SOME CHARACTERIZATIONS OF V-MODULES and RINGS

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Abstract

A module M has the property (V) if for every $K \leq M, K \neq M$ and $m \in M - K$, any submodule L maximal with respect to the property that it contains K but does not contain the element m is maximal in M. It has the property (Ve) if (V) holds for every essential proper submodule K and $m \in M - K$. It is shown that M is a V-module if and only if M has the property (V). $M/\mathrm{Soc}\, M$ is a V-module if and only if M has the property (Ve). Some further characterizations of V-rings and GV-rings are given.

All rings considered are associative, have an identity and all modules are unitary right modules. Let R be a ring and M a module. We write Rad$M$,Z(M), Soc$M$ and E(M) for the radical, the singular submodule, the socle and the injective envelope of M respectively. Let M and N be modules. N is called M-*injective* if for each submodule K of M every homomorphism from K into N can be extended to an R-homomorphism from M into N. M is called a *V-module* by Hirano in [6] (or *cosemisimple* by Fuller [2]) if every proper submodule of M is an intersection of maximal submodules. R is called a *V-ring* if the right module R_R is a V-module. M is a V-module if and only if every simple module is M-injective. Following Hirano [6], M is called a *generalized V-module* or a *GV-module* if every simple singular module is M-injective. If the module R_R is a GV-module R is called a GV-ring.

In this note we give some characterizations of V-modules and GV-modules in terms of certain maximal submodules.

We write $N \leq M$ for N is a submodule of M. A right R-module M is said to have property (V), (Ve) respectively if

(V) For every $K \leq M, K \neq M$ and $m \in M - K$, any submodule L maximal with respect to the property that it contains K but does not contain the element m is maximal in M.

(Ve) (V) holds for every essential proper submodule K and $m \in M - K$.

1. Modules with Properties (V) and (Ve)

In [8] it is proved that R is a right V-ring if and only if the right R-module R has the property (V).

Theorem 1. *Let M be a module. Then the following are equivalent.*
(1) *M is a V-module.*
(2) *M has the property (V).*

Proof. (1)$\Longrightarrow$ (2) Let $K \leq M, K \neq M$, $m \in M - K$ and let L be a maximal submodule with respect to the property that it contains K but does not contain the element m. Then $(mR + L)/L$ is a simple R-module. By (1) it is M-injective and so M/L-injective. Also $(mR + L)/L$ is an essential submodule of M/L. Hence $(mR + L)/L = M/L$. Thus L is a maximal submodule of M.
(2)$\Longrightarrow$ (1) Let X be a simple module, N an essential proper submodule of M, f a non-zero homomorphism from N to X and let $x \in N - kerf$. Let L be a submodule of M maximal with respect to $x \notin L$ and $kerf \leq L$. Then $xR + L = M = N + L$ by (2). Hence $N \cap L$ is maximal in N. Since $kerf$ is a maximal submodule of N, then $N \cap L = kerf$. Thus f extends to M.

Theorem 2. *Let M be a module. Then the following are equivalent.*
(1) *$M/SocM$ is a V-module.*
(2) *M has the property (Ve).*

Proof. (1)$\Longrightarrow$ (2) Let $m \in M$ and let N be an essential submodule of M maximal with respect to $m \notin N$. Then $(mR + N)/N$ is a simple module and essential in M/N. By (1) it is $M/\text{Soc}M$-injective. Since $\text{Soc}M \leq N$, then $(mR + N)/N$ is M/N-injective. Thus $M = mR + N$. This implies that N is a maximal submodule of M.
(2)$\Longrightarrow$ (1) Let X be any simple module. To prove X is $M/\text{Soc}M$-injective, let $N/\text{Soc}M$ be an essential submodule of $M/\text{Soc}M$ and f a non-zero homomorphism from $N/\text{Soc}M$ to X. Set $Kerf = K/\text{Soc}M$ for some $K \leq M$. Then N is essential submodule of M and K is a maximal submodule of N. We consider two cases: Assume K is essential in N. Then K is essential in M. Let $x \in N - K$ and let L be a submodule of M maximal with respect to $x \notin L$ and $K \leq L$. Since K is essential in M, then L is essential in M. By (Ve), L is a maximal submodule of M, and so $M = xR + L = N + L$. Then $N \cap L$ is maximal in N. Hence $K = N \cap L$. Thus $K/\text{Soc}M = (N/\text{Soc}M) \cap (L/\text{Soc}M)$, which is the kernel of f. It follows that f extends to a homomorphism from $M/\text{Soc}M$ to X.

If K is not essential in N, then K is a direct summand of N and $N = K \oplus T$ for some $T \leq N$. Hence $N/K, T$ and X are isomorphic simple modules. It follows that $T \leq$Soc$M$. Since Soc$M \leq K$, then $T = 0$. This is a contradiction which completes the proof.

2. Co-singular Submodule Z$^*(M)$ and V-Rings

A submodule N of M is called *small* in M if whenever $N + L = M$ for some submodule L of M we have $M = L$. A module M is said to be *small* if M is small in E(M) [7]. Let M be an R-module. We set Z$^*(M) = \{m \in M : mR$ is small $\}$. We call Z$^*(M)$ a *co-singular* submodule of M. In this note we consider the classes $\underline{X} = \{R$-module M: Z$^*(M) = 0\}$, $\underline{X}^* = \{R$-module M: whenever $Q \leq P \leq M, P/Q \in \underline{X}$ implies $P/Q = 0\}$, following [5]. Submodules and homomorphic images of small modules are small [7] and $\underline{X}$ is closed under submodules, direct products, direct sums, essential extensions and module extensions. $\underline{X}^*$ is closed under submodules, homomorphic images and direct sums. Any member of $\underline{X}$ is called an $\underline{X}$-module. $\underline{X} \cap \underline{X}^* = 0$, and since Rad$M$ is the sum of all small submodules of M, Rad$M \leq$Z$^*(M)$ and Z$^*(M) = M \cap$Rad$E(M)$. Z$^*(E) =$Rad$E$ for any injective module E. In general Z$^*(M) \neq$Rad$M$ [e.g. Example 11].

Lemma 3. *Let M be a module and $N \leq M$. Then (Z$^*(M) + N)/N$ is a submodule of Z$^*(M/N)$.*

Proof. Let $m \in$Z$^*(M)$. Then mR is small in E(mR) so that $(mR + N)/N$ is small in (E$(mR) + N)/N$. Hence $(m + N)R = (mR + N)/N$ is small. Thus $m + N \in$Z$^*(M/N)$.

Lemma 4. *Let M be a module. Then*
(1) *If M is small then Z$^*(M) = M$,*
(2) *If Z$^*(M) = M$ then $M \in \underline{X}^*$,*
(3) *If M is semisimple injective then $M \in \underline{X}$.*

Proof.(1) Clear from the definitions.
(2) Let M be a module and $Q \leq P \leq M$ be such that Z$^*(M) = M$ and $P/Q \in \underline{X}$. Let $x \in P$. Then xR and $(xR + Q)/Q$ are small and $(xR + Q)/Q \in \underline{X}$. By (1) $(xR + Q)/Q \in \underline{X}^*$. Hence $xR + Q = Q$ and $x \in Q$. Thus $M \in \underline{X}^*$.
(3) Assume M is semisimple injective. Since $\underline{X}$ is closed under direct sums, without loss of generality we may assume M is simple injective. If Z$^*(M) = M$ then M is small in M. This is a contradiction. Hence Z$^*(M) = 0$ and so $M \in \underline{X}$. This completes the proof.

Lemma 5. *For any module M, Z$^*(M) = 0$ if and only if RadE$(M) = 0$.*

Proof. M is essential in E(M).

Proposition 6. *Let R be a ring such that $R/J(R)$ is right Artinian. Then $Z^*(M) = 0$ if and only if M is semisimple injective.*

Proof. Sufficiency is clear from Lemma 4(3). Conversely, suppose that $\mathrm{Z}^*(M) = 0$. Then $0 = \mathrm{Rad}\mathrm{E}(M) = \mathrm{E}(M)\mathrm{J}(R)$. Hence $\mathrm{E}(M)$ is semisimple and so $M = \mathrm{E}(M)$. Thus M is semisimple injective.

Example 7. *Let R be a prime right Goldie ring which is not right primitive (e.g. a commutative domain which is not a field). Then $Z^*(R) = R$.*

Proof. Let $r \in R$ and $E = \mathrm{E}(rR)$. Suppose that $E = rR + L$ for some $L \leq E$. If r is not in L, then E/L is non-zero and a cyclic module so that there exists a maximal submodule P of E with L contained in P. The module $U = E/P$ is simple, and if I is its annihilator in R we know that I is a non-zero ideal of R by our hypothesis. But in this case I contains a non-zero divisor by Goldie's Theorem [4, Proposition 5.9] and then $E = EI$ by [9, Proposition 2.6] so that $E = P$, a contradiction. Hence $r \in L$ and so $E = L$ and rR is small. Thus $\mathrm{Z}^*(R) = R$.

Lemma 8. *Let R be a ring such that $Z^*(R) = R$. Then for every module M, $Z^*(M) = M$.*

Proof. Let M be a module and $m \in M$. Let $r(m)$ denote the right annihilator of m in R. Then $mR \cong R/r(m)$ and $\mathrm{Z}^*(R) = R$ imply that mR is small, and so $m \in \mathrm{Z}^*(M)$.

We combine Example 7 and Lemma 8

Corollary 9. *Let R be a prime right Goldie ring which is not a right primitive ring. Then for every module M, $Z^*(M) = M$.*

Theorem 10. *Let R be a ring. Then the following are equivalent.*
(1) *R is a right GV-ring,*
(2) *Every $\underline{X}^*$-module is projective,*
(3) *Every simple $\underline{X}^*$-module is projective,*
(4) *For every R-module M with $Z^*(M) \neq 0$, $Z^*(M)$ is projective,*
(5) *Every small module is projective,*
(6) *For every R-module M with $Z^*(M) = M$, M contains a non-zero projective submodule,*
(7) *For every R-module M, $Z(M) \cap Z^*(M) = 0$,*
(8) *For every right ideal I of R, $Z(R/I) \cap Z^*(R/I) = 0$,*
(9) *For every R-module M with $Z(M)$ essential in M, $Z^*(M) = 0$,*
(10) *$R/SocR$ is a V-module and $Z(R) \cap Z^*(R) = 0$,*
(11) *Every proper essential right ideal of R is an intersection of maximal right ideals and $Z(R) \cap Z^*(R) = 0$,*
(12) *For every essential right ideal K of R, $Z^*(R/K) = 0$ and $Z(R) \cap Z^*(R) = 0$.*

Proof. **(1)$\Longrightarrow$ (2)** Let $M \in \underline{X}^*$ and $m \in M, m \neq 0$. Let K be a maximal submodule of mR. Then mR/K is injective or projective. If mR/K is injective, then by Lemma 4(3) $mR/K \in \underline{X}$. Hence $mR/K = 0$. Thus it is projective. It follows that K is a direct summand of mR, and so mR is semisimple and so too is M. As before it can be shown that every simple submodule of M is projective.
(2)$\Longrightarrow$ (3) Clear.
(3)$\Longrightarrow$ (4) Since $\mathrm{Z}^*(M)$ is in $\underline{X}^*$ by Lemma 4(2) and every simple module is injective or small; the proof is the same as that of (1$\Longrightarrow$ 2).
(4)$\Longrightarrow$ (5) Let M be a non-zero small module. Then $\mathrm{Z}^*(M) = M$ by Lemma 4(1). Thus M is projective by (4).
(5)$\Longrightarrow$ (6) Let M be a module with $\mathrm{Z}^*(M) = M$. Let $m \in M, m \neq 0$. Since mR is small, then mR is projective by (5).
(6)$\Longrightarrow$ (7) Let $m \in \mathrm{Z}(M) \cap \mathrm{Z}^*(M)$. Then $\mathrm{Z}^*(mR) = mR$. Assume $m \neq 0$. Then by (6), mR contains a non-zero projective submodule L. Hence L is isomorphic to $I/r(m)$ for some right ideal I of R. Thus $r(m)$ is a direct summand of I. But, since $m \in \mathrm{Z}(M), r(m)$ is essential in R, and so in I, then $L = 0$. A contradiction.
(7)$\Longrightarrow$ (8) Clear.
(8)$\Longrightarrow$ (9) Let M be a module with $\mathrm{Z}(M)$ essential in M. Let $x \in \mathrm{Z}^*(M)$. Assume $x \neq 0$. There exists a non-zero $m \in xR \cap \mathrm{Z}(M)$. Then $mR \leq \mathrm{Z}^*(M) \cap \mathrm{Z}(M)$. Hence $mR \cong R/r(m) \leq \mathrm{Z}^*(R/r(m)) \cap \mathrm{Z}(R/r(m))$ which is zero by (8). This is a contradiction.
(9)$\Longrightarrow$ (10) Let X be a simple module, $I/\mathrm{Soc}R$ a right ideal of $R/\mathrm{Soc}R$ and f a non-zero homomorphism from $I/\mathrm{Soc}R$ to X. Set $Kerf = K/\mathrm{Soc}R$ for some right ideal K of R. Then K is a maximal right ideal of I. If K is not essential in I then $I = K \oplus T$ for some $T \leq I$. Hence $T \leq \mathrm{Soc}R \leq K$. This is a contradiction. It follows that K is essential in I, and so I/K is singular. By (9) $\mathrm{Z}^*(I/K) = 0$, and then $\mathrm{Z}^*(X) = 0$. Since X is simple then X is injective and so $R/\mathrm{Soc}R$-injective. It follows that f extends to $R/\mathrm{Soc}R$.
(10)$\Longleftrightarrow$ (11) $R/\mathrm{Soc}R$ is a V-module if and only if every proper essential right ideal of R is an intersection of maximal right ideals [10].
(11)$\Longrightarrow$ (12) Let K be an essential right ideal of R Let $0 \neq x + K \in \mathrm{Z}^*(R/K)$. By (11) there exists a maximal right ideal L of R such that $x \notin L$ and $K \leq L$. Then $(xR + L)/L$ is small and a singular module. Next we prove $(xR + L)/L$ is an injective module. Let I be an essential right ideal of R and f a non-zero homomorphism from I to $(xR + L)/L$. Set $T = Kerf$. Assume T is essential in I. Then T is an essential right ideal in R. By (11) we may find a maximal right ideal J of R so that $T \leq J$ and $I \not\leq J$. Hence $R = I + J$. Since $T \leq I \cap J \leq I$ and $I \not\leq J$, then $T = I \cap J$, and so f extends. If T is not an essential right ideal in I, then $I = T \oplus U$ for some right ideal U of R. Hence U is a simple singular and small module. Thus $U \leq \mathrm{Z}(R) \cap \mathrm{Z}^*(R)$ that is zero. This is a contradiction for f a non-zero mapping. It follows that $(xR + L)/L$ is an injective module. This is a contradiction for $(xR + L)/L$ is a small module. Hence $\mathrm{Z}^*(R/K) = 0$.
(12)$\Longrightarrow$ (1) Let X be a simple singular module and I an essential right ideal of R. Let f be a non-zero homomorphism from I to X with kernel K. Then K is a maximal submodule of I. If K is not essential in I then $I = K \oplus L$ for some $L \leq I$. Then L is a simple singular right ideal of R. Hence $L^2 = 0$ or $L = eR$ for

some idempotent e of R. Assume $L = eR$. Then $r(e) = (1-e)R$ is essential in R. This is a contradiction. Hence $L^2 = 0$, and so $L \leq \mathrm{Rad}R$. Since L is singular then $L \leq \mathrm{Z}(R)$. Since $\mathrm{Rad}R \leq \mathrm{Z}^*(R)$, then by (12), $L = 0$. Hence K is essential in I and so too in R. By (12), $\mathrm{Z}^*(R/K) = 0$ and so $\mathrm{Z}^*(I/K) = 0$. This and I/K simple imply I/K is injective. Since $X \cong I/K$, X is injective. This completes the proof.

Example 11. Let $R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$ be lower triangular matrices over a field F. $J(R) = \begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}$, $Soc(R_R) = \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$ and by [1, Example 4.b] R is a right and left GV-ring and not a V-ring. $\mathrm{Z}^*(R)$ is semisimple by the proof of Theorem 10(1$\Rightarrow$ 2) and $\mathrm{J}(R) \leq \mathrm{Z}^*(R) \leq \mathrm{Soc}R$. Set $K = \begin{bmatrix} 0 & 0 \\ F & F \end{bmatrix}$ and $L = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$. By [3, Exercise 3.B.20-21] K is an injective right ideal and every injective right ideal of R is contained in K. Since the simple right ideal L is injective or small, and L is not in K, then L is small. Hence $\mathrm{Z}^*(R) = \mathrm{Soc}(R_R)$ and $\mathrm{J}(R) \neq \mathrm{Z}^*(R)$.

Theorem 12. *Let R be a ring. Then the following are equivalent.*
(1) *R is a right V-ring.*
(2) *For every R-module M, $Z^*(M) = 0$.*
(3) *For every simple R-module M, $Z^*(M) = 0$.*

Proof.(1)$\Longrightarrow$ (2) By (1), $\mathrm{RadE}(M) = 0$. Hence $\mathrm{Z}^*(M) = 0$.
(2)$\Longrightarrow$ (3) Clear.
(3)$\Longrightarrow$ (1) Let M be a simple module. By (3), $\mathrm{Z}^*(M) = 0$. Since M is simple, then M is injective or small. Assume M is small, then by Lemma 4, $M \in \underline{X}^*$. This is a contradiction. Hence M is injective.

We combine Theorem 1 and Theorem 12

Corollary 13. *Let R be a ring. Then, R_R has the property (V) if and only if $Z^*(M) = 0$ for every R-module M.*

Acknowledgement. I am indepted to Professor P.F.Smith (University of Glasgow) and my supervisor Professor A.Harmancı for his very helpful comments and suggestions.

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