MODULES WITH SMALL CYCLIC SUBMODULES IN THEIR INJECTIVE HULLS

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Abstract

In this paper we study when a unital right module M over a ring R with identity has a special "small image" property we call (S*): namely, M has (S*) if every submodule N of M contains a direct summand K of M such that every cyclic submodule C of N/K is small (meaning "small in its injective hull E(C)"). If xR is small for every element x of a module M, M is said to be *cosingular*. In Theorem 4.4 we prove every right R–module satisfies (S*) if and only if every right R–module is the direct sum of an injective module and a cosingular module. Over a right self–injective ring R, every right R–module satisfies (S*) if and only if R is quasi–Frobenius (Theorem 5.5). It follows that over a commutative ring R, every module satisfies (S*) if and only if R is a direct product of a quasi-Frobenius ring and a cosingular ring.

Key words: small module, self–injective ring, Harada ring, quasi–Frobenius ring.

1 INTRODUCTION AND NOTATION

All rings have identity and all modules are unital right modules.

Let R be a ring and M a right R–module. We write E(M), Rad$M$ and Z(M) for the injective envelope, the radical and the singular submodule of M, respectively. We denote the radical of R by J(R). We use $N \leq M$ to signify that N is a submodule of M. If N is essential in M we write $N \leq_e M$.

A submodule N of M is called a *small submodule* if, whenever $N + L = M$ for some submodule L of M, we have $M = L$; and in this case we write $N << M$. In [1], Leonard defines a module M to be *small* if it is a small submodule of some

R–module and he shows that M is small if and only if M is small in its injective hull. We put

$Z^*(M) = \{m \in M : mR$ is a small module $\}$.

Since Rad(M) is the union of all small submodules of M, we see that

$Z^*(E)$=Rad(E) for any injective module E, and

$Z^*(M) = M \cap$ Rad E(M) $= M \cap$ RadE' for every injective module E' containing M.

Note that if M is a vector space over the rational numbers Q, then M is a semisimple injective Q–module; hence $Z^*(M_{\mathrm{Q}})$ =Rad(M_{Q}) $= 0$. However M is also a module over the integers Z, and as such is torsion–free injective, so that $Z^*(M_{\mathrm{Z}}) = M$. Thus $Z^*(M)$ depends on which ring R one is considering. In practice it is usually clear which ring is being considered.

In this note, we call a module M *cosingular* if $Z^*(M) = M$. A ring R is called *right cosingular* if the (right) R–module R is cosingular.

In Section 2, we give some properties of cosingular modules and some examples of cosingular rings.

Let $\mathcal{K}$ be a class of modules. Then $d^*\mathcal{K}$ is defined in [2] to be the class of modules M such that for every submodule N of M, there exists a direct summand K of M such that (1) K is contained in N and (2) the factor module N/K belongs to $\mathcal{K}$. Some properties of $d^*\mathcal{K}$ have been studied for various special classes $\mathcal{K}$ of modules. In [3], the class of modules M such that, for every submodule N of M, there exists a direct summand K of M contained in N with $N/K \leq$Rad(M/K) is investigated.

For $\mathcal{K}$ the class of cosingular modules, we associate the class $d^*\mathcal{K}$ with property (S^*). That is, a module M satisfies (S^*) if, for every submodule N of M, there exists a direct summand K of M such that K is contained in N and the factor module N/K is cosingular.

In Section 3, we study some properties of modules that satisfy (S^*). We prove that if the ring R satisfies (S^*), then $M/Z^*(M)$ is semisimple for every R–module M (Proposition 3.9).

In Section 4, we deal with properties of a ring R that hold when every R–module satisfies (S^*). It is proved that every R–module satisfies (S^*) if and only if every R–module is a direct sum of an injective module and a cosingular module. In Theorem 4.9 we characterize H–rings (defined at the beginning of section 4) using (S^*).

Finally in Section 5 we characterize QF–rings (Theorem 5.5).

2 COSINGULAR MODULES

Before we define a cosingular module, let us state some useful lemmas.

Lemma 2.1 *Let R be a ring and let $\varphi : M \to M'$ be a homomorphism of R-modules M, M'. Then $\varphi(Z^*(M)) \leq Z^*(M')$.*

Proof If $i : M' \to$E(M') is the inclusion mapping then the homomorphism $i\varphi : M \to$E(M') can be extended to a homomorphism θ :E$(M) \to$E(M'). Now θ(Rad E$(M)) \leq$RadE(M') by [4, Proposition 9.14]. Hence φ(Z$^*(M)) \leq$Z$^*(M')$. □

Lemma 2.2 *Let N be a submodule of an R–module M. Then Z$^*(N) = N \cap$Z$^*(M)$.*

Proof It is clear. □

Lemma 2.3 *Let $M_i (i \in I)$ be any collection of R–modules and let $M = \oplus_{i\in I} M_i$. Then Z$^*(M) = \oplus_{i\in I}$Z$^*(M_i)$.*

Proof By Lemma 2.2, Z$^*(M_i) \leq$ Z $^*(M)$ for all $i \in I$ and hence $\oplus_{i\in I}$ Z$^*(M_i) \leq$ Z$^*(M)$.

Let $\pi_i : M \to M_i$ denote the canonical projection for each $i \in I$. Let $m \in$Z$^*(M)$. Then $m = m_1 + \cdots + m_n$ for some positive integer n and elements $m_j \in M_{i(j)}$ (1$\leq j \leq n$), for distinct $i(1), \ldots, i(n)$ in I. For each $1 \leq j \leq n$,

$$m_j = \pi_{i(j)}(m) \in \pi_{i(j)}(\mathrm{Z}^*(M)) \leq \mathrm{Z}^*(M_{i(j)}),$$

by Lemma 2.1. Thus $m \in \oplus_{i\in I}$Z$^*(M_i)$ and hence Z$^*(M) = \oplus_{i\in I}$Z$^*(M_i)$. □

For any non–empty subset X of an R–module M we set
$\underline{r}_R(X) = \{r \in R : xr = 0$ for all $x \in X\}$.

Lemma 2.4 *Let R be a right Artinian ring with Jacobson radical J and let M be an R–module. Then Z$^*(M) = \{m \in M : m\underline{r}_R(J) = 0\}$.*

Proof See [5, Theorem 3]. □

Definitions 2.5 Let R be a ring and M an R–module. M is called *cosingular* if Z$^*(M) = M$. R is called *right cosingular* if the (right) R–module R is cosingular.

Small modules are cosingular. If R is a right perfect ring, Rad$M$ is the unique largest small submodule of M and so M is small if and only if M is cosingular [6, Chapter 1].

Lemma 2.6 *For any ring R, the class of cosingular R–modules is closed under submodules, homomorphic images and direct sums but not (in general) under essential extensions or extensions.*

Proof The class of cosingular R-modules is closed under submodules by Lemma 2.2, under homomorphic images by Lemma 2.1 and under direct sums by Lemma 2.3.

Let F be a field and let $R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in F \right\}$. Then R is a commutative Artinian ring with Jacobson radical $J = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$. Note that $\underline{r}_R(J) = J$ and that J is an essential ideal of R. By Lemma 2.4, the R-module J is cosingular but its essential extension R_R is not. Moreover, the R-module J and R/J are both cosingular by Lemma 2.4 but the R–module R is not. □

Corollary 2.7 *Let R be a right cosingular ring. Then any (right) R–module is cosingular.*

Proof Let M be an R–module. Let $m \in M$. By Lemma 2.6, $mR = \mathrm{Z}^*(mR) \leq \mathrm{Z}^*(M)$. Thus $\mathrm{Z}^*(M) = M$ and M is cosingular. □

Next we consider some examples of right cosingular rings.

Lemma 2.8 *A ring R is right cosingular if and only if $E = RadE$ for every injective right R–module E.*

Proof It is clear from Corollary 2.7. □

Lemma 2.9 [6] *There does not exist a right perfect right cosingular ring.*

Proof It is clear from Lemma 2.8. □

Theorem 2.10 [7] *Let R be a prime right Goldie ring which is not right primitive (e.g. a commutative domain which is not a field). Then R is a right cosingular ring.*

Proof Let $r \in R$ and $E = \mathrm{E}(rR)$. Suppose that $E = rR + L$ for some $L \leq E$. If r is not in L, then E/L is non–zero and a cyclic module so that there exists a maximal submodule P of E with L contained in P. The module $U = E/P$ is simple, and if I is its annihilator in R we know that I is a non–zero ideal of R by our hypothesis. But in this case I contains a non–zero divisor by Goldie's Theorem [8, Proposition 5.9] and then $E = EI$ by [9, Proposition 2.6] so that $E = P$, a contradiction. Hence $r \in L$ and so $E = L$ and rR is small. Thus R is right cosingular. □

3 MODULES WITH (S*)

We begin with some definitions.

A module M is called a *(D1)–module* if for every submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 << M$ [10, Chapter 4].

Let A and L be submodules of a module M. L is called a *supplement* of A in M if it is minimal with the property $A + L = M$. A submodule K of M is called a *supplement* (in M) if K is a supplement of some submodule of M. It is easy to check that L is a supplement of A in M if and only if $M = A + L$ and $A \cap L$ is small in L.

We say that M has **(S*)** if for every submodule N of M there exists a direct summand K of M such that $K \leq N$ and N/K is cosingular. A ring R satisfies (S*) if the (right) R–module R satisfies (S*).

(D1)–modules satisfies (S*). But the converse does not hold in general. For example, let $R = \mathrm{Z}$. Since $\mathrm{Z}^*(R) = R$, R satisfies (S*). But, since no proper submodule in R has a supplement in R, R is not a (D1)–module [10, p.56].

The following two lemmas follow immediately from the definitions.

Lemma 3.1 *Let M be an R–module. The following statements are equivalent.*
(i) M satisfies (S),*
(ii) For every submodule N of M, M has a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is cosingular,
(iii) For every submodule N of M, N has a decomposition $N = A \oplus B$ such that A is a direct summand of M and B is cosingular.

Lemma 3.2 *Let M be an R–module that satisfies (S*). Then any submodule of M satisfies (S*).*

Lemma 3.3 *Let M be a module that satisfies (S*) and such that $Z^*(M)$ is small in M. Then M is a (D1)–module.*

Proof Let N be a submodule of M. Then there exists a direct summand K of M such that $K \leq N$ and N/K is cosingular. Let L be a submodule of M such that $M = K \oplus L$. Then $N = K \oplus (N \cap L)$. Since $N/K = \mathrm{Z}^*(N/K)$, $N \cap L$ is cosingular. Then by hypothesis $N \cap L << M$. Hence M is a (D1)–module. □

Remark 3.4 In general, when a module M is a (D1)–module, $\mathrm{Z}^*(M)$ is not small in M.

Let $R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$ be lower triangular matrices over a field F. R is right Artinian, $J(R) = \begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}$, $\mathrm{Soc}(R_R) = \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$ and $\mathrm{Z}^*(R_R) = \mathrm{Soc}(R_R)$ by [7, Example 11]. Hence $\mathrm{Z}^*(R_R)$ is not small in R because $\mathrm{Z}^*(R_R) \neq \mathrm{J}(R)$.

Lemma 3.5 *Let M be an R–module that satisfies (S^*). Suppose that there exists a supplement of $Z^*(M)$ in M. Then there is a decomposition $M = A \oplus B$ such that A is a (D1)–module and B is cosingular.*

Proof By hypothesis, there exists a submodule A of M such that $M = A+$ $Z^*(M)$, $A \cap Z^*(M) << A$. Then $Z^*(A) = \mathrm{Rad}A << A$. Since M satisfies (S^*), there exists a direct summand K of M such that $K \leq A$, $A/K = Z^*(A/K)$. Let B be a submodule of M such that $M = K \oplus B$. Then $A = K \oplus (A \cap B)$. Since A/K is cosingular, $A \cap B = Z^*(A \cap B) \leq Z^*(A)$. Since $Z^*(A) << A$ and $A \cap B$ is a direct summand of A then $A \cap B = 0$. Hence $M = A \oplus B$. By Lemma 3.2 and Lemma 3.3, A is a (D1)–module. In addition, we have $M = A+$ $Z^*(M) = A + Z^*(A) + Z^*(B) = A \oplus Z^*(B)$ and hence $Z^*(B) = B$. This completes the proof. □

Corollary 3.6 *Let M be a module that satisfies (S^*). Then there is a decomposition $M = A \oplus B$ such that A is semisimple with $Z^*(A) = 0$ and $Z^*(B) \leq_e B$.*

Proof Let A be a submodule of M maximal with respect to the property $A \cap Z^*(M) = 0$. Since M satisfies (S^*), it follows that there exists a direct summand K of M such that $K \leq A$, A/K is cosingular. Let B be a submodule of M such that $M = K \oplus B$. Then $A = K \oplus (A \cap B)$. Since $A \cap Z^*(M) = 0$, $Z^*(A \cap B) = A \cap B = 0$. Then $M = A \oplus B$. By Lemma 3.2, A is semisimple. Now $Z^*(M) = Z^*(B)$ and $A \oplus Z^*(M)$ is an essential submodule of M. It follows that $Z^*(B) \leq_e B$. □

For the converse of the Corollary 3.6 we have the following example.

Example 3.7 Let F be a field, L an F–vector space of finite dimension and $L^* = \mathrm{Hom}_F(L, F)$. We put

$$R = \begin{bmatrix} F & L^* & F \\ 0 & F & L \\ 0 & 0 & F \end{bmatrix}$$

Then R is right perfect and a QF–3 ring. If $[L : F] \geq 2$, (**) Every indecomposable injective module is hollow, namely every proper submodule is small, does not hold [6, Example 1].

Then there exists an indecomposable injective module M that is not hollow. Then $Z^*(M) = \mathrm{Rad}M << M$. If $Z^*(M) = 0$, M is semisimple since R is right perfect, a contradiction. If M satisfies (S^*), then M is a (D1)–module since $Z^*(M) << M$ by Lemma 3.3. By [10, Corollary 4.9] M is hollow, a contradiction.

Hence there exists a uniform module M with $Z^*(M) \neq 0$ that does not satisfy (S^*).

Now we generalize Corollary 2.7.

Lemma 3.8 *Let R be a ring. Then $MZ^*(R) \leq Z^*(M)$ for any R–module M.*

Proof Let $m \in M$. Define a mapping $\varphi : R \to$E(M) by $\varphi(r) = mr$ for all $r \in R$. Then φ is a homomorphism and φ can be extended to a homomorphism $\theta :$E$(R) \to$E(M). By [4, Proposition 9.14], $\theta($Rad E$(R)) \leq$RadE(M).

Let $a \in$Z$^*(R) = R\cap$RadE(R). Then $ma = \varphi(a) = \theta(a) \in \theta($RadE$(R)) \leq$ RadE(M) and hence $ma \in M\cap$RadE$(M) =$Z$^*(M)$. It follows that mZ$^*(R) \leq$ Z$^*(M)$ and hence MZ$^*(R) \leq$Z$^*(M)$. □

Proposition 3.9 *Let R be a ring that satisfies (S^*). Then $M/Z^*(M)$ is semisimple for every R–module M.*

Proof If $R =$Z$^*(R)$ then $M =$Z$^*(M)$ for every R–module M by Corollary 2.7 or Lemma 3.8. Suppose that Z$^*(R) \neq R$. Let P be a maximal right ideal of R such that Z$^*(R) \leq P$. There exists an idempotent e and a cosingular right ideal C such that $P = eR \oplus C$. Note that $C =$Z$^*(C) \leq$Z$^*(R)$ and hence $P/$Z$^*(R) = (eR+$Z$^*(R))/$Z$^*(R) = (e+$Z$^*(R))(R/$Z$^*(R))$. Thus $P/$Z$^*(R)$ is a direct summand of $R/$Z$^*(R)$.

It follows that every maximal right ideal of $R/$Z$^*(R)$ is a direct summand. Therefore $R/$Z$^*(R)$ is semisimple. Let M be any R–module. By Lemma 3.8, MZ$^*(R) \leq$Z$^*(M)$ and hence $M/$Z$^*(M)$ is an $R/$Z$^*(R)$–module. It follows that $M/$Z$^*(M)$ is semisimple. □

Let Gen(M) denote the class of M–generated modules for any module M.

Proposition 3.10 *Let M be an R–module. The following statements are equivalent.*
(i) $M/Z^*(M)$ is semisimple,
(ii) For every $L \leq M$ there exists a submodule $K \leq M$ such that $L + K = M$ and $L \cap K$ cosingular,
(iii) There exists a decomposition $M = A\oplus B$ such that A is semisimple, $B/Z^*(B)$ is semisimple and $Z^*(B) \leq_e B$,
(iv) For any $N \in$Gen(M), $N/Z^*(N)$ is semisimple,
(v) For any $N \in$Gen(M), for every $L \leq N$ there exists a submodule $K \leq N$ such that $L + K = N$ and $L \cap K$ cosingular,
(vi) For any $N \in$Gen(M), $N = N_1 \oplus N_2$ such that N_1 is semisimple, $N_2/Z^*(N_2)$ is semisimple and $Z^*(N_2) \leq_e N_2$.

Proof (i)⇒(iii) Let A be a maximal submodule with respect to $A\cap$Z$^*(M) = 0$. Then $A\oplus$Z$^*(M)$ is essential in M. Moreover $A \cong (A\oplus$Z$^*(M))/$Z$^*(M)$ is a direct summand in $M/$Z$^*(M)$, hence semisimple and there is a semisimple submodule $B/$Z$^*(M)$ such that $(A + B)/$Z$^*(M) = M/$Z$^*(M)$. Hence $M = A + B$ and $A \cap B \leq A\cap$Z$^*(M) = 0$. Because $A\oplus$Z$^*(M) \leq_e M$, Z$^*(M) \leq_e B$. Note that Z$^*(M) =$Z$^*(A)\oplus$Z$^*(B) =$Z$^*(B)$.
(iii)⇒(i) Since the homomorphic image of a semisimple module is semisimple and $M/$Z$^*(B) \cong A \oplus (B/$Z$^*(B))$, it is clear.
(i)⇒(ii) Since $(L+$Z$^*(M))/$Z$^*(M)$ is a direct summand in $M/$Z$^*(M)$, it is clear.

(ii)⇒(i) Let $L/\mathrm{Z}^*(M) \leq M/\mathrm{Z}^*(M)$, then there exists a submodule $K \leq M$ such that $L+K=M$ and $L \cap K$ is cosingular. Thus $L/\mathrm{Z}^*(M) \oplus K+\mathrm{Z}^*(M)/\mathrm{Z}^*(M) = M/\mathrm{Z}^*(M)$. Hence $M/\mathrm{Z}^*(M)$ is semisimple.
(iv)⇒(i) It is clear.
(i)⇒(iv) Let $N \in$Gen(M). Then there exist a set Λ and an epimorphism $f : M^{(\Lambda)} \to N$. Since $f(\mathrm{Z}^*(M^{(\Lambda)})) \leq \mathrm{Z}^*(N)$ and $M^{(\Lambda)}/\mathrm{Z}^*(M^{(\Lambda)}) \cong (M/\mathrm{Z}^*(M))^{(\Lambda)}$, we get an epimorphism $\overline{f} : (M/\mathrm{Z}^*(M))^{(\Lambda)} \to N/\mathrm{Z}^*(N)$. Hence $N/\mathrm{Z}^*(N)$ is semisimple.
(iv)⇔(v) ⇔(vi) Same as the proof of (i) ⇔ (ii) ⇔ *(iii)* for $N \in$Gen(M). □

4 H–RINGS

H–rings are investigated by several authors for example [5], [6], [11], [12].

Definitions 4.1 An R–module M is called *non–small* if M is not small. Oshiro [12] called a ring R a *right H–ring* (in honor of Harada [6]) if every injective right R–module is (D1), defined at the beginning of section 3.

Theorem 4.2 [12, Theorem 2.11] *The following statements are equivalent for a ring R.*
(i) R is a right H–ring,
(ii) R is right Artinian and every non–small R–module contains a non–zero injective submodule,
(iii) R is right perfect and for any exact sequence $\phi : P \to E \to 0$ *where E is injective and ker*ϕ *is small in P, P is injective,*
(iv) Every R–module is a direct sum of an injective module and a small module.

Before giving the characterization of H–rings first we are interested in the condition that every right R–module satisfies (S*).

Proposition 4.3 *Let R be a ring. An injective R–module M satisfies (S*) if and only if every submodule of M is a direct sum of an injective module and a cosingular module.*

Proof Suppose that M satisfies (S*). Let N be a submodule of M. There exist submodules K, K' of M such that $M = K \oplus K'$, $K \leq N$ and N/K is cosingular. Then $N = K \oplus (N \cap K')$ where K is injective and $N \cap K'$ is cosingular because $N \cap K' \cong N/K$.

Conversely, suppose that every submodule of M is a direct sum of an injective module and a cosingular module. Let L be any submodule of M. Then $L = L_1 \oplus L_2$ for some injective module L_1 and cosingular module L_2. Clearly L_1 is a direct summand of M and $L/L_1 = \mathrm{Z}^*(L/L_1)$ because $L/L_1 \cong L_2$. □

Theorem 4.4 *The following statements are equivalent for a ring R.*
(i) Every right R–module satisfies (S),*
(ii) Every injective right R–module satisfies (S),*
(iii) Every right R–module is a direct sum of an injective module and a cosingular module.

Proof (i)⇔(ii) It is clear because every submodule of a module with (S*) also has (S*). (ii)⇔ (iii) by Proposition 4.3. □

If the right R–module R satisfies (S*), then every right R–module need not satisfy (S*). The following example is given in [12, Chapter 5]. It is also discussed in [11, 2.3.4 and 2.3.5].

Example 4.5 Let $Q = k[x, y]/(x^2, y^2)$ where k is a field. Then Q is a local QF–ring by [12, Remark on p. 336]. Let J=J(Q), S=Soc(Q_Q)(=Soc($_QQ$)), $\overline{Q} = Q/S$ and $\overline{a} = a + S$ for any a in Q. We define W as follows:

$$W = \begin{bmatrix} Q & \overline{Q} \\ J & \overline{Q} \end{bmatrix} = \left\{ \begin{bmatrix} a & \overline{b} \\ d & \overline{c} \end{bmatrix} : a, b, c \in Q, d \in J \right\}.$$

W is a ring by the usual addition and multiplication of matrices. We put $1_W = \begin{bmatrix} 1 & \overline{0} \\ 0 & \overline{1} \end{bmatrix}$, $e = \begin{bmatrix} 1 & \overline{0} \\ 0 & \overline{0} \end{bmatrix}$, and $f = \begin{bmatrix} 0 & \overline{0} \\ 0 & \overline{1} \end{bmatrix}$ in W. Then 1_W is the identity element of W and $\{e, f\}$ is a set of orthogonal primitive idempotents and $1 = e+f$. Oshiro showed that W is a right and left Artinian but not right H-ring. Then there exists an injective right W–module E such that E is not a (D1)–module. But since W is right perfect, W_W satisfies (S*) and Z*(E) =Rad(E) $<<$ E. If E satisfies (S*), by Lemma 3.3, E must be a (D1)–module, a contradiction. Hence every right W–module does not satisfy (S*).

Definitions 4.6 A module M is called *extending* if it satisfies (C1): Every submodule of M is essential in a summand of M. M is called *quasi–continuous* if it satisfies (C1), and if M_1 and M_2 are summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a summand of M [10]. M is called $\sum$*–injective (respectively,* $\sum$*–extending)* if every direct sum of copies of M is injective (respectively, extending) [13] or [14].

Proposition 4.7 *Assume that every right R–module satisfies (S*). Then $R = A \oplus B$ is the direct sum of a $\sum$–injective right ideal A and a cosingular right ideal B.*

Proof Assume that every right R–module satisfies (S*). Then R has a decomposition $R = A \oplus B$ where A is injective and B is cosingular by Theorem 4.4. Since A is an injective right ideal, Z*(A) =Rad(A). Hence Z*($A^{(\Lambda)}$) =Rad($A^{(\Lambda)}$) for every index set Λ. Again, by Theorem 4.4, $A^{(\Lambda)} = N \oplus K$ where N is injective and K is cosingular. Note that

$Z^*(A^{(\Lambda)}) = Z^*(N) \oplus K = \mathrm{Rad}(A^{(\Lambda)}) = \mathrm{Rad}N \oplus \mathrm{Rad}K$.

Then Rad$K = K$. By [4, Proposition 17.14], $K = 0$. Thus $A^{(\Lambda)}$ is injective for every index set Λ. This completes the proof. □

Example 4.8 The converse of the Proposition 4.7 does not hold in general.

Let W be as defined in Example 4.5. Then $W_W \cong eW \oplus eJ(W)$ and eW is injective [12, Section 5] or [11, the proof of Theorem 2.3.5]. Since W is right Artinian, W is right Noetherian by the Hopkins–Levitzki Theorem. Hence eW is $\sum$–injective by [4, Proposition 18.13]. On the other hand, $eJ(W)$ is cosingular because $eJ(W) = J(eW) \leq Z^*(W)$. But, as was shown in Example 4.5, every right W–module does not satisfy (S*).

Now we give a characterization of H–rings.

Theorem 4.9 *The following statements are equivalent for a ring R.*
(i) R is a right H–ring,
(ii) R is right perfect and every right R–module satisfies (S),*
(iii) For every injective right R-module M, Rad$M << M$ and every right R–module satisfies (S).*

Proof (i)⇒(ii) If R is a right H–ring, then every injective R–module satisfies (S*). Hence (ii) holds by Theorem 4.4.
(ii)⇒(iii) It is clear.
(iii)⇒(i) Let M be an injective R–module. Then Rad$M = Z^*(M)$. By hypothesis $Z^*(M) << M$. Since M satisfies (S*), M is a (D1)–module by Lemma 3.3. Hence R is a right H–ring. □

If every right R–module satisfies (S*), then R need not be H–ring in general. For example, let $R = \mathbb{Z}$. Since $Z^*(R) = R$, every R–module satisfies (S*) by Corollary 2.7 or Lemma 3.8. But since no proper submodule in R has a supplement in R, R is not a right H–ring.

5 QF–RINGS

In this section our aim is to use (S*) property to characterize QF–rings. Next we give three lemmas used in the characterization of QF–rings.

Lemma 5.1 *Let P_i $(1 \leq i \leq n)$ be a finite collection of projective injective R–modules satisfying (S*) and let $P = P_1 \oplus \cdots \oplus P_n$. Then P satisfies (S*).*

Proof By induction on n it is sufficient to prove the result when $n = 2$. Let $P = P_1 \oplus P_2$ and let $\pi_i : P \longrightarrow P_i$ $(i = 1, 2)$ denote the canonical projections. Let N be a submodule of P. By hypothesis, the submodule $\pi_1(N) = Q_1 \oplus L_1$

for some direct summand Q_1 of P_1 and cosingular submodule L_1 of P_1. Let $\sigma : \pi_1(N) \longrightarrow Q_1$ denote the canonical projection. Then $\sigma\pi_1 : N \longrightarrow Q_1$ is an epimorphism with kernel $H = \{m \in N : \pi_1(m) \in L_1\}$. Note that Q_1 is a projective module and hence $N = N_1 \oplus H$ for some submodule $N_1 \cong Q_1$. Repeating the same argument for $\pi_2(H)$ we see that $H = N_2 \oplus N'$ for some submodule N_2 isomorphic to a direct summand of P_2 and submodule N' where $N' = \{m \in N : \pi_1(m) \in L_1, \pi_2(m) \in L_2\}$ for some cosingular submodule L_2 of P_2.

Now $N = N_1 \oplus N_2 \oplus N'$ where $N_1 \oplus N_2$ is injective and hence a direct summand of P. Moreover, $N' \leq L_1 \oplus L_2$ so that N' is cosingular by Lemma 2.6. It follows that P satisfies (S*). □

Corollary 5.2 *Let R be a right self–injective ring that satisfies (S*). Then R is semiperfect.*

Proof By Lemma 5.1, every finitely generated free right R–module satisfies (S*) and, by Lemma 3.2, so too does every finitely generated projective R–module.

Let M be a finitely generated R–module. Let P be a finitely generated projective R–module and let $\varphi : P \to M$ be an epimorphism with kernel K. There exist submodules Q, Q' of P such that $P = Q \oplus Q'$, $Q \leq K$ and K/Q is cosingular. Now $K = Q \oplus (K \cap Q')$ and hence $M \cong P/K \cong Q'/(K \cap Q')$ where $K \cap Q'$ is cosingular.

Now $K \cap Q' = \mathrm{Z}^*(K \cap Q') \leq \mathrm{Z}^*(Q') = \mathrm{Rad}Q' << Q'$ since Q' is injective and finitely generated. Thus M has a projective cover and R is semiperfect. □

Lemma 5.3 *Let R be a ring with $Z^*(R_R) = J(R)$ and assume that the right R-module $E(R^{(\mathrm{N})})$ satisfies (S*). Then the right R–module R is $\sum$–injective, hence R is a QF–ring.*

Proof Let $F = R \oplus R \oplus \ldots$ be the free right R–module which is the direct sum of a countably infinite number of copies of R, i.e. $F = R^{(\mathrm{N})}$. By hypothesis $\mathrm{E}(F)$ satisfies (S*). By Proposition 4.3, $F = X \oplus Y$ for some injective submodule X and cosingular submodule Y. Note that

$$Y = \mathrm{Z}^*(Y) \leq \mathrm{Z}^*(F) = \mathrm{J} \oplus \mathrm{J} \oplus \ldots = F\mathrm{J},$$

by Lemma 2.3, where J is the Jacobson radical of R. Note that $(F/X) = (F/X)\mathrm{J}$. But $F/X \cong Y$ so that F/X is projective and hence $F/X = 0$ by [4, Proposition 17.14]. Thus F is injective. By [14, Proposition 20.3A], R_R is $\sum$–injective. Hence R is a QF–ring [13, 18.1]. □

Lemma 5.4 *Let R be a semiperfect ring. If $Z^*(R_R) = Z(R_R)$ then $Z^*(R_R) = J(R)$. The converse holds when R is right or left perfect right quasi-continuous.*

Proof Let R be a semiperfect ring and assume $Z^*(R_R) = Z(R_R)$. Then there exists an idempotent e of R such that $eR \leq Z(R_R)$ and $(1-e)R \cap Z(R_R)$ is small in R by [10, Corollary 4.42]. Since $Z(R_R)$ does not contain any non–zero idempotents, it follows that $Z(R_R) \leq J(R)$ (see also [15, the proof of Lemma 4(viii)]). Hence $Z^*(R_R) = J(R)$.

For converse, assume that $Z^*(R_R) = J(R)$. Since R is right or left perfect right quasi–continuous, $Z(R_R) = J(R)$ by [16, Lemma 6]. Hence $Z^*(R_R) = Z(R_R)$. □

Oshiro [12] also called a ring R a *right co–H–ring* if every projective right R–module is extending. R is a right co–H–ring if and only if R is right $\sum$–extending [13, 11.13].

Theorem 5.5 *The following statements are equivalent for a ring R.*

(1) R is a QF–ring,
(2) R is a right self–injective ring and every right R–module satisfies (S),*
(3) R is a right self-injective ring and $E(R^{(\mathrm{N})})$ satisfies (S),*
(4) $Z^*(R_R) = J(R)$ and either of the following conditions hold.
  (a) every right R-module satisfies (S) or*
  (b) $E(R^{(\mathrm{N})})$ satisfies (S) or*
  (c) R is a right co–H–ring or
  (d) R is a right H–ring,
(5) $Z^*(R_R) = Z(R_R)$ and either of the following conditions hold.
  (a) every right R–module satisfies (S) or*
  (b) $E(R^{(\mathrm{N})})$ satisfies (S) or*
  (c) R is a right co–H–ring or
  (d) R is a right H–ring.

Proof For (1) ⇒ (2), suppose that R is a QF–ring. Then R is right self–injective and by [12, Theorem 4.3] and Theorem 4.9, every right R–module satisfies (S*).
The implications (2) ⇒ (3), (1) ⇒ (4a) ⇒ (4b), (1) ⇒ (4c) are all clear.
The implications (3) ⇒ (1) and (4b) ⇒ (1) follow from Lemma 5.3.
For (4c) ⇒ (4b), since R is a right perfect ring, every projective R–module satisfies (S*) by [10, Theorem 4.41]. On the other hand, the family of all projective R–modules is closed under taking essential extensions by [12, Theorem 3.18]. Hence $E(R^{(\mathrm{N})})$ satisfies (S*).
The equivalency (4d) ⇔ (1) follows from [6, p.673 Corollary]. The implications (1) ⇒ (5a) ⇒ (5b) and (1) ⇒ (5d) are clear.
For (5b) ⇒ (1), let $F = R^{(\mathrm{N})}$. By hypothesis $E(F)$ satisfies (S*). By Proposition 4.3, $F = X \oplus Y$ for some injective submodule X and cosingular submodule Y. Note that $Z^*(X) \oplus Y = Z^*(F) = Z(F) = Z(X) \oplus Z(Y)$. Then $Y = Z(Y)$. Since Y is projective, $Y = 0$. Hence $F = X$ is injective. Then R_R is $\sum$–injective by [13, 2.4].
The implication (1) ⇒ (5c) follows from [12, Theorem 4.3]. The implications (5c) ⇒ (4c) and (5d) ⇒ (4d) follow from Lemma 5.4. □

Corollary 5.6 *Let R be a commutative ring. Then every R–module satisfies (S*) if and only if R is the direct sum $R_1 \oplus R_2$ of a QF–ring R_1 and a cosingular ring R_2.*

Proof Suppose that every R–module satisfies (S*). By Theorem 4.4 there exist ideals R_1 and R_2 of R such that $R = R_1 \oplus R_2$, R_1 is injective and R_2 is cosingular. Thus R_1 is a self–injective ring and R_2 is a cosingular ring. It can easily be checked that every R_1–module satisfies (S*). By Theorem 5.5, R_1 is a QF–ring.

Conversely, suppose that $R = R_1 \oplus R_2$ is the direct sum of a QF–ring R_1 and a cosingular ring R_2. Let M be any R–module. Then $M = MR_1 \oplus MR_2$. By Lemma 3.8, $MR_2 \leq Z^*(M)$ and by Theorem 5.5 and Theorem 4.4, $MR_1 = A \oplus B$ for some injective submodule A and cosingular submodule B. Hence $M = A \oplus A'$ where A is injective and A' is cosingular. By Theorem 4.4, every R–module satisfies (S*). □

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