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SEMIPERFECT MODULES WITH RESPECT TO A PRERADICAL

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In this article, we consider the module theoretic version of I-semiperfect rings R for an ideal I which are defined by Yousif and Zhou (2002). Let M be a left module over a ring R, $N \in \sigma[M]$, and τ_M a preradical on $\sigma[M]$. We call N τ_M-semiperfect in $\sigma[M]$ if for any submodule K of N, there exists a decomposition $K = A \oplus B$ such that A is a projective summand of N in $\sigma[M]$ and $B \leq \tau_M(N)$. We investigate conditions equivalent to being a τ_M-semiperfect module, focusing on certain preradicals such as Z_M, Soc, and δ_M. Results are applied to characterize Noetherian QF-modules (with $Rad(M) \leq Soc(M)$) and semisimple modules. Among others, we prove that if every R-module M is Soc-semiperfect, then R is a Harada and a co-Harada ring.

Key Words: Harada and co-Harada module; Noetherian QF-module; Projective module; Projective cover; Semiperfect module; Semisimple module.

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1. INTRODUCTION

Sandomierski (1969) proved that a ring R is semiperfect if and only if every simple left R-module has a projective cover. The concept of a semiperfect ring has been generalized to semiperfect modules by Mares (1963). Mares calls a module M *semiperfect* if M is projective and every quotient of M has a projective cover. Azumaya (1974) proved that a projective module M is semiperfect if and only if every proper submodule of M is contained in a maximal submodule and every simple homomorphic image of M has a projective cover. Semiperfect modules were originally defined for projective modules by Mares, but it has been extended to arbitrary modules in Kasch (1982).

Let M be a module. Wisbauer (1991) calls a module N in $\sigma[M]$ *semiperfect in* $\sigma[M]$ if every factor module of N has a projective cover in $\sigma[M]$. By Wisbauer (1991, 41.14 and 42.1), if a module P in $\sigma[M]$ is projective in $\sigma[M]$, then P is semiperfect

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in $\sigma[M]$ if and only if for every submodule K of P there exists a decomposition $K = A \oplus B$ such that A is a summand of P and $B \ll P$.

Recently, Yousif and Zhou (2002) have defined right I-semiperfect rings for an ideal I of a ring R as a generalization of semiperfect rings. They consider the cases when I is the right singular ideal or the right socle or $\delta(R_R)$ (defined in Zhou, 2000).

In this article, we define τ_M-semiperfect modules N in $\sigma[M]$ for any preradical τ_M, and consider the cases when $\tau_M(N)$ is the M-singular submodule or the socle or $\delta_M(N)$. In Section 2, we give conditions equivalent to being a τ_M-semiperfect module in $\sigma[M]$ under some assumptions. We prove that if M is projective in $\sigma[M]$ and $Rad\, M \ll M$, then M is Z_M-semiperfect in $\sigma[M]$ if and only if M is semiperfect in $\sigma[M]$ and $Z_M(M) = Rad(M)$. Also we characterize δ_M-semiperfect modules in $\sigma[M]$ by using projective δ-covers in $\sigma[M]$. After defining projective *Soc*-covers in $\sigma[M]$, we characterize *Soc*-semiperfect modules in $\sigma[M]$. In Section 3, we give a characterization of Noetherian QF-modules (with $Rad(M) \leq Soc(M)$) and semisimple modules in terms of τ_M-semiperfect modules.

Throughout this article, R denotes an associative ring with identity, and modules M are unitary left R-modules. R-Mod denotes the category of all left R-modules. For a module M, $Rad(M)$, and $Soc(M)$ are the Jacobson radical and the socle of M. We write $J(R)$ for the Jacobson radical of R. We use $N \leq_e M$ ($N \ll M$) to signify that N is an essential (small) submodule of M. For a (direct) summand K of M, we write $K \leq^{\oplus} M$.

Recall that $\sigma[M]$ denotes the full subcategory of R-Mod whose objects are isomorphic to a submodule of an M-generated module for any R-module M (Wisbauer, 1991). In case of $M = R$, $\sigma[M] = R$-Mod. $\sigma[M]$ is closed under direct sums, submodules, and factor modules. If a module P is P-projective, then it is called *self-projective*. A module P in $\sigma[M]$ is called *projective in* $\sigma[M]$ if it is N-projective for every $N \in \sigma[M]$. If P is finitely generated, then it is M-projective if and only if it is projective in $\sigma[M]$. A projective module P in $\sigma[M]$ together with an epimorphism $\pi : P \to N$ with $Ker(\pi) \ll P$ is called *a projective cover of N in* $\sigma[M]$ (Wisbauer, 1991).

We say that "a submodule A of N is a projective summand of N in $\sigma[M]$" whenever A is a summand of N which is projective in $\sigma[M]$.

A module $N \in \sigma[M]$ is called *M-singular* if $N \cong L/K$ for an $L \in \sigma[M]$ and $K \leq_e L$. The largest M-singular submodule of N is denoted by $Z_M(N)$. If $Z_M(N) = 0$, N is called *non-M-singular*. Note that any simple module is M-singular or M-projective (Dung et al., 1994, Proposition 4.2).

A functor τ_M from $\sigma[M]$ to itself is called a *preradical on* $\sigma[M]$ if it satisfies the following properties:

i) $\tau_M(N)$ is a submodule of N, for every $N \in \sigma[M]$;
ii) If $f : N' \to N$ is a homomorphism in $\sigma[M]$, then $f(\tau_M(N')) \leq \tau_M(N)$ and $\tau_M(f)$ is the restriction of f to $\tau_M(N')$.

For example Rad, Soc, and Z_M are preradicals. In case $M = R$, we write $\tau(N)$ instead of $\tau_M(N)$. Note that if K is a summand of $N \in \sigma[M]$, then $K \cap \tau_M(N) = \tau_M(K)$.

2. τ_M-SEMIPERFECT MODULES

In this section, M will be any R-module and τ_M any preradical on $\sigma[M]$ unless otherwise stated.

Proposition 2.1. *The following are equivalent for a module N in $\sigma[M]$:*

(1) *For every submodule K of N, there is a decomposition $K = A \oplus B$ such that A is a projective summand of N in $\sigma[M]$ and $B \leq \tau_M(N)$;*
(2) *For every submodule K of N, there is a decomposition $N = A \oplus B$ such that A is projective in $\sigma[M]$, $A \leq K$ and $K \cap B \leq \tau_M(N)$.*

Proof. This is obvious. □

Definition 2.2. A module $N \in \sigma[M]$ is said to be τ_M*-semiperfect in* $\sigma[M]$ if it satisfies one of the conditions of Proposition 2.1. If $\sigma[M] = R$-Mod, then it is said that N is τ-semiperfect.

M is semisimple if and only if M is 0-semiperfect in $\sigma[M]$, if and only if every module N in $\sigma[M]$ is τ_M-semiperfect in $\sigma[M]$ by Wisbauer (1991, 20.3). Let M be a projective module in $\sigma[M]$ with $Rad(M) \ll M$. Then M is Rad-semiperfect in $\sigma[M]$ if and only if M is semiperfect in $\sigma[M]$.

A module N in $\sigma[M]$ is called τ_M*-semiregular in* $\sigma[M]$ if for every finitely generated submodule K of N, there exists a decomposition $K = A \oplus B$ such that A is a summand of N which is projective in $\sigma[M]$ and $B \leq \tau_M(N)$. Clearly, if N is τ_M-semiperfect in $\sigma[M]$, then it is τ_M-semiregular in $\sigma[M]$. The converse does not hold in general (see Yousif and Zhou, 2002, Example 2.7(5)). τ_M-semiregular modules in R-Mod are studied in Alkan and Özcan (2004) by taking $\tau_M(N)$ as a fully invariant submodule F of N. Note that any fully invariant submodule F of a module M defines a preradical (see Raggi et al., 2005).

Zhou (2000) introduces the concept "δ-small submodule" as a generalization of a small submodule. Here we consider this definition in the category $\sigma[M]$.

Let N be a module in $\sigma[M]$ and K a submodule of N. K is called δ*-M-small* in N (notation $K \ll_{\delta_M} N$) if $K + L \neq N$ for any proper submodule L of N with N/L M-singular.

The properties of δ-small submodules that are listed in Zhou (2000, Lemma 1.3) also hold in $\sigma[M]$. We write them for convenience.

Lemma 2.3. *Let M be a module.*

a) *For modules K and L with $K \leq L \leq M$, we have $L \ll_{\delta_M} M$ if and only if $K \ll_{\delta_M} M$ and $L/K \ll_{\delta_M} M/K$.*
b) *For submodules K and L of M, $K + L \ll_{\delta_M} M$ if and only if $K \ll_{\delta_M} M$ and $L \ll_{\delta_M} M$.*
c) *If $K \ll_{\delta_M} M$ and $f : M \to L$ is a homomorphism, then $f(K) \ll_{\delta_M} L$.*
In particular, if $K \ll_{\delta_M} M \leq L$, then $K \ll_{\delta_M} L$.
d) *If $K \leq L \leq^{\oplus} M$ and $K \ll_{\delta_M} M$, then $K \ll_{\delta_M} L$.*

The following lemma can be seen by a proof similar to Zhou (2000, Lemma 1.2).

Lemma 2.4. *Let K be a submodule of a module N in $\sigma[M]$. Then $K \ll_{\delta_M} N$ if and only if $N = X \oplus Y$ for a projective semisimple submodule Y in $\sigma[M]$ with $Y \leq K$, whenever $X + K = N$.*

Now we consider the following submodule of a module N in $\sigma[M]$ (see also Zhou, 2000)

$$\delta_M(N) = \bigcap\{K \leq N : N/K \text{ is } M\text{-singular simple}\}.$$

By Zhou (2000, Lemma 1.5), δ_M is a preradical on $\sigma[M]$. Also $\delta_M(N)$ is the sum of all δ-M-small submodules of N, and hence $Rad(N) \leq \delta_M(N)$. If every proper submodule of N is contained in a maximal submodule of N, then $\delta_M(N) \ll_{\delta_M} N$.

Let $N \in \sigma[M]$. Consider the condition:

(S_1) for every summand K of N, there exists a decomposition $N = A \oplus B$ such that $A \leq K \cap \tau_M(N)$ and $B \cap K \cap \tau_M(N) \ll_{\delta_M} N$.

If $\tau_M(N) \ll_{\delta_M} N$, then $\tau_M(N)$ satisfies (S_1).

Lemma 2.5 (Wisbauer, 1991, 41.14). *Let M be a self-projective module. Suppose $M = P + K$ where P and K are submodules of M and $P \leq^{\oplus} M$. Then there exists a submodule $Q \leq K$ such that $M = P \oplus Q$.*

Theorem 2.6. *Let M be a module and $\overline{M} = M/\tau_M(M)$. Consider the following conditions*:

(1) *For every submodule K of M, there exists a decomposition $K = A \oplus B$ such that A is a summand of M and $B \leq \tau_M(M)$;*

(2) (i) *$\overline{M}$ is semisimple.*

(ii) *If $M/\tau_M(M) = A/\tau_M(M) \oplus B/\tau_M(M)$, then there exists a decomposition $M = P \oplus Q$ such that $\overline{P} = \overline{A}$ and $\overline{Q} = \overline{B}$.*

Then (1) $\Rightarrow$ (2i). *If M is self-projective, then* (1) $\Rightarrow$ (2ii). *If M is self-projective and $\tau_M(M)$ satisfies* (S_1), *then* (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Let $\overline{K} \leq \overline{M}$. Then there is a decomposition $M = A \oplus B$ such that $A \leq K$ and $K \cap B \leq \tau_M(M)$. So we get $\overline{M} = \overline{A} \oplus \overline{B}$. This proves (i).

Now assume M is self-projective and $\overline{M} = \overline{A} \oplus \overline{B}$. Then there is a decomposition $M = C \oplus D$ such that $C \leq A$ and $A \cap D \leq \tau_M(M)$. This implies that $\overline{M} = \overline{C + B}$. By Lemma 2.5, $M = C \oplus Q$ where $Q \leq B$. Then (ii) follows because $\overline{C} = \overline{A}$ and $\overline{Q} \leq \overline{B}$.

(2) $\Rightarrow$ (1) Assume M is self-projective and $\tau_M(M)$ satisfies (S_1). Let K be a submodule of M. By hypothesis, $\overline{M} = \overline{K} \oplus \overline{B}$ for some submodule B of M with $\tau_M(M) \leq B$. Then there exists a decomposition $M = P \oplus Q$ such that $\overline{P} = \overline{K}$ and $\overline{Q} = \overline{B}$. Hence $M = K + Q + \tau_M(M)$ and so $M = K + Q + (P \cap \tau_M(M))$. By ($S_1$) and the modularity, there exists a decomposition $P \cap \tau_M(M) = X \oplus S$, where X is a summand of M and $S \ll_{\delta_M} M$. Then $M = K + Q + X + S = (K + Q + X) \oplus D$ for a submodule $D \leq S$ by Lemma 2.4. Let $T = K + Q + X$. Then there is a

decomposition $T = (Q \oplus X) \oplus A$, where $A \leq K$ by Lemma 2.5. Since $(Q + X + D) \cap K \leq (Q + \tau_M(M)) \cap (K + \tau_M(M)) = \tau_M(M)$, (1) is proven. □

By the proof of Theorem 2.6, we have the following corollary.

Corollary 2.7. *Let M be a module and $\overline{M} = M/\tau_M(M)$. Consider the following conditions*:

(1) *M is τ_M-semiperfect in $\sigma[M]$*;
(2) (i) *$\overline{M}$ is semisimple.*
 (ii) *If $M/\tau_M(M) = A/\tau_M(M) \oplus B/\tau_M(M)$, then there exists a decomposition $M = P \oplus Q$ such that $\overline{P} = \overline{A}$ and $\overline{Q} = \overline{B}$.*

Then (1) ⇒ (2*i*). *If M is self-projective, then* (1) ⇒ (2ii). *If M is projective in $\sigma[M]$ and $\tau_M(M)$ satisfies (S_1), then* (2) ⇒ (1).

Let I be an ideal of a ring R. If for every idempotent $g + I$ in R/I there is an idempotent $e \in R$ such that $g + I = e + I$, then it is said that *idempotents can be lifted modulo I* (Anderson and Fuller, 1974).

For an ideal I of R, we may define a preradical $I : \sigma[M] \to \sigma[M]$ by $I(N) = IN$ for a module $N \in \sigma[M]$. Then we have the following corollary.

Corollary 2.8. *Let I be an ideal of a ring R satisfying (S_1). Then the following conditions are equivalent*:

(1) *R is left I-semiperfect*;
(2) *R/I is semisimple and idempotents can be lifted modulo I.*

Remark 2.9. Indeed, an ideal I of a ring R must satisfy the condition (S_1) for the above equivalence. In Alkan and Özcan (2004, Proposition 3.1), it is proven that $Z(_RR)$ satisfies (S_1) if and only if $Z(_RR) \leq J(R)$. Hence $Z(_RR)$ does not satisfy (S_1) in general. For example, Bergman's example (see Yousif and Zhou, 2002, Example 2.8 and Chatters and Hajarnavis, 1980, Example 1.36) shows that there exists a ring R with $J(R) = 0, Z(_RR) \neq 0$. Also for this ring, $R/Z(_RR)$ is semisimple and idempotents can be lifted modulo $Z(_RR)$.

Theorem 2.10. *Let M be projective in $\sigma[M]$ and $M = M_1 \oplus M_2$ a direct sum of modules M_1, M_2 such that M_i is τ_M-semiperfect in $\sigma[M]$ for $i = 1, 2$. Then M is τ_M-semiperfect in $\sigma[M]$.*

Proof. Let $L \leq M$. We show that there exists a decomposition $M = A \oplus B$ such that $A \leq L$ is projective in $\sigma[M]$ and $L \cap B \leq \tau_M(M)$.

Case (1). If $M_1 \cap (L + M_2) = 0$, then $L \leq M_2$. Since M_2 is τ_M-semiperfect, there exists $B_1 \leq L$ such that $M_2 = B_1 \oplus B_2$ and $L \cap B_2 \leq \tau_M(M_2)$ for some submodule B_2 of M_2. Hence $M = M_1 \oplus B_1 \oplus B_2$ and $L \cap (M_1 \oplus B_2) = L \cap B_2 \leq \tau_M(M_2) \leq \tau_M(M)$.

Case (2). If $M_1 \cap (L + M_2) \neq 0$, then M_1 has a decomposition $M_1 = A_1 \oplus A_2$ such that $A_1 \leq M_1 \cap (L + M_2)$ and $M_1 \cap (L + M_2) \cap A_2 = A_2 \cap (L + M_2) \leq \tau_M(M_1) \leq \tau_M(M)$. Then $M = A_1 \oplus A_2 \oplus M_2 = L + (M_2 \oplus A_2)$.

Assume $M_2 \cap (L + A_2) = 0$. Since $L \cap A_2 \leq A_2$ and A_2 is τ_M-semiperfect, A_2 has a decomposition $A_2 = C_1 \oplus C_2$ such that $C_1 \leq L \cap A_2$ and $L \cap A_2 \cap C_2 = L \cap C_2 \leq \tau_M(M_1)$. Then $M = (A_1 \oplus C_1) \oplus (C_2 \oplus M_2) = L + (C_2 + M_2)$. Since M is self-projective, there exists $L' \leq L$ such that $M = L' \oplus C_2 \oplus M_2$. Since $M_2 \cap (L + A_2) = 0$, we have $L \cap (C_2 \oplus M_2) = L \cap C_2 \leq \tau_M(M_1)$.

Assume $M_2 \cap (L + A_2) \neq 0$. Then M_2 has a decomposition $M_2 = B_1 \oplus B_2$ such that $B_1 \leq M_2 \cap (L + A_2)$ and $B_2 \cap (L + A_2) \leq \tau_M(M_2)$. Then $M = L + (A_2 + B_2) = (A_1 \oplus B_1) \oplus (A_2 \oplus B_2)$. Since M is self-projective, there exists $L' \leq L$ such that $M = L' \oplus A_2 \oplus B_2$.

To show that $L \cap (A_2 \oplus B_2) \leq \tau_M(M)$, take $0 \neq l = a + b \in L \cap (A_2 \oplus B_2)$, where $l \in L$, $a \in A_2$, $b \in B_2$. Then $l - b = a \in A_2 \cap (L + M_2) \leq \tau_M(M)$ and $l - a = b \in B_2 \cap (L + A_2) \leq \tau_M(M)$ and so $l \in \tau_M(M)$. Hence M is τ_M-semiperfect in $\sigma[M]$. □

Corollary 2.11. *Let M be projective in $\sigma[M]$. Then M is τ_M-semiperfect in $\sigma[M]$ if and only if every finitely M-generated projective module is τ_M-semiperfect in $\sigma[M]$.*

Proof. Let N be a finitely M-generated projective module. Then N is isomorphic to a summand of a finite direct sum of copies of M. Since Theorem 2.10 holds for any finite direct sum of modules, N is τ_M-semiperfect. □

Hence for an ideal I of R, R is left I-semiperfect if and only if every finitely generated projective module M is IM-semiperfect. In particular, a ring R is left Z- (Soc-, δ-) semiperfect if and only if every finitely generated projective module is Z- (respectively Soc-, δ-) semiperfect (see also Yousif and Zhou, 2002, Theorems 2.3 and 2.5).

From now on we consider some well-known preradicals and we obtain some results by using their own properties. First we start with the M-singular preradical.

Theorem 2.12. *Let M be projective in $\sigma[M]$ and $Rad(M) \ll M$. Then the following are equivalent:*

(1) *M is Z_M-semiperfect in $\sigma[M]$;*
(2) *M is semiperfect in $\sigma[M]$ and $Rad(M) = Z_M(M)$.*

If M is finitely generated this is also equivalent to:

(3) *For any maximal submodule K of M, $K = A \oplus B$ such that A is a projective summand of M in $\sigma[M]$ and $B \leq Z_M(M)$.*

Proof. (1) ⇒ (2) Since M is Z_M-semiregular in $\sigma[M]$ and since every cyclic submodule of $Rad(M)$ is small in M, it can be seen that $Rad(M) \leq Z_M(M)$. For the converse, let $x \in Z_M(M)$. To show that $x \in Rad(M)$, let $L \leq M$ be such that $M = Rx + L$. By (1), L has a decomposition $L = P \oplus S$, where P is a projective summand of M in $\sigma[M]$, and S is M-singular. Then $Rx + S \leq Z_M(M)$. $M = Rx + S + P$

and then M/P is M-singular. Since M is projective in $\sigma[M]$ and $P \leq^{\oplus} M$, M/P is projective in $\sigma[M]$. But this implies that $M = P$. Hence $M = L$ and so $Rx \ll M$. Since $Rad(M) \ll M$, M is semiperfect in $\sigma[M]$.

(2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (2) Assume M is finitely generated and projective in $\sigma[M]$. First we claim that $M/Z_M(M)$ is semisimple. Let $K/Z_M(M)$ be a maximal submodule of $M/Z_M(M)$. Then there is a decomposition $M = A \oplus C$ such that A is projective in $\sigma[M]$, $A \leq K$ and $K \cap C \leq Z_M(M)$. Then $K = A \oplus (K \cap C)$ and $K \cap C = Z_M(C)$. Since $K \cap (C + Z_M(M)) = K \cap (C + Z_M(A)) = Z_M(A) + (K \cap C) = Z_M(A) + Z_M(C) = Z_M(M)$, $K/Z_M(M)$ is a summand of $M/Z_M(M)$. So $M/Z_M(M)$ is semisimple. It follows that $Rad(M) \leq Z_M(M)$.

Now if $Rad(M) \neq Z_M(M)$, then there exists an element $x \in Z_M(M)$ such that $x \notin Rad(M)$. Then there exists a maximal submodule K of M such that $x \notin K$. This implies that $M = Rx + K$. By (3), $K = A \oplus B$ such that A is a projective summand of M in $\sigma[M]$ and $B \leq Z_M(M)$. Then $M = Rx + A + Z_M(M) = A + Z_M(M)$. Let C be a submodule of M such that $M = A \oplus C$. Then $C \cong M/A \cong Z_M(M)/Z_M(A)$ is M-singular and projective in $\sigma[M]$. Hence $M = A$, a contradiction. So $Rad(M) = Z_M(M)$.

To see that M is semiperfect in $\sigma[M]$, let K be a maximal submodule of M. Then M has a decomposition $M = A \oplus B$ such that $A \leq K$ and $K \cap B \leq Z_M(M) = Rad(M) \ll M$. This implies that $M = K + B$ and $K \cap B \ll B$. By Wisbauer (1991, 41.6(1) and 42.3(1)), M is semiperfect in $\sigma[M]$. □

The next proposition is proven in Zhou (2000, Corollary 1.7) when $N = M = R$.

Proposition 2.13. *Let $N \in \sigma[M]$ be a projective module in $\sigma[M]$. Then*

$$Rad(N/Soc(N)) = \delta_M(N)/Soc(N).$$

In particular, $\delta_M(N) = N$ if and only if N is semisimple.

Proof. Since N is projective in $\sigma[M]$, $\delta_M(N)$ is the intersection of all essential maximal submodules of N. Then $Soc(N) \leq \delta_M(N)$. Let $\overline{n} \in Rad(N/Soc(N))$. If $n \notin \delta_M(N)$, then there exists an essential maximal submodule K of N such that $n \notin K$. But $\overline{n} \in K/Soc(N)$, a contradiction. Conversely, let $\overline{n} \in \delta_M(N)/Soc(N)$ and assume that $\overline{n} \notin Rad(N/Soc(N))$. Then there exists a maximal submodule $L/Soc(N) \leq N/Soc(N)$ such that $\overline{n} \notin L/Soc(N)$ and so $n \notin L$. Then $N = L + Rn$ with $Rn \leq \delta_M(N)$. So $Rn \ll_{\delta_M} N$. By Lemma 2.4, $N = L \oplus Y$, where $Y \leq Rn$ is semisimple. Since $Soc(N) \leq L$, it must be that $Y = 0$. So $L = N$, a contradiction. □

Note that there exists a module M and $N \in \sigma[M]$ such that N is not projective in $\sigma[M]$ and $Soc(N)$ is not contained in $\delta_M(N)$. For example, let $M = \mathbb{Z}$ and $N = \mathbb{Z}_p$ where p is prime.

Let $N \in \sigma[M]$. A homomorphism $f : P \to N$ is called a *projective δ-cover in $\sigma[M]$* of the module N if $P \in \sigma[M]$ is projective in $\sigma[M]$ and f is an epimorphism with $\text{Ker}(f) \ll_{\delta_M} P$. If $\sigma[M] = R$-Mod, then f is called a *projective δ-cover* (Zhou, 2000).

By a proof similar to Zhou (2000, Lemma 2.4), we have the following lemma.

Lemma 2.14. *Let $N \in \sigma[M]$ be a projective module in $\sigma[M]$ and $K \leq N$. Then the following are equivalent*:

(1) *N/K has a projective δ-cover in $\sigma[M]$*;
(2) *$N = N_1 \oplus N_2$ for some N_1 and N_2 with $N_1 \leq K$ and $N_2 \cap K \ll_{\delta_M} N$.*

Now we need to prove some propositions to give a characterization of δ_M-semiperfect modules in $\sigma[M]$.

Proposition 2.15. *If S is a simple module in $\sigma[M]$ which has a projective δ-cover in $\sigma[M]$, then S is N-projective for every module N in $\sigma[M/\delta_M(M)]$.*

Proof. Let $f : P \to S$ be a projective δ-cover of S in $\sigma[M]$. Then $Ker(f) \leq \delta_M(P)$ and is a maximal submodule of P. If $\delta_M(P) = P$, then P is semisimple by Proposition 2.13. This implies that $P/Ker(f) \cong S$ is projective in $\sigma[M]$ and hence projective in $\sigma[M/\delta_M(M)]$.

If $Ker(f) = \delta_M(P)$, then $P/\delta_M(P) \cong S$. Now we claim that $P/\delta_M(P)$ is $M/\delta_M(M)$-projective. Let $T \leq M/\delta_M(M)$ and $\theta : P/\delta_M(P) \to (M/\delta_M(M))/T$ be a homomorphism and $\mu : M/\delta_M(M) \to (M/\delta_M(M))/T$ be the canonical epimorphism. Since P is $M/\delta_M(M)$-projective, there exists $\alpha : P \to M/\delta_M(M)$ such that $\mu\alpha = \theta\pi$ where $\pi : P \to P/\delta_M(P)$ is the canonical epimorphism. Since $\delta_M(M/\delta_M(M)) = 0$, $\delta_M(P) \leq Ker(\alpha)$. Now define $\beta : P/\delta_M(P) \to M/\delta_M(M)$ such that $\beta(p + \delta_M(P)) = \alpha(p)$, where $p \in P$. Then $\mu\beta\pi = \mu\alpha = \theta\pi$. Since π is epic, $\mu\beta = \theta$. Hence $P/\delta_M(P)$ is $M/\delta_M(M)$-projective. Since $P/\delta_M(P)$ is finitely generated, it is N-projective for every module N in $\sigma[M/\delta_M(M)]$. □

Proposition 2.16. *Let $N \in \sigma[M]$. If every factor module of N has a projective δ-cover in $\sigma[M]$, then every proper submodule of N is contained in a maximal submodule.*

Proof. Let U be a proper submodule of N and $f : P \to N/U$ a projective δ-cover of N/U in $\sigma[M]$. If $\delta_M(P) \neq P$, then P has an essential maximal submodule V. Then $Ker(f) \leq \delta_M(P) \leq V$. This implies that $f(V)$ is a maximal submodule of N/U. If $\delta_M(P) = P$, P and hence N/U is semisimple. It follows that N/U has a maximal submodule. □

Let N be an M-generated module. Then there exists an epimorphism $M^{(\Lambda)} \to N$ for a suitable index set Λ. This induces an epimorphism $(M/\delta_M(M))^{(\Lambda)} \to N/\delta_M(N)$. It follows that $N/\delta_M(N) \in \sigma[M/\delta_M(M)]$.

Proposition 2.17. *Let N be an M-generated module. If every proper submodule of N is contained in a maximal submodule and every simple factor module of N has a projective δ-cover in $\sigma[M]$ then $N/\delta_M(N)$ is semisimple.*

Proof. Let $\overline{N} = N/\delta_M(N)$ and $C = Soc(\overline{N})$. If $C \neq \overline{N}$, then there exists a maximal submodule D of $\overline{N}$ such that $C \leq D \leq \overline{N}$. Then $\overline{N}/D$ is a simple factor module of

$\overline{N}$ whence of N, therefore, has a projective δ-cover in $\sigma[M]$. Since $\overline{N} \in \sigma[M/\delta_M(M)]$, $\overline{N}/D$ is projective in $\sigma[M/\delta_M(M)]$ by Proposition 2.15. Then D is a summand of $\overline{N}$. So $\overline{N} = D \oplus D'$ for some D'. This implies that $D' \leq C \leq D$, a contradiction. □

Proposition 2.18. *Let N be an M-generated and a finitely generated module. If every simple factor module of N has a projective δ-cover in $\sigma[M]$, then every factor module of N has a projective δ-cover in $\sigma[M]$.*

Proof. By Proposition 2.17, $\overline{N} = N/\delta_M(N)$ is semisimple. Then it is a finite direct sum of simple modules S_i, $i = 1, \ldots, n$. Let $f_i : P_i \to S_i$ be a projective δ-cover of S_i in $\sigma[M]$. Then $f := \bigoplus_{i=1}^n f_i : \bigoplus_{i=1}^n P_i \to \overline{N}$ is a projective δ-cover of $\overline{N}$ in $\sigma[M]$ by a proof similar to Lemma 2.3(b). Let $P = \bigoplus_{i=1}^n P_i$. Let $g : N \to \overline{N}$ be the canonical epimorphism. Since P is projective in $\sigma[M]$, there exists a homomorphism $h : P \to N$ such that $gh = f$. Then we have that $N = h(P) + \delta_M(N)$. Since $\delta_M(N) \ll_{\delta_M} N$, there exists a semisimple projective submodule X in $\sigma[M]$ such that $N = h(P) \oplus X$ by Lemma 2.4. Then $h : P \to h(P)$ is a projective δ-cover in $\sigma[M]$. This implies that N has a projective δ-cover in $\sigma[M]$. The hypotheses of the theorem are also satisfied for any factor module of N. Hence every factor module of N has a projective δ-cover in $\sigma[M]$. □

The following theorem characterizes δ_M-semiperfect modules in $\sigma[M]$ and also we will use it to give a characterization of *Soc*-semiperfect modules.

Theorem 2.19. *Let $N \in \sigma[M]$ be projective in $\sigma[M]$ and $\delta_M(N) \ll_{\delta_M} N$. Then the following are equivalent*:

(1) *N is δ_M-semiperfect in $\sigma[M]$*;
(2) *Every factor module of N has a projective δ-cover in $\sigma[M]$.*

If N is finitely generated, this is also equivalent to:

(3) *For every countably generated submodule L of N, N/L has a projective δ-cover in $\sigma[M]$.*

If N is finitely generated and M-generated, this is also equivalent to:

(4) *Every simple factor module of N has a projective δ-cover in $\sigma[M]$.*

Proof. By Lemma 2.14 and Proposition 2.18, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1) By Lemma 2.14, N is δ_M-semiregular in $\sigma[M]$. Now we show that $\overline{N} = N/\delta_M(N)$ is Noetherian. Assume not. Then there exists a strict ascending chain $K_1 \subset K_2 \subset \cdots$ of $\overline{N}$. Let $\overline{a}_1 \in K_1$, $\overline{a}_2 \in K_2 \backslash R\overline{a}_1$, $\overline{a}_3 \in K_3 \backslash (R\overline{a}_1 + R\overline{a}_2), \ldots$. Then there exists a strict ascending chain $R\overline{a}_1 \subset R\overline{a}_1 + R\overline{a}_2 \subset \cdots$ of $\overline{N}$. Let $N_k = R\overline{a}_1 + \cdots + R\overline{a}_k$ $(k \geq 1)$. Since every finitely generated submodule of $\overline{N}$ is a summand, $N_i \leq^{\oplus} N_{i+1}$ for all $i \geq 1$. Let $L = Ra_1 + Ra_2 + \cdots$. Then by Lemma 2.14, $L = E \oplus D$, where E is a summand of N and $D \leq \delta_M(N)$. Since N is finitely generated, there exists $k \geq 1$ such that $\overline{E} \leq R\overline{a}_1 + \cdots + R\overline{a}_k$. Then we have $N_{k+1} \leq \overline{E} = N_k = \overline{L}$. This gives a contradiction. Hence $N/\delta_M(N)$ is Noetherian. By Alkan and Özcan (2004, Corollary 2.13), N is δ_M-semiperfect in $\sigma[M]$. □

Let N be an R-module in $\sigma[M]$. We call a homomorphism $f : P \to N$ a *projective Soc-cover* of N in $\sigma[M]$ if P is projective in $\sigma[M]$ and f is an epimorphism with $Ker(f) \leq Soc(P)$. If $\sigma[M] = R$-Mod, then f is called *projective Soc-cover* of N. Then we have

Lemma 2.20. *Let $N \in \sigma[M]$ be such that $N = \bigoplus_{i\in K} N_i$. If each $f_i : P_i \to N_i$ $(i \in K)$ is a projective Soc-cover in $\sigma[M]$, then $\bigoplus_{i\in K} f_i : \bigoplus_{i\in K} P_i \to N$ is a projective Soc-cover in $\sigma[M]$.*

Although the proof of the following lemma is very similar to the proof of Zhou (2000, Lemma 2.3) it is given for completeness.

Lemma 2.21. *Let $N \in \sigma[M]$ and $f : Q \to N$ a projective Soc-cover in $\sigma[M]$. If $P \in \sigma[M]$ is a projective module in $\sigma[M]$ and $g : P \to N$ is an epimorphism, then there exist decompositions $Q = A \oplus B$ and $P = X \oplus Y$ such that*

(1) *$A \cong X$,*
(2) *$f_{|A} : A \to N$ is a projective Soc-cover in $\sigma[M]$,*
(3) *$g_{|X} : X \to N$ is a projective Soc-cover in $\sigma[M]$,*
(4) *B is a projective semisimple module in $\sigma[M]$ with $B \subseteq Ker(f)$ and $Y \subseteq Ker(g)$.*

Proof. Since P is projective in $\sigma[M]$, there exists $h : P \to Q$ such that $g = fh$. Thus $fh(P) = N = f(Q)$ and so $Q = h(P) + Ker(f)$. Let $A = h(P)$. Since $Ker(f) \subseteq Soc(Q)$, there exists a submodule B in $Ker(f)$ such that $Q = A \oplus B$. Thus B is a projective semisimple submodule in $\sigma[M]$. $f(Q) = f(A) = N$ and $Ker(f_{|A}) = A \cap Ker(f) \subseteq A \cap Soc(Q) = Soc(A)$. Thus $f_{|A} : A \to N$ is a projective Soc-cover in $\sigma[M]$. Since A is projective in $\sigma[M]$, there exists a homomorphism $\alpha : A \to P$ such that $h\alpha = 1_A$. Thus $P = X \oplus Y$ with $Y = Ker(h)$ and $X = \alpha(A)$. This gives $X \cong A$. On the other hand, $Ker(g_{|X}) = \alpha(Ker(f_{|A}))$ and so $Ker(g_{|X}) \subseteq X \cap Soc(P) = Soc(X)$. Also $g(X) = fh(X) = fh(X + Y) = fh(P) = g(P) = N$. Thus $g_{|X} : X \to N$ is a projective Soc-cover in $\sigma[M]$. □

Lemma 2.22. *Let $P \in \sigma[M]$ be a projective module in $\sigma[M]$ and $N \leq P$. Then the following are equivalent:*

(1) *P/N has a projective Soc-cover in $\sigma[M]$;*
(2) *$P = P_1 \oplus P_2$ for some P_1 and P_2 with $P_1 \subseteq N$ and $P_2 \cap N \subseteq Soc(P)$.*

Proof. (1) ⇒ (2) Consider a projective Soc-cover $f : Q \to P/N$ in $\sigma[M]$. Let $g : P \to P/N$ be the canonical epimorphism. By Lemma 2.21, there exists a decomposition $P = X \oplus Y$ such that $g_{|X} : X \to P/N$ is a projective Soc-cover in $\sigma[M]$ and $Y \subseteq Ker\, g = N$. Thus $X \cap N = Ker(g_{|X}) \subseteq Soc(X) \subseteq Soc(P)$. Let $P_1 = Y$ and $P_2 = X$.

(2) ⇒ (1) This is obvious. □

Theorem 2.23. *Let $N \in \sigma[M]$ be projective in $\sigma[M]$. Then the following are equivalent:*

(1) *N is Soc-semiperfect in $\sigma[M]$;*
(2) *Every factor module of N has a projective Soc-cover in $\sigma[M]$.*

If N is finitely generated, this is equivalent to:

(3) *For every countably generated submodule L of N, N/L has a projective Soc-cover in $\sigma[M]$.*

If N is finitely generated and M-generated, this is equivalent to:

(4) *Every simple factor module of N has a projective Soc-cover in $\sigma[M]$.*

Proof. (1) $\Leftrightarrow$ (2) is by Lemma 2.22. (2) $\Rightarrow$ (4) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) Assume N is finitely generated and projective in $\sigma[M]$. By hypothesis, N is *Soc*-semiregular in $\sigma[M]$. By Alkan and Özcan (2004, Theorem 2.12), every finitely generated submodule of $N/Soc(N)$ is a summand. Then $Soc(N) = \delta_M(N)$ by Proposition 2.13. Since N is finitely generated, the claim follows from Theorem 2.19.

(4) $\Rightarrow$ (1) Assume N is finitely generated, M-generated and projective in $\sigma[M]$. First we claim that $N/Soc(N)$ is semisimple. Let $K/Soc(N)$ be a maximal submodule of $N/Soc(N)$. Then $N = A \oplus B$ such that $A \leq K$ is projective in $\sigma[M]$ and $K \cap B \leq Soc(N)$ by Lemma 2.22. This implies that $K/Soc(N)$ is a summand of $N/Soc(N)$. Hence $N/Soc(N)$ is semisimple. By Proposition 2.13, $\delta_M(N) = Soc(N)$. On the other hand, every simple factor module of N has a projective δ-cover in $\sigma[M]$ by Lemma 2.14. Hence N is *Soc*-semiperfect in $\sigma[M]$ by Theorem 2.19. □

By Lemma 2.22 and Theorem 2.23, we have a characterization of *Soc*-semiperfect rings. The proof of (2) $\Rightarrow$ (3) of the following corollary is similar to that of Zhou (2000, Theorem 3.6 (1$\Rightarrow$2)).

Corollary 2.24. *The following are equivalent for a ring R*:

(1) *R is left Soc-semiperfect*;
(2) *Every simple R-module has a projective Soc-cover*;
(3) *Every R-module has a projective Soc-cover*;
(4) *Every projective R-module is Soc-semiperfect*;
(5) *For every countably generated left ideal I, R/I has a projective Soc-cover.*

Baccella (2002) proved that for any ring R, every idempotent modulo $Soc({}_RR)$ can be lifted to R. We will prove this result for modules under some conditions and give other characterization of *Soc*-semiperfect modules.

Proposition 2.25. *Let N be a module in $\sigma[M]$ with $N/Soc(N)$ semisimple. Then $Soc(N)$ is projective in $\sigma[M]$ if and only if $Z_M(N) = 0$.*

Proof. Since $N/Soc(N)$ is semisimple, we have $Soc(N) \leq_e N$. So $Z_M(N) = 0$ if and only if $Z_M(N) \cap Soc(N) = 0$, if and only if $Z_M(Soc(N)) = 0$, if and only if $Soc(N)$ is non-M-singular, if and only if $Soc(N)$ is projective in $\sigma[M]$. □

Theorem 2.26. *Let $N \in \sigma[M]$ be M-generated and finitely generated. If N and $Soc(N)$ are projective in $\sigma[M]$, then the following are equivalent*:

(1) *N is Soc-semiperfect in $\sigma[M]$*;
(2) *$N/Soc(N)$ is semisimple.*

Proof. (2) $\Rightarrow$ (1) We show that every simple factor module of N has a projective Soc-cover in $\sigma[M]$. Then by Theorem 2.23, N is Soc-semiperfect in $\sigma[M]$. Let A be a maximal submodule of N. We have two cases:

(i) If $Soc(N) \not\subseteq A$, then there exists a simple submodule S such that $A \oplus S = N$. Then N/A is projective in $\sigma[M]$ and so has a projective Soc-cover in $\sigma[M]$.

(ii) If $Soc(N) \subseteq A$, then by (2), there exists a submodule B of N such that $A + B = N$ and $A \cap B = Soc(N)$. Consider the homomorphism $\alpha : A \oplus B \to N$ with $\alpha(a, b) = a + b$. Then α is an epimorphism and also $Ker(\alpha) = \{(a, -a) : a \in A \cap B\} \cong A \cap B = Soc(N)$. Then $A \oplus B \cong N \oplus Soc(N)$ is projective in $\sigma[M]$. Let $f : B \to N/A$ with $f(b) = b + A$. Then $Ker(f) = A \cap B = Soc(N) = Soc(B)$. Thus B is a projective Soc-cover of N/A in $\sigma[M]$. □

3. EVERY MODULE IN $\sigma[M]$ IS τ_M-SEMIPERFECT IN $\sigma[M]$

In this section, we characterize modules M for which every module in $\sigma[M]$ is δ_M, Soc, Z_M-semiperfect in $\sigma[M]$.

Let M be a module. A preradical τ_M on $\sigma[M]$ is called a *left exact preradical* if for any submodule K of $N \in \sigma[M]$, $\tau_M(K) = K \cap \tau_M(N)$ (see Stenström, 1975).

For example, Soc and Z_M are left exact preradicals on $\sigma[M]$.

Lemma 3.1. *Let τ_M be a left exact preradical on $\sigma[M]$. Then the following are equivalent*:

(1) *In $\sigma[M]$, every injective module is τ_M-semiperfect in $\sigma[M]$*;
(2) *In $\sigma[M]$, every module is τ_M-semiperfect in $\sigma[M]$.*

Proof. (2) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (2) Let N be a module in $\sigma[M]$ and $K \leq N$. Since $\widehat{N}$, the *M-injective hull* of N, is τ_M-semiperfect by (1), there is a decomposition $K = A \oplus B$ such that A is a projective summand of $\widehat{N}$ in $\sigma[M]$ and $B \leq \tau_M(\widehat{N})$. Then A is a projective summand of N in $\sigma[M]$ and $B \leq N \cap \tau_M(\widehat{N}) = \tau_M(N)$. So N is τ_M-semiperfect in $\sigma[M]$. □

Now we recall some definitions. A module M is called *extending* (or CS, or (C_1)) if every submodule is essential in a summand of M. M is called $\sum$*-extending* if every direct sum of copies of M is extending. M is called *lifting* (or (D_1)) if for every submodule N of M, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B \ll M$. A module N in $\sigma[M]$ is called an *M-small module* if $N \ll \widehat{N}$. Following Oshiro (1984), a ring R is called a *left H-ring* (in honour of Harada) if every injective left R-module is lifting. For a module M, Harada modules are considered by Jayaraman and Vanaja. They call M *a Harada module* if every injective module in $\sigma[M]$ is lifting. M is a Harada module if and only if every module in $\sigma[M]$ is a direct sum of an injective in $\sigma[M]$ and an M-small module (Jayaraman and Vanaja, 2000, Theorem 2.8).

Oshiro defines a ring R a *left co-H-ring* if every projective left R-module is extending. Jayaraman and Vanaja call a module M a *co-Harada module* if it is projective in $\sigma[M]$ and is $\sum$-extending. If M is finitely generated and self-projective, then M is a co-Harada module if and only if every M-generated module is a direct sum of a module in $\mathrm{Add}\,M$ and an M-singular module, where $\mathrm{Add}\,M$ is the full subcategory of $\sigma[M]$ whose objects are summands of direct sum of copies of M (Dung et al., 1994, Corollary 11.11). Note that $\mathrm{Add}\,R$ is just the class of all projective R-modules.

If for any injective module E in $\sigma[M]$, $Rad(E) \ll E$, then any direct sum of M-small modules is M-small. For, let $N = \bigoplus_{i\in I} N_i$ where each N_i is M-small. Then $N_i \leq Rad(\widehat{N_i})$ for each i. It follows that $N = \bigoplus_{i\in I} N_i \leq \bigoplus_{i\in I} Rad(\widehat{N_i}) = Rad(\bigoplus_{i\in I} \widehat{N_i}) \leq Rad(\widehat{N})$ (Rayar, 1982).

Theorem 3.2. *Let M be finitely generated and self-projective. If every module in $\sigma[M]$ is Soc-semiperfect in $\sigma[M]$, then M is a co-Harada-module and a Harada-module.*

Proof. Let N be an M-generated module in $\sigma[M]$. By hypothesis, N is a direct sum of a projective module in $\sigma[M]$ and an M-singular module. Since N is M-generated, N is a direct sum of a module in $\mathrm{Add}\,M$ and an M-singular module. Hence M is a co-Harada module. Since $M/Soc(M)$ is semisimple by Corollary 2.7, M is Noetherian by Dung et al. (1994, 5.15 and 18.7).

Now we claim that M is a Harada module. Let $N \in \sigma[M]$. Since $\widehat{N}$ is *Soc*-semiperfect in $\sigma[M]$, N has a decomposition $N = A \oplus B$ such that A is a summand of $\widehat{N}$ which is projective in $\sigma[M]$ and $B \leq Soc(\widehat{N})$. Any simple module is either M-injective or M-small. Then B has a decomposition $B = B_1 \oplus B_2$ where B_1 is a direct sum of injective simple modules in $\sigma[M]$, and B_2 is a direct sum of M-small simple modules. Since M is Noetherian, B_1 is injective in $\sigma[M]$. Since M is perfect in $\sigma[M]$, B_2 is M-small. Hence by Jayaraman and Vanaja (2000, Theorem 2.8), M is a Harada module. □

Oshiro (1983) proved that R is a left H-ring if and only if R is a right co-H-ring. Then we have

Corollary 3.3. *Let R be a ring. If every R-module is Soc-semiperfect then R is a (right and left) co-H-ring and a (right and left) H-ring.*

If M is a Noetherian injective cogenerator in $\sigma[M]$, then it is called a *Noetherian Quasi-Frobenius* or *QF-module* (Wisbauer, 1991). For a finitely generated self-projective module M, M is a Noetherian QF-module if and only if every injective module in $\sigma[M]$ is projective in $\sigma[M]$ (Wisbauer, 1991, 48.14). A module M is called a *self-generator* if it generates all its submodules. Note that a projective self-generator in $\sigma[M]$ is a generator in $\sigma[M]$. For a finitely generated self-projective module M which is self-generator, M is a Noetherian QF-module if and only if M

is a Harada (co-Harada) module with $Z_M(M) = Rad(M)$ (Jayaraman and Vanaja, 2000, Theorem 3.11).

Theorem 3.4. *Let M be a finitely generated self-projective module which is a self-generator in $\sigma[M]$. Then the following are equivalent*:

(1) *M is a Noetherian QF-module with $Rad(M) \leq Soc(M)$*;
(2) *$Rad(M) \leq Z_M(M)$ and every module in $\sigma[M]$ is Soc-semiperfect in $\sigma[M]$.*

Proof. (1) $\Rightarrow$ (2) Let N be an injective module in $\sigma[M]$. Then N is projective in $\sigma[M]$ (Wisbauer, 1991). By Jayaraman and Vanaja (2000, Theorem 3.11) and (1), $Z_M(M) = Rad(M) \leq Soc(M)$. Since N is M-generated and projective in $\sigma[M]$, N is isomorphic to a summand of $M^{(\Lambda)}$ for an index set Λ. This implies that $Z_M(N) = Rad(N) \leq Soc(N)$. Since M is perfect in $\sigma[M]$, N is semiperfect in $\sigma[M]$ by Wisbauer (1991, 43.2). Hence N is *Soc*-semiperfect in $\sigma[M]$.

(2) $\Rightarrow$ (1) If every module in $\sigma[M]$ is *Soc*-semiperfect in $\sigma[M]$, then M is a co-Harada module by Theorem 3.2. Since M is *Soc*-semiperfect in $\sigma[M]$, $Z_M(M) \leq Soc(M)$ by the definition. Let S be a simple M-singular submodule of M. If $S \not\leq Rad(M)$, then S is a summand of M. This is a contradiction. So $Z_M(M) \leq Rad(M)$. By (2), $Z_M(M) = Rad(M)$. Hence M is a Noetherian QF-module. □

Corollary 3.5. *The following are equivalent for a ring R*:

(1) *R is a QF-ring with $J(R)^2 = 0$*;
(2) *$J(R) \leq Z({}_RR)$ and every R-module is Soc-semiperfect.*

The following example shows that the assumption "$J(R) \leq Z({}_RR)$" in Corollary 3.5 is not removable.

Example 3.6. There exists a ring R such that every R-module is *Soc*-semiperfect but $J(R) \not\leq Z({}_RR)$.

Proof. Let $R = \left[\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right]$ where F is a field. Then $J(R) = \left[\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right]$, $Soc({}_RR) = \left[\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}\right]$ and $Z({}_RR) = Z(R_R) = 0$. So $J(R) \not\leq Z({}_RR)$. Since R is an Artinian serial ring with $J(R)^2 = 0$, R is a co-H-ring and an H-ring by Oshiro (1984, Theorem 4.5). Now we claim that every R-module is *Soc*-semiperfect. Let M be an R-module and $N \leq M$. Since R is an Artinian serial ring with $J(R)^2 = 0$, M is lifting by Vanaja and Purav (1992). Then there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B \ll M$. So $N = A \oplus (N \cap B)$. Since $J(R) \leq Soc({}_RR)$, $N \cap B \leq Rad(M) = J(R)M \leq Soc({}_RR)M \leq Soc(M)$. Since R is a co-H-ring, A has a decomposition $A = A_1 \oplus A_2$ such that A_1 is projective and A_2 is singular. By Dung et al. (1994, 13.6 and 7.16), every singular R-module is semisimple. Let $C := A_2 \oplus (N \cap B)$. Hence $N = A_1 \oplus C$, where A_1 is projective summand of M and $C \leq Soc(M)$. □

Also note that there exists a QF-ring R such that $J(R) \not\leq Soc({}_RR)$. For example, let $R = \mathbb{Z}_8$. Then $J(R) = 2R$ and $Soc({}_RR) = 4R$. Hence over a QF-ring not every R-module need to be *Soc*-semiperfect.

Theorem 3.7. *Let M be a finitely generated self-projective module which is a self-generator in $\sigma[M]$. Then the following are equivalent*:

(1) *M is a Noetherian QF-module*;
(2) *Every module in $\sigma[M]$ is Z_M-semiperfect in $\sigma[M]$.*

Proof. (1) $\Rightarrow$ (2) Let $N \in \sigma[M]$ be injective in $\sigma[M]$. Then N is projective in $\sigma[M]$. By the proof of Theorem 3.4 (1 $\Rightarrow$ 2), $Z_M(N) = Rad(N)$. Since M is perfect in $\sigma[M]$ we have that N is Z_M-semiperfect in $\sigma[M]$. By Lemma 3.1, every module in $\sigma[M]$ is Z_M-semiperfect in $\sigma[M]$.

(2) $\Rightarrow$ (1) By (2), every module in $\sigma[M]$ is a direct sum of a projective module in $\sigma[M]$ and an M-singular module. Hence M is a co-Harada module by Dung et al. (1994, Corollary 11.11). By Theorem 2.12, $Z_M(M) = Rad(M)$. Hence (1) holds by Jayaraman and Vanaja (2000, Theorem 3.1). □

Corollary 3.8. *The following are equivalent for a ring R*:

(1) *R is a QF-ring*;
(2) *Every R-module is Z-semiperfect.*

Theorem 3.9. *Let M be a module. The following are equivalent*:

(1) *M is semisimple*;
(2) *Every module in $\sigma[M]$ is δ_M-semiperfect in $\sigma[M]$*;
(3) *Every module in $\sigma[M]$ is δ_M-semiregular in $\sigma[M]$.*

Proof. If M is semisimple, then every module N in $\sigma[M]$ is semisimple and projective in $\sigma[M]$. Hence (1) $\Rightarrow$ (2). (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) By the proof of Alkan and Özcan (2004, Theorem 4.2), in $\sigma[M]$ every simple module is projective. Hence M is semisimple by Wisbauer (1991, 20.3). □

Corollary 3.10. *The following are equivalent for a ring R*:

(1) *R is semisimple*;
(2) *Every R-module is δ-semiperfect*;
(3) *Every R-module is δ-semiregular.*

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