq \mathcal{M}$. Thus

$f_n(M) \subseteq \mathcal{M}$. But $(1-f_n)(M) \subseteq \mathcal{M}$, then $M \subseteq (1-f_n)(M) + f_n(M) \subseteq \mathcal{M}$, a contradiction. Now, since $z_n(M) \subseteq g_n(M)$, then $g_n(M) \nsubseteq P$ for all n.

Since $T = Q_{max}(E)$, by [13, Lemma 9.7] there exist orthogonal idempotents $e_n \in T$ such that $g_nT = e_nT$. We claim that $e_n(M) \nsubseteq TP$ for all n. Assume to the contrary that $e_n(M) \subseteq TP$ for some n. Then $g_nE(M) = e_ng_nE(M) \subseteq TP \cap M = P$, and so $g_n(M) \subseteq P$, a contradiction. This enables us to define the canonical projection $\rho : \widehat{M} \to \widehat{M}/TP$ with $\rho(e_n(M)) \neq 0$ for all n. Now let $\rho(e_1(m)) \in \rho(e_1(M)) \cap \sum_{i\neq 1} \rho(e_i(M))$, then $e_1(m) = e_2(m_2) + \cdots + e_k(m_k) + x$ with $x \in TP$. Since e_n's are orthogonal idempotents, $e_1(m) = e_1(x) \in TP$. Thus $\rho(e_1(m)) = 0$. This implies that $\{\rho(e_n(M))\}$ is an independent family of submodules of $\widehat{M}/TP$. Notice that we have a monomorphism $\eta : M/P \to \widehat{M}/TP$ given by $\eta(m+P) = m + TP$. Since $g_n = e_ng_n$, $g_nE(M) \subseteq e_n(M)$ and we have that $g_nE(M) \nsubseteq P$. So $\{\rho(g_nE(M))\}$ is an inde pendent family of nonzero submodules of $\eta(M/P) \cong M/P$. This is a contradiction because M/P is artinian.

Thus $\widehat{M}$ has finite uniform dimension, then so does M. Now, M is a prime Goldie module by [9, Theorem 2.8].

Since M is co-semisimple Goldie, the only prime submodule in M is 0 by [10, Proposition 4.6 and Corollary 4.8]. Since M has a proper primitive submodule by Proposition 5.5 and proper primitive submodules are prime, 0 is primitive. According to the hypothesis, M is artinian. Any retractable semiprime artinian module is semisimple by [9, Theorem 1.17]. Thus M is semisimple. $\square$

Corollary 5.12. *Let M be finitely generated, quasi-projective, co-semisimple, and fully bounded. If every primitive factor module of M is artinian, then M is regular.*

Proof. Let M be prime. By hypothesis, M is bounded, and so non-M-singular by Lemma 5.10. According to Lemma 5.9 and Theorem 5.11, M is semisimple, hence it is regular.

Assume that M is not prime and P is a prime submodule of M. Then M/P is a nonzero prime module. Since M/P satisfies all the conditions of the hypothesis, it is semisimple artinian by the argument above. This implies that every prime factor module of M is regular. Since M is co-semisimple, every proper fully invariant submodule of M is semiprime. Thus M is regular by Corollary 2.4. $\square$

This corollary will enable us to prove a module theoretic version of Theorem 2.7 of [31].

Recall that a module M is called *duo* (see [18]) if every submodule of M is fully invariant in M.

Proposition 5.13. *Let M be finitely generated quasi-projective. If M is quasi-duo, then the following are equivalent.*

1) M is co-semisimple.

2) M is regular, duo, and a generator in $\sigma[M]$.

3) M is regular and a generator in $\sigma[M]$.

4) M is fully idempotent and a generator in $\sigma[M]$.

Proof. (1) $\Rightarrow$ (2). Since M is co-semisimple and projective in $\sigma[M]$, it is a generator in $\sigma[M]$ by [29, 23.8]. Let $N \leq M$, and write $N = \cap_{i\in I} N_i$ for some maximal submodules N_i of M. Then $f(N_i) \subseteq N_i$ for any endomorphism f of M. This implies that $f(N) \subseteq N$, hence M is duo. Since M is quasi-projective, every factor module of M is duo by [18, Proposition 1.4]. On the other hand, obviously every duo module is bounded. It follows that M is fully bounded. Thus M is regular by Proposition 3.9 and Corollary 5.12.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4). They are obvious.

(4) $\Rightarrow$ (1). It follows by Proposition 3.9 and Theorem 5.6. □

Proposition 5.14. *Let M be projective in $\sigma[M]$. Suppose that M is prime but not primitive, and every essential submodule of M is a finite intersection of maximal submodules. Then M is non-M-singular.*

Proof. We will show that M is bounded. Let $N \leq_e M$. By hypothesis, there exists a finite family of maximal submodules $\mathcal{M}_1, \ldots, \mathcal{M}_n$ such that $N = \bigcap_{i=1}^n \mathcal{M}_i$. Hence there exists a monomorphism $M/N \hookrightarrow \bigoplus_{i=1}^n M/\mathcal{M}_i$. Let $P_i = Ann_M(M/\mathcal{M}_i)$ for $i = 1, \ldots, n$. Since M is prime but not primitive, each $P_i \neq 0$ and $P_i \leq_e M$, so $P_1 \cap \cdots \cap P_n \neq 0$ and

$$(P_1 \cap \cdots \cap P_n)_M \bigoplus_{i=1}^n M/\mathcal{M}_i = 0.$$

This implies that

$$(P_1 \cap \cdots \cap P_n)_M M/N = 0.$$

Hence $P_1 \cap \cdots \cap P_n \subseteq N$. Since $P_1 \cap \cdots \cap P_n$ is essential and fully invariant in M, we have that M is bounded. Thus the proof is completed by Lemma 5.10. □

Theorem 5.15. *Let M be finitely generated and quasi-projective. Suppose that every primitive factor module of M is artinian, and every essential submodule is a finite intersection of maximal submodules. If M is co-semisimple, then M is regular.*

Proof. Suppose that M is primitive. By hypothesis, it is Artinian. Since M is co-semisimple, it is semiprime by Proposition 5.4. Hence, M is semisimple by [9, Theorem 1.17].

Suppose that M is not primitive. If M is prime, then M is non-M-singular by Proposition 5.14. So M is semisimple artinian by Lemma 5.9 and Theorem 5.11.

Assume that M is not prime. We will use Corollary 2.4 to show the regularity of M. Since M is co-semisimple, every nonzero fully invariant factor module of M is semiprime. Now take a proper prime submodule P of M and consider the prime module M/P. We claim that M/P is regular. If M/P is primitive, then M/P is semisimple Artinian by hypothesis and Proposition 4.3. Assume that M/P is not primitive. Note the fact that M/P satisfies all of the conditions in the hypothesis. Indeed, every primitive factor module of M/P is artinian, M/P is co-semisimple, and for any essential submodule N/P of M/P, N is an intersection of maximal submodules $M_1, \ldots, M_n$ of M since $N \leq_e M$. This implies that N/P is the finite intersection of maximal submodules $M_1/P, \ldots, M_n/P$ of M/P. Hence, M/P is semisimple artinian as was in the case of M above. As a result, in all cases, every prime factor module of M is regular. Thus M is regular by Corollary 2.4. $\square$

Corollary 5.16. *Let R be a ring such that every left primitive factor ring is artinian. Assume that every essential left ideal of R is a finite intersection of maximal left ideals. If R is a left V-ring, then R is von Neumann regular.*

We end the paper with the following remark.

Remark 5.17. Recall that a ring R is fully idempotent if every two-sided ideal is idempotent. In [2, Lemma 4.3], Baccella proved the following:

"A prime fully idempotent ring is right and left nonsingular."

But this lemma is false. Consider the ring R constructed by G.M. Bergman which is presented in detail in [11, pp. 27]. This ring is a prime (in fact, primitive), uniform ring and has a unique proper two-sided ideal U. The ideal U is idempotent and $U = Z_r(R)$, the right singular ideal of R. Thus R is a prime fully idempotent ring and it is not right nonsingular. Moreover, it can be seen that R is not left nonsingular. This implies that R is neither right nor left V-ring by [24, Proposition 4.5].

On the other hand, in [3, Theorem], it was proved that if R is a ring whose right primitive factor rings are artinian, then R is a right V-ring iff R is fully right idempotent iff R is von Neumann regular. But in that proof, it was used the fact that any prime right V-ring is right nonsingular by citing [2, Lemma 4.3]. Therefore the truth of [3, Theorem] is not certain now. Accordingly, we proved in Corollary 5.12 that if R is a right V-ring and right fully bounded ring whose right primitive factor rings are artinian, then R is von Neumann regular.

In the literature some authors frequently use "any prime right V-ring is right nonsingular" based on [2, Lemma 4.3]. But now it turns out to be a problem and we do not have a proof or a counterexample. Some approaches to this are Lemma 5.10 and Proposition 5.14. The next proposition is another approximation.

Proposition 5.18. *Let M be projective in $\sigma[M]$. If M is prime, co-semisimple and $Soc(M) \neq 0$, then M is non-M-singular and primitive.*

Proof. Assume that $\mathcal{Z}(M) \neq 0$. Since M is prime, any nonzero fully invariant submodule of M is essential in M. Then $Soc(M) \cap \mathcal{Z}(M) \neq 0$. So there exists a simple M-singular submodule S of M. Since S is M-injective, it is a direct summand of M. Thus S is projective in $\sigma[M]$ and M-singular, a contradiction.

Let S be a simple submodule of M. Since $Ann_M(S)_M S = 0$ and M is prime, $Ann_M(S) = 0$. Thus M is primitive. $\square$

Corollary 5.19. *If R is a prime left V-ring such that $Soc(_R R) \neq 0$, then R is a left nonsingular left primitive ring.*

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