Modules Having $*$-Radical

A. Çiğdem Özcan

June 9, 2005

Abstract

ABSTRACT. Let R be a ring with identity and M a right R-module. Let $\mathrm{E}(M)$ denote the injective hull of M and $\mathrm{Z}^*(M) := M \cap \mathrm{RadE}(M)$. We say M has $*$-radical if $\mathrm{Z}^*(M) = \mathrm{Rad}M$. In this note we characterize rings in terms of modules having $*$-radical. First we prove that R is a right V-ring (GV-ring) if and only if every (singular) right R-module has $*$-radical. After that we show that R is a right H-ring if and only if every right R-module that has $*$-radical is lifting and, R is a semiprimary QF-3 ring if and only if R is right perfect and every projective right R-module that has $*$-radical is injective (extending). Finally we obtain that R is a QF-ring if and only if every right R-module that has $*$-radical is projective if and only if $\mathrm{Z}^*(R) = \mathrm{J}(R)$ and every projective right R-module that has $*$-radical is injective (extending).

1 Preliminaries

Throughout this paper we assume that R is an associative ring with unit and all R-modules cosidered are unitary right R-modules. Let M be an R-module. We write $\mathrm{E}(M)$, $\mathrm{Rad}M$, $\mathrm{Soc}(M)$ and $\mathrm{Z}(M)$ for the injective envelope, the Jacobson radical, the socle and the singular submodule of M, respectively. $\mathrm{J}(R)$ is the Jacobson radical of R. A submodule N of M is indicated by writting $N \leq M$. The notation $N \leq_e M$ is reserved for essential submodules.

DEFINITION. A ring R is called a right *V-ring* if every right ideal of R is an intersection of maximal right ideals. R is called a right *GV-ring* if every simple singular right R-module is injective [12].

R is a right V-ring iff every simple right R-module is injective iff $\mathrm{Rad}M = 0$ for every right R-module M. [7]

DEFINITION. A module M is called *extending* if every submodule of M is essential in a summand of M. A module M is called *quasi-continuous* if it is extending and for summands M_1 and M_2 of M such that $M_1 \cap M_2 = 0$, $M_1 \oplus M_2$ is a summand of M. M is called *continuous* if it is extending and for a submodule A of M which is isomorphic to a summand of M, A is a summand of M. Note that quasi-injective modules are continuous (see, for example [15]).

1991 *Mathematics Subject Classification.* Primary 16L60; Secondary 16D50, 16D60, 16D80.

M is called $\sum$-*extending (-injective)* if every direct sum of copies of M is extending (-injective) (see for example [6] or [8]).

DEFINITION. Let N be a submodule of a module M. N is called a *small submodule* if whenever $N + L = M$ for some submodule L of M we have $L = M$ and in this case we write $N << M$. M is called *lifting* if for every submodule N of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 << M$ (see, for example [15]). Oshiro [18] called a ring R a right *H-ring* if every injective right R-module is lifting. He also called a ring R a right *co-H-ring* if every projective right R-module is extending.

A ring R is called *semilocal* if $R/\mathrm{J}(R)$ satisfies the minimum condition on right ideals. A ring R is *semiprimary* if R is semilocal and $\mathrm{J}(R)$ is nilpotent. A ring R is called a right *QF-3 ring* if R has injective projective faithful right ideal. We call R is a right *QF-3^+ ring* if $\mathrm{E}(R_R)$ is projective. Jans [13] showed that among rings with minimal condition on right ideals, the classes of QF-3 and QF-3^+ rings coincide.

A ring R is a semiprimary QF-3 ring when R is a semiprimary left and right QF-3 ring. The class of semiprimary QF-3 rings is a generalization of the class of QF-rings (Quasi-Frobenius rings). The class of H-rings and co-H-rings are generalizations of semiprimary QF-3 rings. Tachikawa [23, Proposition 3.3] proved that a semiprimary QF-3 ring is a right and left QF-3^+-ring.

DEFINITION. An R-module M is said to be *small* if it is a small submodule of some R-module and it is said to be *non-small* if it is not a small module. M is a small module if and only if M is small in its injective hull [14]. We put

$$\mathrm{Z}^*(M) = \{m \in M : mR \text{ is small }\} \quad [11].$$

Since $\mathrm{Rad}(M)$ is the union of all small submodules in M, $\mathrm{Rad}M \leq \mathrm{Z}^*(M)$, and

$$\mathrm{Z}^*(M) = M \cap \mathrm{Rad}\ \mathrm{E}(M) = M \cap \mathrm{Rad}E'$$

for every injective module $E' \supseteq M$. Note that simple modules are either injective or small. If M is a small module then $\mathrm{Z}^*(M) = M$.

In this note we say a module M has $*$-*radical* if $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$. A ring R has $*$-radical if R_R has $*$-radical. Clearly injective modules have $*$-radical. But modules that have $*$-radical are not injective in general (Example 4.1). In the light of this result we define the following properties in this note.

(T1) Every module has $*$-radical.
(T2) Every singular module has $*$-radical.
(T3) Every projective module has $*$-radical.
(T4) Every module that has $*$-radical is projective.
(T5) Every module that has $*$-radical is injective.
(T6) Every projective module that has $*$-radical is injective.
(T7) Every projective module that has $*$-radical is extending.

At once it can be easily seen that (T1) $\Longrightarrow$ (T2) and (T3); (T5) $\Longrightarrow$ (T6) $\Longrightarrow$ (T7).

In the second part of this note we prove that R is a right V-ring $\Longleftrightarrow$ (T1) holds $\Longleftrightarrow$ Every quasi-injective module has $*$-radical $\Longleftrightarrow$ Every quasi-projective module has $*$-radical $\Longleftrightarrow$ (T3) holds and R is a right GV-ring. And (T2) holds $\Longleftrightarrow$ R is a right GV-ring.

In the third part we prove that (T4) holds $\Longleftrightarrow$ R is a QF-ring. Also we give some other results about (T3).

In the last part of this study we prove that R is a right H-ring if and only if every module that has $*$-radical is lifting if and only if R is a right perfect ring and (T5) holds. After that we show that (T7) holds $\Longleftrightarrow$ Every projective module that has $*$-radical is quasi-injective $\Longleftrightarrow$ Every projective module that has $*$-radical is continuous $\Longleftrightarrow$ Every projective module that has $*$-radical is quasi-continuous. If R is a right QF-3^+ ring, (T6) $\Longleftrightarrow$ (T7). And R is a semiprimary QF-3 ring $\Longleftrightarrow$ (T6) holds and R is right perfect $\Longleftrightarrow$ (T7) holds and R is right perfect. Finally we give a characterization of QF-rings by using these properties.

2 Properties (T1) and (T2)

First we give the following useful lemmas.

Lemma 2.1 *Let R be a ring and let $\varphi : M \longrightarrow M'$ be a homomorphism of R-modules M, M'. Then $\varphi(Z^*(M)) \leq Z^*(M')$.*

Proof If $i : M' \longrightarrow \mathrm{E}(M')$ is the inclusion mapping, then the homomorphism $i\varphi : M \longrightarrow \mathrm{E}(M')$ can be lifted to a homomorphism $\theta : \mathrm{E}(M) \longrightarrow \mathrm{E}(M')$. Now $\theta(\mathrm{Rad\ E}(M)) \leq \mathrm{RadE}(M')$ by [1, Proposition 9.14]. Then $\varphi(\mathrm{Z}^*(M)) \leq \mathrm{Z}^*(M')$. $\square$

Lemma 2.2 *Any direct summand of a module that has $*$-radical has $*$-radical.*

Proof Let M be a module that has $*$-radical and N a direct summand of M. Let $x \in \mathrm{Z}^*(N)$. Then $xR << \mathrm{E}(N) \leq \mathrm{E}(M)$. It follows that $x \in \mathrm{Z}^*(M) = \mathrm{Rad}(M)$ and then $xR << M$. Since N is a direct summand of M, $xR << N$. Hence $\mathrm{Z}^*(N) = \mathrm{Rad}(N)$. $\square$

Proposition 2.3 *The following are equivalent for any ring R.*
(i) R is a right V-ring,
(ii) R satisfies (T1),
(iii) Every quasi-injective right R-module has $*$-radical,
(iv) Every quasi-projective right R-module has $*$-radical,
(v) R satisfies (T3) and is a right GV-ring.

Proof We first note that R is a right V-ring $\Longleftrightarrow$ for every right R-module M, $\mathrm{Z}^*(M) = 0$ [19, Theorem 12].
(i)$\Longrightarrow$ (ii) As Rad$M \leq \mathrm{Z}^*(M)$ for any R-module M, it is clear. (ii)$\Longrightarrow$ (iii) Clear. (iii)$\Longrightarrow$ (i)Let M be a simple R-module. Then Rad$M = \mathrm{Z}^*(M) = 0$, i.e. M is injective. (i)$\Longrightarrow$ (iv) Clear. (iv) $\Longrightarrow$ (v) Let M be a simple singular R-module. Since M is quasi-projective, Rad$M$=$\mathrm{Z}^*(M) = 0$. Then M is injective. (v)$\Longrightarrow$ (i) Let M be a simple R-module. If M is singular M is injective. If M is projective, by (T3), Rad$M$=$\mathrm{Z}^*(M) = 0$. Again M is injective. $\square$

Proposition 2.4 *The following are equivalent for any ring R.*
(i) R is a right GV-ring,
(ii) R satisfies (T2).

Proof R is a right GV-ring $\Longleftrightarrow$ $\mathrm{Z}(M) \cap \mathrm{Z}^*(M) = 0$ for any right R-module M [19, Theorem 10].
(i) $\Longrightarrow$ (ii) Let M be a singular R-module. Then $\mathrm{Z}^*(M) = 0$. Hence $\mathrm{Z}^*(M) = \mathrm{Rad} M$.
(ii)$\Longrightarrow$ (i) Let M be a simple singular R-module. By hypothesis, $\mathrm{Z}^*(M) = \mathrm{Rad} M = 0$. Since M is simple, M is injective. □

Example 2.5 There exists a ring R with $*$-radical, but R has a right R-module which does not have $*$-radical. Let R be the endomorphism ring of an infinite dimensional (left) vector space V over a field F. Then R is a von Neumann regular right self-injective ring but not a right V-ring, because V_R is a simple small module (see [25, 23.6]). Then $\mathrm{Z}^*(R_R) = \mathrm{J}(R) = 0$ but $0 = \mathrm{J}(V_R) \neq \mathrm{Z}^*(V_R) = V_R$.

3 Properties (T3) and (T4)

Example 3.1 *Every projective module does not have $*$-radical in general.*

Proof Let $R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$ be lower triangular matrices over a field F. Then $J(R) = \begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}$ and $\mathrm{Soc}(R_R) = \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$. By [19, Example 11], $\mathrm{Soc}(R_R) = \mathrm{Z}^*(R_R) \neq \mathrm{J}(R)$. □

By Proposition 2.3, V-rings satisfy (T3). Also QF-rings satisfy (T3) because over a QF-ring R, every projective right R-module is injective [8, 24.8]. If R satisfies (T3), then R is not necessarily a V-ring nor a QF-ring. Because there are many examples of QF-rings which are not V-rings and V-rings which are not QF-rings.

Note that any projective module that has $*$-radical is non-small. Because projective modules do not equal to their radicals. Hence small rings, for example commutative domains (see [22]), do not satisfy (T3).

In [21], Rayar showed that R is a QF-ring iff every R-module is a direct sum of an injective and a singular module iff every R-module is a direct sum of a projective and a small module. Now,

Proposition 3.2 *Let R be a right Noetherian or a semilocal ring. If R satisfies (T3) then every semisimple right R-module is a direct sum of an injective module and a singular module.*

Proof Let M be a semisimple module. As any simple module is projective or singular then M has a decomposition $M = N \oplus K$ where N is the direct sum of projective simples and K is the direct sum of singular simples. Then K is singular. Also by (T3), $\mathrm{Z}^*(N) = \mathrm{Rad} N = 0$. Hence N is the direct sum of injectives. If R is right Noetherian, by [8, 20.1 Theorem], N is injective. If R is semilocal then N is also injective by [20, Theorem 4]. □

For the converse of the Proposition 3.2 we give the following example.

Example 3.3 [2, Example 12.18] Let S be Z localised at 2Z and set

$$R = \left\{ \begin{bmatrix} a & 2b \\ c & d \end{bmatrix} : a, b, c, d \in S, a - d \in 2S \right\}$$

with the usual matrix operations, then R is a prime left and right Noetherian local ring which is not an integral domain. J=J(R)= 2Se_{11}+2Se_{12}+Se_{21}+ 2Se_{22} then R/J≅ Z/2Z.

Let M be a semisimple R-module and N a simple submodule of M. As R is local, $N \cong R$/J; and as Z is uniform, N is singular. This implies that M is singular.

On the other hand since R is a prime right Goldie ring which is not primitive, $Z^*(M) = M$ for every right R-module M [19]. So R does not satisfy (T3) because $Z^*(R_R) = R$. □

Harada proved that over a right perfect ring R, R is a right QF-3^+ ring if and only if any non-small indecomposable projective R-module is injective [11, Theorem 1.3]. He also proved that if R is a right Artinian right QF-3^+ ring with $Z^*(R)$ =J(R) then it is a QF-ring. Now we give the following result over a right perfect ring.

Theorem 3.4 *Let R be a right perfect right QF-3^+ ring and assume that R satisfies (T3). Then R is a QF-ring.*

Proof Let $R = e_1R \oplus \ldots \oplus e_nR$ where $\{e_1, ..., e_n\}$ is an orthogonal set of idempotents with each e_iR is local indecomposable projective (see [1] and [15]). By (T3), $Z^*(e_iR)$ =J(e_iR) for all i. Then each e_iR is non-small. Hence each e_iR is injective by [11, Theorem 1.3]. This implies that R is right self-injective.

Now we claim that R is a semiprimary ring. Since R is extending and has no infinite set of orthogonal idempotents, R has acc on right annihilator ideals. Z(R) and hence J(R) is nilpotent by [10, Theorem 3.31]. This implies that R is a semiprimary ring.

Since R is semiprimary and a right QF-3^+ ring R is a semiprimary QF-3 ring. Then E(R) $= R$ is $\sum$-injective by [5], i.e. R is a QF-ring. □

Note that a ring R is a QF-ring if and only if every injective right R-module is projective by [8, 24.8].

Theorem 3.5 *The following are equivalent for any ring R.*
(i) R is a QF-ring,
(ii) R satisfies (T4).

Proof (ii)⟹ (i) Let M be an injective R-module. Then $Z^*(M)$ =Rad$M$. Hence M is projective. This implies that R is a QF-ring.
(i)⟹ (ii) Let M be an R-module with $Z^*(M)$=Rad$M$. By [21], M has a decomposition $M = P \oplus S$ where P is projective and S is small. Then $Z^*(S)$=Rad$S = S$. Since R is right perfect, $S = 0$. Hence M is projective. □

Corollary 3.6 *(T4)⟹ (T3).*

4 Properties (T5), (T6) and (T7)

In this section we characterize QF-rings, H-rings and semiprimary QF-3 rings.
Example 4.1 *Every module that has ∗-radical need not be injective.*

Proof Let R be the ring of polynomials in countably many indeterminates $\{x_i\}$ over $Z_2 = Z/2Z$ where we impose the following relations:
(i) $x_k^3 = 0$ for all k,
(ii) $x_k x_j = 0$ for all $k \neq j$ and,
(iii) $x_k^2 = x_j^2$ for all k, j.
R is commutative, semiprimary, local, continuous but not self-injective by [17]. $J(R) = (x_1, x_2, ...)$ is the unique maximal ideal in R. Since $J(R) \leq Z^*(R)$, $Z^*(R) = J(R)$ or $Z^*(R) = R$. If $Z^*(R) = R$ then for any injective module M, $Z^*(M) = \text{Rad}(M) = M$. This contradicts that R is a perfect ring. Hence $Z^*(R) = J(R)$ but R is not self-injective. □

Theorem 4.2 [18, Theorem 2.11] *The following statements are equivalent for any ring R.*
(i) R is a right H-ring,
(ii) R is right Artinian and every non-small R-module contains a non-zero injective submodule,
(iii) R is right perfect and for any exact sequence $\phi : P \longrightarrow E \longrightarrow 0$ where E injective and kerϕ is small in P, P is injective,
(iv) Every R-module is a direct sum of an injective module and a small module.
When this is so, then R is a semiprimary QF-3 ring.

Lemma 4.3 *Let R be a ring which satisfies (T5). Then for any exact sequence $\phi : P \longrightarrow E \longrightarrow 0$ where E is injective and ker$\phi << P$, P is injective.*

Proof Let $\phi : P \longrightarrow E \longrightarrow 0$ be an exact sequence where E is injective and $\ker\phi << P$. Then $\phi(\text{Rad}P) = \text{Rad}E \leq \phi(Z^*(P)) \leq Z^*(E) = \text{Rad}E$ by [1, Proposition 9.15] and Lemma 2.1, and so $\phi(\text{Rad}P) = \phi(Z^*(P))$. Since $\ker\phi \leq \text{Rad}P$, $\text{Rad}P = Z^*(P)$. By hypothesis, P is injective. □

Theorem 4.4 *The following statements are equivalent for any ring R.*
(i) R is a right H-ring,
(ii) R is right perfect and satisfies (T5),
(iii) Every right R-module that has $*$-radical is lifting.

Proof (i)$\Longrightarrow$(ii) R is right perfect by Theorem 4.2. Let M be a module that has $*$-radical. $M = N \oplus K$ where N is injective and K is small by Theorem 4.2. Then $K = Z^*(K) \leq Z^*(M) = \text{Rad}M$. Since R is right perfect, $\text{Rad}M << M$. It follows that $K << M$. So $M = N$ is injective.
(ii)$\Longrightarrow$(i) By Lemma 4.3 and Theorem 4.2.
(ii)$\Longrightarrow$(iii) Let M be a right R-module that has $*$-radical. By (ii), M is injective. Then M is lifting by Theorem 4.2.
(iii)$\Longrightarrow$(i) It is clear. □

Lemma 4.5 *R satisfies (T7) if and only if for every R-module M that has $*$-radical and has a projective cover P, P is $\sum$-extending.*

Proof ($\Longleftarrow$) It is clear.
($\Longrightarrow$) Let M be a module that has $*$-radical and $f : P \longrightarrow M$ an epimorphism with

$\ker f << P$. Then by the proof of Lemma 4.3, $Z^*(P) = \mathrm{Rad}P$. Hence $Z^*(P^{(\Lambda)}) = \mathrm{Rad}(P^{(\Lambda)})$ for any index set Λ. Since any direct sum of projective modules is projective, $P^{(\Lambda)}$ is projective. By (T7), P is $\sum$-extending. □

Proposition 4.6 *The following are equivalent for any ring R.*
(i) R satisfies (T7),
(ii) Every projective R-module that has $*$-radical is quasi-continuous,
(iii) Every projective R-module that has $*$-radical is continuous,
(iv) Every projective R-module that has $*$-radical is quasi-injective.

Proof (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) Clear.
(i)$\Longrightarrow$ (iv) Let M be a projective R-module that has $*$-radical. Then M is $\sum$-extending by Lemma 4.5. By [4, 3.6], M has a decomposition $M = \oplus M_i (i \in I)$ where each M_i is finitely generated, quasi-injective and indecomposable. In addition, M_i's have local endomorphism ring by [25, 19.9] and then M_i's are local by [25, 19.7]. Since M_i's are non-small and local, every monomorphism $M_i \longrightarrow M_j (i \neq j)$ is an isomorphism. Hence by [6, Corollary 8.9], M is quasi-injective. □

Now we deal with the relationship between (T6) and (T7).

Proposition 4.7 *Assume that R is a right QF-3^+ ring and satisfies (T7). Then R satisfies (T6).*

Proof Let M be a projective R-module that has $*$-radical. Then $M \oplus \mathrm{E}(R_R)$ is projective by hypothesis and [15, Corollary 4.36]. Since $\mathrm{E}(R_R)$ is injective, $Z^*(M \oplus \mathrm{E}(R_R)) = \mathrm{Rad}(M \oplus \mathrm{E}(R_R))$. By Proposition 4.6 , $M \oplus \mathrm{E}(R_R)$ is quasi-injective. Hence M is injective. □

Example 4.8 *If R is (right and left) perfect right QF-3^+ then R need not satisfy (T7).*

Proof Let R be any (right and left) perfect ring such that $\mathrm{E}(R_R)$ is projective but $\mathrm{E}({}_RR)$ is not (for the existence of such a ring see [16]). Let M be a direct sum of countably many copies of $\mathrm{E}(R_R)$. Then M is not quasi-injective by [26, Lemma 3.1]. But M is projective and has $*$-radical. Hence R_R does not satisfy (T7) by Proposition 4.6. □

We do not know whether (T7) is equivalent to (T6) for any ring R. Now we give some results over a perfect ring.

Colby and Rutter [5, Theorem 1.3] proved that a ring R is semiprimary QF-3 if and only if R is right perfect and the projective cover of every injective R-module is injective if and only if R is right perfect and injective envelope of every projective R-module is projective. After that Vanaja [24, Theorem 1.5] showed that R is semiprimary QF-3 if and only if R is right perfect and any projective R-module whose indecomposable direct summands are non-small is extending.

Now, let R be a semiperfect ring and M a projective R-module that has $*$-radical. Then M has a decomposition $M \cong \oplus M_\alpha$ $(\alpha \in \Lambda)$ where each M_α is indecomposable local (see [1, 27.11], [1, 27.6] and [25, 19.7]). By Lemma 2.2, $Z^*(M_\alpha) = \mathrm{Rad}(M_\alpha)$ and then M_α is non-small for all α.

Theorem 4.9 *The following are equivalent for any ring R.*
(i) R is a semiprimary QF-3 ring,
(ii) R satisfies (T6) and is right perfect,
(iii) R satisfies (T7) and is right perfect.

Proof (ii) $\Longrightarrow$ (iii) It is clear.
(i)$\Longrightarrow$ (ii) Let M be a projective module that has $*$-radical. By above remark, $M \cong \oplus M_\alpha$ $(\alpha \in \Lambda)$ where each M_α is indecomposable and non-small. Since R is a right QF-3$^+$ ring, all M_α is injective. $M \cong \oplus M_\alpha$ is a direct summand of $\mathrm{E}(R_R)^{(\Lambda)}$. Then as $\mathrm{E}(R_R)$ is $\sum -$injective M is injective.
(iii)$\Longrightarrow$ (i) Let M be a projective module which every indecomposable summands are non-small. Then $M \cong \oplus M_\alpha$ $(\alpha \in \Lambda)$ where each M_α is indecomposable non-small and local. Then $\mathrm{Z}^*(M_\alpha) = \mathrm{Rad}(M_\alpha)$ $(\alpha \in \Lambda)$. This implies that $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$. By (T7), M is extending. Thus by [24, Theorem 1.5], we get the result. □

Example 4.10 *If R satisfies (T6), R need not satisfy (T5).*

Proof Let $R = \begin{bmatrix} \mathrm{R} & 0 & 0 \\ \mathrm{R} & \mathrm{Q} & 0 \\ \mathrm{R} & \mathrm{R} & \mathrm{R} \end{bmatrix}$ where R is the real numbers and Q is the rational numbers. R is a semiprimary QF-3 ring but not right Noetherian [5, 1.4 Remarks]. By Theorem 4.9, R satisfies (T6) and by Theorem 4.2 and Theorem 4.4, R does not satisfy (T5). □

Proposition 4.11 *Assume that R is semiperfect. If R satisfies (T6) then any non-small indecomposable projective R-module is injective. The converse holds when, in addition, R is right Noetherian.*

Proof Let M be a non-small indecomposable projective R-module. Since R is semiperfect, M is local. This implies that $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$. By (T6), M is injective.

For the converse, let M be a projective R-module that has $*$-radical. Again $M \cong \oplus M_\alpha$ $(\alpha \in \Lambda)$ where each M_α is non-small indecomposable projective. By assumption, M_α's are injective. As R is right Noetherian, M is injective. □

Another relationship between (T6) and "any non-small indecomposable projective module is injective" is given over a right GV-ring. In [19, Theorem 10] it is also proved that R is a right GV-ring if and only if every small module is projective.

Proposition 4.12 *If R is a right GV-ring and satisfies (T6) then any non-small indecomposable projective module is injective.*

Proof Let M be a non-small indecomposable projective module. We claim that $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$. If not, let $x \in \mathrm{Z}^*(M) - \mathrm{Rad}(M)$. Then there exists a maximal submodule B of xR such that $xR/B \leq_d M/B$. Then $M/B = xR/B \oplus L/B$ for some L. Since xR is small, then xR/B is small. By [19, Theorem 10], xR/B is projective. This implies that M/L is simple projective. Hence $L \leq_d M$. If $L = 0$, $M/B = xR/B$ and then $B \leq_d M$. If $B = 0$, $M = xR$ which is contradicted by M is non-small. If $B = M$, $xR = B$, a contradiction. If $L = M$, again $xR = B$, a contradiction. Hence $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$. By (T6), M is injective. □

Theorem 4.13 [18, Theorem 3.18], [6, 11.13] *The following are equivalent for any ring R.*
(i) R is a right co-H-ring,
(ii) Every R-module is expressed as a direct sum of a projective module and a singular module,
(iii) The family of all projective R-modules is closed under taking essential extensions,
(iv) R is right $\sum$-extending,
When this is so, then R is a semiprimary QF-3 ring.

Theorem 4.14 [18, Theorem 4.3] *The following are equivalent for any ring R.*
(i) R is a QF-ring,
(ii) R is a right H-ring with Z(R) =J(R),
(iii) R is a right co-H-ring with Z(R) =J(R).

Lemma 4.15 *Let R be a semiperfect ring. If $Z^*(R_R)$ =$Z(R_R)$ then $Z^*(R_R)$ =J(R). The converse holds when R is right or left perfect right quasi-continuous.*

Proof Let R be a semiperfect ring and assume $\mathrm{Z}^*(R_R) = \mathrm{Z}(R_R)$. Then there exists an idempotent e of R such that $eR \leq \mathrm{Z}(R_R)$ and $(1-e)R \cap \mathrm{Z}(R_R)$ is small in R by [15, Corollary 4.42]. Since $\mathrm{Z}(R_R)$ does not contain any non-zero idempotents, it follows that $\mathrm{Z}(R_R) \leq \mathrm{J}(R)$. Hence $\mathrm{Z}^*(R_R) = \mathrm{J}(R)$.

For converse, assume that $\mathrm{Z}^*(R_R) = \mathrm{J}(R)$. Since R is right or left perfect right quasi-continuous $\mathrm{Z}(R_R) = \mathrm{J}(R)$ by [3, Lemma 6]. Hence $\mathrm{Z}^*(R_R) = \mathrm{Z}(R_R)$. □

Theorem 4.16 *The following are equivalent for any ring R.*

(1) R is a QF-ring,
(2) $Z^*(R_R)$ =J(R) and
(a) R satisfies (T5) or
(b) R satisfies (T6) or
(c) R satisfies (T7) or
(d) R is a right co-H-ring or
(e) R is a right H-ring,
(3) $Z^*(R_R)$ =$Z(R_R)$ and
(a) R is semiperfect and
(i) R satisfies (T5) or
(ii) R satisfies (T6) or
(iii) R satisfies (T7) or
(d) R is a right co-H-ring or
(e) R is a right H-ring.

Proof (1⟹2a) Since R is right self-injective, $\mathrm{Z}^*(R_R) = \mathrm{J}(R)$. By Theorem 4.4, R satisfies (T5).
(2a⟹2b⟹2c) Clear.
(2c⟹2d) By Lemma 4.5, R is $\sum$-extending. Hence R is a right co-H-ring.
(2d⟹1) Let $F = R^{(\mathbb{N})}$ be the free right R-module which is the direct sum of a countably infinite number of copies of R. By Theorem 4.13, E(F) is projective. Since R is right perfect, E(F) is lifting. Then $\mathrm{E}(F) = X \oplus Y$ where $X \leq F$ and $F \cap Y << \mathrm{E}(F)$. Hence $F = X \oplus (F \cap Y)$. As $\mathrm{Z}^*(F)=\mathrm{Rad}F$ and $F \cap Y \leq_d F$,

$Z^*(F\cap Y)$=Rad$(F\cap Y) = F\cap Y$. Since $F\cap Y$ is projective, this is a contradiction. Hence $F = X$ is injective. By [8, Proposition 20.3A], R_R is $\sum$-injective. By [6, 18.1], R is a QF-ring.
(2e$\Longleftrightarrow$1) By [11, p.673 Corollary].
(1$\Longrightarrow$3a(i)) As R is self-injective, Z(R_R)=J(R)=Z*(R_R).
(3a(i)$\Longrightarrow$3a(ii)$\Longrightarrow$3a(iii)) Clear.
(3a(iii)$\Longrightarrow$3d) As Z*(R_R) =Z(R_R) and R is semiperfect, Z*(R_R) =J(R) by Lemma 4.15. Hence R is $\sum$-extending by Lemma 4.5.
(3d$\Longrightarrow$1)As by Lemma 4.15, Z*(R_R) =J(R) the proof is completed by the proof of (2d$\Longrightarrow$1).
(3e$\Longleftrightarrow$1) By Lemma 4.15 and [11, p.673 Corollary]. □

References

[1] F.W. Anderson, K.R. Fuller, Ring and Categories of Modules, (1974), Springer-Verlag, Berlin-Heidelberg-NewYork.

[2] A.W. Chatters, C.R. Hajarnavis, Rings with chain conditions, (1980), Pitman, London.

[3] J.Clark, D.V. Huynh, When self-injective semiperfect ring Quasi-frobenius?, *Journal of Alg.*,165 (1994), 531-542.

[4] J.Clark, R.Wisbauer, Polyform and projective $\sum$-extending modules, Algebra Colloquim, 5:4 (1998), 391-408.

[5] R.R. Colby, E.A.Rutter, Generalizations of QF-3 algebras, *Trans. Amer. Math. Soc.*, 153 (1971), 371-386.

[6] N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, Extending modules, (1994), Pitman RN Mathematics 313, Longman, Harlow.

[7] C. Faith, Lectures on injective modules and quotient rings (Lecture Notes in Math. 49), (1967), Springer-Verlag, Berlin Heidelberg-NewYork.

[8] C. Faith, Algebra II, Ring Theory, (1976), Springer Grundl. 191.

[9] K.R. Fuller, Relative projectivity and injectivity classes determined by simple modules, *J. London Math.Soc*, 5 (1972), 423-431.

[10] K.R. Goodearl, Ring Theory, (1976), Pure and Applied Math., No:33, Marcel-Dekker.

[11] M. Harada, Non-small modules and non-cosmall modules, In Ring Theory: Proceedings of the 1978 Antwerp Conference, F.Van Oystaeyen, ed.NewYork: Marcel Dekker.

[12] Y. Hirano, Regular modules and V-modules, *Hiroshima Math.J.*, 11 (1981), 125-142.

[13] J.P.Jans, Projective-injective modules, *Pacific J.Math.* 9 (1959), 1103-1108.

[14] W.W. Leonard, Small Modules, *Proc.Amer.Math.Soc.* 17 (1966), 527-531.

[15] S.H. Mohamed and B.J. Müller, Continuous and discrete modules, (1990), London Math.Soc. LN.147, Cambridge University Press, NewYork Sydney.

[16] B. Müller, Dominant dimension of semiprimary rings, *J.Reine Angew Math.* 232 (1968), 173-179.

[17] W.K. Nicholson and M.F. Yousif, Continuous rings with chain conditions, *Journal of Pure and Applied Algebra*, 97 (1994), 325-332.

[18] K. Oshiro, Lifting modules, extending modules and their applications to QF-rings, *Hokkaido Math. J.*, 13 (1984), 310-338.

[19] A.Ç.Özcan, Some characterizations of V-modules and rings, *Vietnam J.Math.*, 26(3) (1998), 253-258.

[20] A.Ç.Özcan and A. Harmancı, Characterization of some rings by functor $Z^*(.)$, *Turkish J.Math.*, 21(3) (1997), 325-331.

[21] M. Rayar, On small and cosmall modules, *Acta Math.Acad.Sci.Hungar*,39(4) (1982), 389-392.

[22] M. Rayar, A note on small rings, *Acta Math.Hung.*, 49(3-4) (1987), 381-383.

[23] H.Tachikawa, On left QF-3 rings, *Pacific J.Math.*, 32 (1970), 255-268.

[24] N. Vanaja, Characterization of rings using extending and lifting modules, *Ring Theory*, (1993), 329-342 World Sci.Publishing, River Edge.

[25] R. Wisbauer, Foundations of Module and Ring Theory, (1991), Gordon and Breach, Reading.

[26] K.Yamagata, The exchange property and direct sums of indecomposable injective modules, *Pacific J.Math*, 55 (1974), 301-317.

DEPARTMENT OF MATHEMATICS, HACETTEPE UNIVERSITY, 06532 BEYTEPE, ANKARA TURKEY.
E-mail address: ozcan@hacettepe.edu.tr