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On a class of modules

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Abstract. Let R be a ring with identity and M a unital right R-module. Let $\mathrm{Z}^*(M) = \{m \in M : mR \ll \mathrm{E}(mR)\}$. In this study we consider the property (T): For every right R-module M with $\mathrm{Z}^*(M) = \operatorname{Rad} M$, M is injective. We give a characterization of the property (T) when R is a prime PI-ring. Also, over a right Noetherian ring R we prove that if R satisfies (T) then every right R-module is the direct sum of an injective module and a Max-module.

1. Introduction and notations

All rings have identity and all modules are unital right modules.

Let R be a ring and M a right R-module. We write $\mathrm{E}(M)$, $\operatorname{Rad} M$ and $\operatorname{Soc}(M)$ for the injective envelope, the radical and the socle of M, respectively. For the right annihilator of M in R we write $\operatorname{ann}(M)$. As usual, $\mathbb{N}$, $\mathbb{C}$ represent the sets of natural numbers and complex numbers. A submodule N of M is indicated by writing $N \leq M$. The notation $N \leq_e M$ is reserved for essential submodules.

Let N be a submodule of M. N is called a *small submodule* if whenever $N + L = M$ for some submodule L of M we have $M = L$, and in this case we write $N \ll M$. In [7] LEONARD defined a module M to be *small* if it is a small submodule of some R-module. He showed that M is small if and only if M is small in its injective hull. We put

$$\mathrm{Z}^*(M) = \{m \in M : mR \text{ is small}\} \quad [5].$$

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Since $\mathrm{Rad}(M)$ is the union of all small submodules in M, $\mathrm{Z}^*(E) = \mathrm{Rad}(E)$ for any injective module E and

$$\mathrm{Z}^*(M) = M \cap \mathrm{Rad}\,\mathrm{E}(M) = M \cap \mathrm{Rad}\,E'$$

for an injective $E' \supseteq M$.

In this note we consider the following property:

(T) For every right R-module M with $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$, M is injective.

Clearly, semisimple rings satisfy (T). We will prove that the following are equivalent for a prime PI-ring R:

i) R satisfies (T),

ii) For every left R-module with $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$, M is injective,

iii) R is a hereditary Noetherian ring.

After that we show that over a right Noetherian ring R, if R satisfies (T) then every right R-module is the direct sum of an injective module and a Max-module. Also, if R is a prime right Goldie ring which is not primitive then the converse of the above result holds.

2. Results

We start with the following

Lemma 1. *For any module M, $\mathrm{Z}^*(M)$ is a submodule of M and $\mathrm{Rad}(M) \leq \mathrm{Z}^*(M)$.*

Proof. Elementary. □

Let R be a ring with identity and M be a unital right R-module. An ideal P of R is called *right primitive* if there exists a simple right R-module U such that P is the annihilator of U in R.

Lemma 2. *Suppose that $M = MP$ for every right primitive ideal P. Then $M = \mathrm{Rad}\,M$.*

Proof. Suppose that M contains a maximal submodule N and let $P = \mathrm{ann}(M/N)$. Then $M = MP \leq N$, a contradiction. □

The ring is called *right bounded* if every essential right ideal contains a two-sided ideal which is essential as a right ideal. Moreover, R is *fully right bounded* if R/P is a right bounded ring for every prime ideal P of R. The abbreviation right FBN or FBN is commonly used for a right Noetherian right fully bounded or a Noetherian fully bounded ring, respectively. A ring R is a *PI-ring* if R satisfies a polynomial identity.

Lemma 3. *Suppose that R is right FBN or a PI-ring. Then the ring R/P is (right) Artinian for every right primitive ideal P of R.*

Proof. (See for example [4, Proposition 8.4].) $\square$

Remark. Certain group rings and certain universal enveloping algebras R have the property that the ring R/P is Artinian (because R/P is prime, R/P is right Artinian implies R/P is also left Artinian) for every right primitive ideal P (see [8]). Of course, simple right Noetherian rings which are not (right) Artinian, for example the Weyl algebras $\mathrm{A}_n(\mathbb{C})$ ($n \in \mathbb{N}$), do not have this property.

Lemma 4. *Let R be a ring such that R/P is an Artinian ring for every right primitive ideal P of R. Then $M = \operatorname{Rad} M$ if and only if $M = MP$ for every right primitive ideal P of R.*

Proof. The sufficiency follows by Lemma 2. Conversely, suppose that $M = \operatorname{Rad} M$, i.e. M has no maximal submodule. Let P be any right primitive ideal of R. Then M/MP is a right module over the simple Artinian ring R/P so that M/MP is semisimple. Because M, and hence M/MP, does not have a maximal submodule, it follows that $M = MP$. $\square$

Let M be an injective module. Then $M = Mc$ for every regular (i.e. non-zero divisor) element c in R. A right R-module N is called *divisible* if $N = Nc$ for every regular c. Thus injective modules are divisible [9, Proposition 2.6].

Lemma 5 [6, Proposition 3.5]. *Suppose that R is a ring such that every divisible right R-module is injective. Then R is right hereditary.*

Remark. Let R be a semiprime right Goldie ring. Then any torsion free (i.e. non-singular) divisible right R-module is injective.

Lemma 6. *Let R be a prime right or left Goldie ring. Let M be a divisible right R-module. Then $M = MI$ for every non-zero ideal I of R.*

Proof. For any non-zero ideal I of R there exists a regular element c of R such that $c \in I$. Hence $M = Mc \leq MI \leq M$, i.e. $M = MI$. $\square$

Lemma 7. *Let R be a prime right Noetherian ring. Then $M = MI$ for every non-zero ideal I of R if and only if $M = MP$ for every non-zero prime ideal P of R.*

Proof. The necessity is clear. Conversely, suppose that $M = MP$ for every non-zero prime ideal P of R. Let I be any non-zero ideal of R. Then there exists a positive integer n and prime ideals P_i $(1 \leq \mathrm{i} \leq n)$ such that $P_1 \ldots P_n \leq I \leq P_1 \cap \cdots \cap P_n$. Then $M = MP_n = MP_{n-1}P_n = \cdots = MP_1 \ldots P_n \leq MI \leq M$, i.e. $M = MI$. □

Lemma 8. *Let R be a left bounded left Goldie prime ring. Then the right R-module M is divisible if and only if $M = MI$ for every non-zero ideal I of R.*

Proof. The necessity follows by Lemma 6. Conversely, suppose that $M = MI$ for every non-zero ideal I of R. Let c be any regular element of R. Then there exists a non-zero ideal J such that $J \leq Rc$. Now $M = MJ \leq MRc = Mc \leq M$, i.e. $M = Mc$. □

Prime PI-rings are right and left bounded and right and left Goldie, so Lemma 7 and Lemma 8 give at once:

Corollary 9. *Let R be a prime PI-ring. Then M is divisible if and only if $M = MI$ for every non-zero ideal I of R. If, in addition, R is right Noetherian, then M is divisible if and only if $M = MP$ for every non-zero prime ideal P of R.*

Lemma 10. *Let R be a prime (right and left) FBN-ring which is not Artinian and for which every non-zero prime ideal is right primitive (maximal in this case). Then the right R-module M satisfies $M = \operatorname{Rad} M$ if and only if M is divisible.*

Proof. By Lemmas 4, 7 and 8. □

We refer to [2, Chapter 6] for the definition of *Krull Dimension*.

Proposition 11. *Let R be a prime PI-ring of right Krull dimension 1. Then the right R-module M satisfies $M = \operatorname{Rad} M$ if and only if M is divisible.*

Proof. Suppose that $S = \operatorname{Soc} R_R \neq 0$. Then S contains a regular element c, because R is prime right Goldie, and $R \cong cR \leq S$ gives that R

is right Artinian, contradicting the fact that R has right Krull dimension 1. Thus $S = 0$.

Let E be any essential right ideal of R. There exists a non-zero ideal I of R such that $I \leq E$. There exists a regular element d such that $d \in I$. Now R/dR is Artinian and hence the right R-module R/I is Artinian (this is because the (right) Krull dimension of R/dR is 0). By the Hopkins–Levitzki Theorem, the right Artinian ring R/I is right Noetherian. Thus R/E is a Noetherian right R-module. It follows that R satisfies the ascending chain condition on essential right ideals and hence the ring R/S is a right Noetherian ring by [2, 5.15]. Thus R is a right Noetherian ring. By [8, 13.6.15 Theorem] R is also left Noetherian.

It is now clear that Lemma 10 can be applied to give the result. □

Proposition 12. *Let R be a prime right Goldie ring which is not primitive. Then $\mathrm{Z}^*(M) = M$ for every right R-module M. In addition if R satisfies* (T)*, then every divisible right R-module is injective.*

Proof. Let M be a right R-module, $x \in M$ and $E = \mathrm{E}(xR)$. Suppose that $E = xR + L$ for some $L \leq E$. If x is not in L, then E/L is non-zero and a cyclic module so that there exists a maximal submodule P of E with L contained in P. The module $U = E/P$ is simple, and if I is its annihilator in R we know that I is a non-zero ideal of R by our hypothesis. But in this case I contains a non-zero divisior by Goldie's Theorem [4, Proposition 5.9] and then $E = EI$ by [9, Proposition 2.6] so that $E = P$, a contradiction. Hence $x \in L$ and so $E = L$ and xR is small. Thus $\mathrm{Z}^*(M) = M$.

Now assume that R satisfies (T). Let M be a divisible right R-module and N a maximal submodule of M. Then $0 \neq \mathrm{ann}(M/N) \leq_e R$. There exists a non-zero regular element $d \in \mathrm{ann}(M/N)$. Now $M = Md \leq N$ and so $M = N$. Hence $\mathrm{Rad}\, M = M$. By hypothesis M is injective. □

Remark. Let R be a prime PI-ring. Suppose in addition that R is right hereditary. Because R is right Goldie it follows that R is right Noetherian [1, Corollary 8.25] and hence also left Noetherian [8, 13.6.15 Theorem]. By [8, 6.2.8 Corollary] R has right Krull dimension at most 1. Note also that R is left hereditary because R is right and left Noetherian [1, Corollary 8.18].

Theorem 13. *The following are equivalent for a prime PI-ring R:*

(i) *For every right R-module M with $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$, M is injective,*

(ii) *For every left R-module M with $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$, M is injective,*

(iii) *R is a hereditary Noetherian ring.*

PROOF. (i) $\Longrightarrow$ (iii) We claim that every divisible right R-module is injective. Let M be divisible. If $M = \operatorname{Rad} M$ then $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$ and hence M is injective. Suppose $M \neq \operatorname{Rad} M$ and let N be a maximal submodule of M. If $\operatorname{ann}(M/N) = 0$ then R is primitive. By Kaplansky's Theorem, R is semisimple Artinian. Hence M is injective. If $\operatorname{ann}(M/N) \neq 0$, then, by Proposition 12, M is injective. Thus, by Lemma 5, R is right hereditary. Hence by the above remark R is a hereditary Noetherian ring.

(iii) $\Longrightarrow$ (i) Let M be a right R-module and suppose $\mathrm{Z}^*(M) = \operatorname{Rad} M$. By Proposition 12, M has no maximal submodule. By the above remark, R has (right or left) Krull dimension 1. By Proposition 11, M is divisible. Hence M is injective by Theorem 3.37 in [3] and Theorem 3.4 in [6].

(ii) $\Longleftrightarrow$ (iii) Symmetrical. □

We call a module M a *Max-module* if for every non-zero submodule N of M, N has a maximal submodule. For any module M we define the radical series of M to be the chain of submodules

$$M = M_0 \geq M_1 \geq \cdots \geq M_\alpha \geq M_{\alpha+1} \geq \cdots$$

where for any ordinal $\alpha \geq 0$, $\operatorname{Rad} M_\alpha = M_{\alpha+1}$ and $M_\alpha = \bigcap_{0 \leq \beta < \alpha} M_\beta$ if α is a limit ordinal. Since M is a set, there exists an ordinal $\rho \geq 0$ such that $M_\rho = M_{\rho+1} = \dots$.

Proposition 14 [10, Proposition 2.2]**.** *A module M is a Max-module if and only if $M_p = 0$.*

Theorem 15. *Let R be a right Noetherian ring. If R satisfies* (T) *then every right R-module is the direct sum of an injective module and a Max-module.*

PROOF. Let M be any right R-module. Let $\mathcal{S}$ denote the collection of injective submodules of M (note that $0 \in \mathcal{S}$). Let $\{C_\lambda : \lambda \in \Lambda\}$ be a chain in $\mathcal{S}$ and let $C = \bigcup C_\lambda$ ($\lambda \in \Lambda$). Since R is right Notherian, Baer's Lemma gives that C is injective. Thus $C \in \mathcal{S}$. By Zorn's Lemma $\mathcal{S}$ has a maximal member M_1. Because M_1 is injective, we have $M = M_1 \oplus M_2$ for some submodule M_2 of M. Let N be a non-zero submodule of M_2. By the choice of M_1, $M_1 \oplus N$, hence N is not injective. Thus $\operatorname{Rad} N \neq \mathrm{Z}^*(N)$ by hypothesis and it follows that $N \neq \operatorname{Rad} N$. Therefore N has a maximal submodule. Thus M_2 is a Max-module. □

Theorem 16. *Let R be a prime right Goldie ring which is not primitive. Assume that every right R-module is the direct sum of an injective module and a Max-module. Then R satisfies* (T).

PROOF. Suppose that every right R-module is the direct sum of an injective module and a Max-module. Let M be any right R-module such that $\mathrm{Z}^*(M) = \mathrm{Rad}\, M$. Then by Proposition 12, $\mathrm{Rad}\, M = M$. Let $M = X \oplus Y$ where X is injective and Y is a Max-module. Now $X \oplus Y = M = \mathrm{Rad}\, M = \mathrm{Rad}\, X \oplus \mathrm{Rad}\, Y$ so that $Y = \mathrm{Rad}\, Y$ and Y does not contain a maximal submodule. This implies that $Y = 0$ and $M = X$, i.e. M is injective. □

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