The Torsion Theory Generated By M-Small Modules

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Abstract

Let M be a right R-module and $\mathcal{M}$ the class of all M-small modules. We consider the torsion theories $\tau_{\mathcal{M}} = (\mathcal{T}_M, \mathcal{F}_{\mathcal{M}})$, $\tau_V = (\mathcal{T}_V, \mathcal{F}_V)$ and $\tau_P = (\mathcal{T}_P, \mathcal{F}_P)$ in $\sigma[M]$ where $\tau_{\mathcal{M}}$ is the torsion theory generated by $\mathcal{M}$, τ_V is the torsion theory cogenerated by $\mathcal{M}$ and τ_P is the dual Lambek torsion theory where P denotes a projective cover of M in $\sigma[M]$. We study some conditions for $\tau_{\mathcal{M}}$ to be cohereditary, stable or split, and we prove that $\text{Rej}(M, \mathcal{M}) = M \Leftrightarrow \mathcal{F}_P = \mathcal{M}(= \mathcal{T}_{\mathcal{M}} = \mathcal{F}_V) \Leftrightarrow \mathcal{T}_P = \mathcal{T}_V \Leftrightarrow \text{Gen}_M(P) \subseteq \mathcal{T}_V$.

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Introduction

Let R be an associative ring with identity and M a right R-module. An R-module N is *subgenerated* by M if N is isomorphic to a submodule of an M-generated module. $\sigma[M]$ denotes the full subcategory of Mod-R whose objects are all R-modules subgenerated by M. Let $N \in \sigma[M]$. An injective module E in $\sigma[M]$ together with an essential monomorphism $\varepsilon : N \to E$ is called an injective hull of N in $\sigma[M]$ or an *M-injective hull* of N and is usually denoted by $\widehat{N}$. E(M) is the R-injective hull of M. (see [17] or [4])

We use the notation $N \leq_e M$ for an essential submodule N of M. A module N in $\sigma[M]$ is called *M-singular* (or *singular in* $\sigma[M]$) if $N \cong L/K$ for an $L \in \sigma[M]$ and $K \leq_e L$ (see [4]). In case $M = R$, instead of R-singular, we just say *singular*. Every module $N \in \sigma[M]$ contains a largest M-singular submodule which is denoted by $Z_M(N)$. Simple modules are M-singular or M-projective.

Let K be a submodule of M. K is called *small* in M if $K+L\neq M$ holds for every proper submodule L of M and denoted by $K\ll M$. We write Rad$M$, which is the sum of all small submodules in M, for the radical of M. An R-module N in $\sigma[M]$ is called M-*small* (or *small in* $\sigma[M]$) if $N\cong K\ll L$ for $K,L\in\sigma[M]$. In case $M=R$, instead of R-small, we just say *small.* We denote the class of all M-small modules by $\mathcal{M}$. An R-module N is M-small if and only if $N\ll\widehat{N}$. Every simple R-module is M-injective or M-small [7, 5.1.4]. $\mathcal{M}$ is closed under submodules, factor modules and finite direct sums [7].

Let M be a module and $\mathcal{C}$ a class of modules in $\sigma[M]$ closed under isomorphisms and submodules. For any $N\in\sigma[M]$ the *trace* of $\mathcal{C}$ in N is denoted by $\mathrm{Tr}(\mathcal{C},N)=\sum\{\mathrm{Im}f: f\in\mathrm{Hom}(C,N), C\in\mathcal{C}\}$. Let

$\mathcal{F}=\{F\in\sigma[M]:\forall C\in\mathcal{C},\mathrm{Hom}(C,F)=0\}$

$\mathcal{T}=\{T\in\sigma[M]:\forall F\in\mathcal{F},\mathrm{Hom}(T,F)=0\}$.

Then $\tau=(\mathcal{T},\mathcal{F})$ is a torsion theory *generated* by $\mathcal{C}$. Also it can be seen that

$\mathcal{F}=\{F\in\sigma[M]:\mathrm{Tr}(\mathcal{C},F)=0\}$

$\mathcal{T}=\{T\in\sigma[M]:\forall U<V\leq T,\mathrm{Tr}(\mathcal{C},V/U)\neq 0\}$.

Since $\mathcal{C}$ is closed under isomorphisms and submodules, τ is a hereditary torsion theory (see [3, II 1.3]). τ is called *stable* if $\mathcal{T}$ is closed under essential extensions in $\sigma[M]$, i.e. if every essential extension $E\in\sigma[M]$ of a torsion module $N\in\sigma[M]$ is again torsion. τ is *splitting* if every R-module N has a decomposition $N=N_1\oplus N_2$ such that $N_1\in\mathcal{T}$ and $N_2\in\mathcal{F}$. $\tau_{\mathcal{C}}(N)=\mathrm{Tr}(\mathcal{T},N)$ is a torsion radical and $\mathrm{Tr}(\mathcal{C},N)\leq_e\tau_{\mathcal{C}}(N)$. Also $\tau_{\mathcal{C}}(N)=\sum\{K\leq N:\forall U\leq V\leq K, V/U\notin\mathcal{F}\}$ [6].

Small modules are dual of singular modules. In this respect the dual of the Goldie torsion theory is the torsion theory generated by small modules which is introduced by Ramamurthi [14]. In [11] and [8] instances are given where this torsion theory is cohereditary or stable or splits.

In this paper we consider the dual Goldie torsion theory in $\sigma[M]$, the torsion theory generated by M-small modules for a right R-module M. We give some equivalent conditions for this torsion theory to be cohereditary, stable or split and investigate the coincidence of this torsion theory and the torsion theory cogenerated by M-small modules which is studied by Talebi and Vanaja [16]. Also we consider the dual Lambek torsion theory in $\sigma[M]$ for a module M having projective cover. Finally we give equivalent conditions for a module M to be a GCO-module which is a generalization of a GV-module.

Now define $\mathrm{Z}^*_M(N)=\{n\in N: nR \text{ is an } M\text{-small module}\}$ for an R-module M and $N\in\sigma[M]$. In case $M=R$, we write $\mathrm{Z}^*(N)$ instead of $\mathrm{Z}^*_R(N)$ which is studied in [5], [11] and [12]. Let $N\in\sigma[M]$. Then $\mathrm{Rad}N\leq\mathrm{Z}^*_M(N)\leq\mathrm{Z}^*(N)$ and $\mathrm{Z}^*_M(N)=\mathrm{Rad}\widehat{N}\cap N$. For any submodule $K\leq N$, $\mathrm{Z}^*_M(K)=K\cap\mathrm{Z}^*_M(N)$. If $f:N\to K$ is a homomorphism of modules N,K in $\sigma[M]$, then $f(\mathrm{Z}^*_M(N))\leq\mathrm{Z}^*_M(K)$. Let N_i $(i\in I)$ be any collection of modules in $\sigma[M]$. Then $\mathrm{Z}^*_M(\oplus_{i\in I}N_i)=\oplus_{i\in I}\mathrm{Z}^*_M(N_i)$. If M is semisimple, then $\mathrm{Z}^*_M(N)=0$ for any $N\in\sigma[M]$. [13]

It is easy to see that

$$\mathrm{Z}^*_M(N) = \mathrm{Tr}(\mathcal{M}, N).$$

Then the torsion theory in $\sigma[M]$ generated by $\mathcal{M}$ is $\tau_{\mathcal{M}} = (\mathcal{T}_{\mathcal{M}}, \mathcal{F}_{\mathcal{M}})$ where
$\mathcal{T}_{\mathcal{M}} = \{N \in \sigma[M] : \forall U < V \leq N, \mathrm{Z}^*_M(V/U) \neq 0\}$
$\mathcal{F}_{\mathcal{M}} = \{N \in \sigma[M] : \mathrm{Z}^*_M(N) = 0\}$.

Since $\mathcal{M}$ is closed under isomorphisms and submodules, $\tau_{\mathcal{M}}$ is a hereditary torsion theory. If M is semisimple, then $\mathcal{F}_{\mathcal{M}} = \sigma[M]$ and $\mathcal{T}_{\mathcal{M}} = \{0\}$.

Note that for $N \in \sigma[M]$, $\mathrm{Z}^*_M(N) \leq_e \tau_{\mathcal{M}}(N)$.

Let $\tau_{dG} = (\mathcal{T}_{dG}, \mathcal{F}_{dG})$ be the Dual Goldie Torsion Theory in Mod-R. It is easy to see that

$$\mathcal{T}_{\mathcal{M}} \subseteq \mathcal{T}_{dG} \text{ and } \mathcal{F}_{dG} \cap \sigma[M] \subseteq \mathcal{F}_{\mathcal{M}}.$$

Let $\mathcal{C}$ be a class of modules in $\sigma[M]$. For any N in $\sigma[M]$ the *reject* of $\mathcal{C}$ in N is denoted by $\mathrm{Rej}(N, \mathcal{C}) = \cap\{\mathrm{Ker}g \mid g \in \mathrm{Hom}(N, C),\ C \in \mathcal{C}\}$. The torsion theory *cogenerated* by a class $\mathcal{C}$ of modules in $\sigma[M]$ is $\tau_c = (\mathcal{T}_c, \mathcal{F}_c)$ where
$\mathcal{T}_c = \{T \in \sigma[M] : \forall C \in \mathcal{C}, \mathrm{Hom}(T, C) = 0\}$
$\mathcal{F}_c = \{F \in \sigma[M] : \forall T \in \mathcal{T}_c, \mathrm{Hom}(T, F) = 0\}$.

If $\mathcal{C}$ is closed under isomorphisms and submodules then
$\mathcal{T}_c = \{T \in \sigma[M] : \mathrm{Rej}(T, \mathcal{C}) = T\}$
$\mathcal{F}_c = \{F \in \sigma[M] : \forall 0 \neq U \leq F,\ \mathrm{Rej}(U, \mathcal{C}) \neq U\}$.

When is $\tau_{\mathcal{M}}$ Stable or Splitting?

Proposition 1 *Let M be a module. $\tau_{\mathcal{M}}$ is stable if and only if every M-injective module N in $\sigma[M]$ has a decomposition $N = N_1 \oplus N_2$ such that $N_1 \in \mathcal{T}_{\mathcal{M}}$ and $N_2 \in \mathcal{F}_{\mathcal{M}}$.*

Proof $(\Rightarrow)$ Assume that $\tau_{\mathcal{M}}$ is stable. Let N be an M-injective module. Then $N = \widehat{N}$. Let K be a submodule of N such that $N = \widehat{\tau_{\mathcal{M}}(N)} \oplus K$. By assumption $\widehat{\tau_{\mathcal{M}}(N)} \in \mathcal{T}_{\mathcal{M}}$. Since $\tau_{\mathcal{M}}(N) = \widehat{\tau_{\mathcal{M}}(N)}$, $K \in \mathcal{F}_{\mathcal{M}}$.
$(\Leftarrow)$ Let $N \in \mathcal{T}_{\mathcal{M}}$. It is enough to show that $\widehat{N} \in \mathcal{T}_{\mathcal{M}}$. Let $\widehat{N} = N_1 \oplus N_2$ where $N_1 \in \mathcal{T}_{\mathcal{M}}, N_2 \in \mathcal{F}_{\mathcal{M}}$. $N_2 \cap N \in \mathcal{T}_{\mathcal{M}} \cap \mathcal{F}_{\mathcal{M}} = 0$ implies that $N_2 = 0$. So $\widehat{N} \in \mathcal{T}_{\mathcal{M}}$.
□

Hence if $\tau_{\mathcal{M}}$ is splitting then it is stable. Note that for a module M if $N/\mathrm{Rad}N$ is semisimple then $N/\mathrm{Z}^*_M(N)$ and hence $N/\tau_{\mathcal{M}}(N)$ is semisimple for any $N \in \sigma[M]$.

Proposition 2 *Let M be a module and $N \in \sigma[M]$ be such that $N/\tau_{\mathcal{M}}(N)$ is semisimple. Then every $\mathcal{F}_{\mathcal{M}}$-module is N-injective.*

Proof By [6, Corollary 2.3]. □

Proposition 3 *Let M be a module. If $M/\tau_{\mathcal{M}}(M)$ is semisimple, then every $\mathcal{F}_{\mathcal{M}}$-module is semisimple and M-injective.*

Proof Let $K \in \mathcal{F}_{\mathcal{M}}$. By Proposition 2, K is M-injective, i.e. injective in $\sigma[M]$. Let $X \leq K$. Then $X \in \mathcal{F}_{\mathcal{M}}$ and by Proposition 2 X is M-injective. Since $K \in \sigma[M]$, X is K-injective. Hence X is a direct summand of K. This implies that K is semisimple. □

Proposition 4 *Let M be a module such that $M/\tau_{\mathcal{M}}(M)$ is semisimple. Then every module N in $\sigma[M]$ has a decomposition $N = N_1 \oplus N_2$ such that $N_1 \in \mathcal{F}_{\mathcal{M}}$ and $\tau_{\mathcal{M}}(N_2) \leq_e N_2$.*

Proof Let $N \in \sigma[M]$ and N_1 a submodule maximal with respect to $N_1 \cap \tau_{\mathcal{M}}(N) = 0$. Then $N_1 \oplus \tau_{\mathcal{M}}(N) \leq_e N$ and $\tau_{\mathcal{M}}(N_1) = N_1 \cap \tau_{\mathcal{M}}(N) = 0$, i.e. $N_1 \in \mathcal{F}_{\mathcal{M}}$. By hypothesis N_1 is M-injective and then N-injective. So there exists a submodule N_2 such that $N = N_1 \oplus N_2$. Since $\tau_{\mathcal{M}}(N_1) = 0$, $\tau_{\mathcal{M}}(N) = \tau_{\mathcal{M}}(N_2)$. Then $(N_1 \oplus \tau_{\mathcal{M}}(N_2)) \cap N_2 \leq_e N_2$. This implies that $\tau_{\mathcal{M}}(N_2) \leq_e N_2$. □

Let M be a module. A module N is said to be *M-generated* (resp. *M-cogenerated*) if there exist an index set I and an epimorphism from $M^{(I)}$ to N (resp. a monomorphism from N to $\prod_{\Lambda}^{M} M_\lambda, M_\lambda = M$, a direct product of copies of M in $\sigma[M]$ [17, 15.1]). For any $N \in \sigma[M]$, the class of all objects in $\sigma[M]$ which are generated (resp. cogenerated) by N is denoted by $\mathrm{Gen}_M(N)$ (resp. $\mathrm{Cog}_M(N)$).

Theorem 5 *Let M be a module such that $M/\tau_{\mathcal{M}}(M)$ is semisimple. Consider the following conditions.*
(1) $\tau_{\mathcal{M}}$ is splitting,
(2) $\tau_{\mathcal{M}}$ is stable,
(3) every $\mathcal{F}_{\mathcal{M}}$-module is projective in $\sigma[M]$,
(4) every module $N \in \sigma[M]$ has a decomposition $N_1 \oplus N_2$ such that N_1 is a $\mathcal{T}_{\mathcal{M}}$-module and N_2 is semisimple,
(5) every simple M-injective module in $\sigma[M]$ is projective in $\sigma[M]$,
(6) every M-singular module in $\sigma[M]$ is a $\mathcal{T}_{\mathcal{M}}$-module,
(7) M cogenerates all M-injective simple modules in $\sigma[M]$.

Then (1)-(6) are all equivalent, (5)⇒(7) and if M is projective in $\sigma[M]$, then (7)⇒(5).

Proof (1 $\Rightarrow$ 2) By Proposition 1.
(2 $\Rightarrow$ 1) By Proposition 4.
(2 $\Rightarrow$ 3) Assume that $\tau_{\mathcal{M}}$ is stable. Let $N \in \mathcal{F}_{\mathcal{M}}$. By hypothesis N is semisimple M-injective. Let S be a simple M-singular submodule of N. Then $S \cong K/L$ where $L \leq_e K \in \sigma[M]$. Let $H := \tau_{\mathcal{M}}(K)$. Since $H + L/L \leq \tau_{\mathcal{M}}(K/L) = 0$, $H \leq L$. Let X be a submodule of K maximal with respect to $H \cap X = 0 = \tau_{\mathcal{M}}(X)$. Then $H \oplus X \leq_e K$. Now $\widehat{H} \oplus X = \widehat{K}$ and then $K = X \oplus (\widehat{H} \cap K)$. Since $\mathcal{T}_{\mathcal{M}}$ is closed under essential extensions, $K \cap \widehat{H} \in \mathcal{T}_{\mathcal{M}}$. This implies that $H = K \cap \widehat{H}$. Then $K = X \oplus H$ and so $L = (X \cap L) \oplus H$. Since X is semisimple, $X = (X \cap L) \oplus T$ for some T. Hence $K = X \oplus H = (X \cap L) \oplus T \oplus H = L \oplus T$. This is a contradiction to that $L \leq_e K$. Now S is M-projective, that is projective in $\sigma[M]$. It follows that N is projective in $\sigma[M]$.
(3 $\Rightarrow$ 1) Let $N \in \sigma[M]$. Since $N/\tau_{\mathcal{M}}(N) \in \mathcal{F}_{\mathcal{M}}$, it is projective. Let K be a submodule of N such that $N = \tau_{\mathcal{M}}(N) \oplus K$. Then $\tau_{\mathcal{M}}(N) \cap K = \tau_{\mathcal{M}}(K) = 0$, i.e. $K \in \mathcal{F}_{\mathcal{M}}$.
(1 $\Rightarrow$ 4) Clear by Proposition 3.
(4 $\Rightarrow$ 3) Let N be an $\mathcal{F}_{\mathcal{M}}$-module. To show that N is projective consider the epimorphism $f : X \to N$ where $X \in \sigma[M]$. Let $X = X_1 \oplus X_2$ where X_1 is a $\mathcal{T}_{\mathcal{M}}$-module and X_2 is semisimple. Then $X_1/X_1 \cap \mathrm{Ker}f \cong X_1 + \mathrm{Ker}f/\mathrm{Ker}f \leq X/\mathrm{Ker}f \cong N$ implies that $X_1/X_1 \cap \mathrm{Ker}f \in \mathcal{T}_{\mathcal{M}} \cap \mathcal{F}_{\mathcal{M}} = 0$. Then $X_1 \leq \mathrm{Ker}f \leq X$. Now $\mathrm{Ker}f = X_1 \oplus (X_2 \cap \mathrm{Ker}f)$, and $X_2 = L \oplus (X_2 \cap \mathrm{Ker}f)$ for some $L \leq X_2$. Then $X = \mathrm{Ker}f \oplus L$. Hence $\mathrm{Ker}f$ is a direct summand of X, i.e. f splits. This implies that N is projective in $\sigma[M]$.
(3 $\Rightarrow$ 5) Simple M-injective modules are $\mathcal{F}_{\mathcal{M}}$-module.
(5 $\Rightarrow$ 3) Let $N \in \mathcal{F}_{\mathcal{M}}$. Then N is semisimple M-injective by Proposition 3. Since every simple summand of N is projective by (5), N is projective.
(3 $\Rightarrow$ 6) Let N be an M-singular module in $\sigma[M]$. To show that $N \in \mathcal{T}_{\mathcal{M}}$, let $F \in \mathcal{F}_{\mathcal{M}}$ and $f : N \to F$ a homomorphism. Then $N/\mathrm{ker}f \cong f(N) \leq F \in \mathcal{F}_{\mathcal{M}}$. By hypothesis, $N/\mathrm{ker}f$ is projective in $\sigma[M]$. Since $N/\mathrm{ker}f$ is M-singular, we have that $f = 0$.
(6 $\Rightarrow$ 5) Let N be a simple M-injective module in $\sigma[M]$. Then $N \in \mathcal{F}_{\mathcal{M}}$. The simple module N is M-singular or M-projective. If N is M-singular, then N is a $\mathcal{T}_{\mathcal{M}}$-module, a contradiction. So N is M-projective. Since N is finitely generated, N is projective in $\sigma[M]$.
(5 $\Rightarrow$ 7) Let N be a simple M-injective module in $\sigma[M]$. By (5) N is projective. Then N is a submodule of a direct sum of copies of M by [17, 18.4]. Since N is simple, N is isomorphic to a submodule of M.
(7 $\Rightarrow$ 5) Assume that M is projective in $\sigma[M]$. Let $N \in \sigma[M]$ be a simple M-injective module. Since N is cogenerated by M, N is isomorphic to a direct summand of M. Hence N is projective in $\sigma[M]$. $\square$

A module M is called a *V-module* (or *co-semisimple*) if every simple module (in $\sigma[M]$) is M-injective. M is a V-module if and only if $\mathrm{Rad}(M/K) = 0$ for every $K \leq M$.

A module M is called a *Kasch module* if $\widehat{M}$ is an (injective) cogenerator in $\sigma[M]$, i.e. if every module in $\sigma[M]$ is $\widehat{M}$-cogenerated, [1]. M is a Kasch module if and only if any simple module in $\sigma[M]$ is cogenerated by M [1, Proposition 2.6].

Theorem 6 *Let M be a module. Then $\tau_{\mathcal{M}}$ is splitting if one of the following holds.*
(1) M is a V-module,
(2) Every $\mathcal{F}_{\mathcal{M}}$-module is projective in $\sigma[M]$.
(3) M is local and every simple module in $\sigma[M]$ is M-generated.
(4) M is a projective Kasch module and $M/\tau_{\mathcal{M}}(M)$ is semisimple.

Proof (1) M is a V-module if and only if $\mathcal{F}_{\mathcal{M}} = \sigma[M]$ by [13, Theorem 3].
(2) By the proof of Theorem 5.
(3) If $M/\mathrm{Rad}M$ is M-small simple, then $M \in \mathcal{T}_{\mathcal{M}}$. Hence every module N in $\sigma[M]$ is in $\mathcal{T}_{\mathcal{M}}$, i.e. $\mathcal{T}_{\mathcal{M}} = \sigma[M]$.

Assume that $M/\mathrm{Rad}M$ is simple M-injective. Now we show that M is a V-module. Let N be a simple module in $\sigma[M]$. Let f be an epimorphism $M^{(\Lambda)} \to N$. Then $M^{(\Lambda)}/\mathrm{Ker}f \cong N$ is simple. It follows that $\mathrm{Rad}M \leq \mathrm{Ker}f$. Since $(M + \mathrm{Ker}f)/\mathrm{Ker}f$ is a homomorphic image of $M/\mathrm{Rad}M$ which is M-injective simple, $(M + \mathrm{Ker}f)/\mathrm{Ker}f$ is simple M-injective. Since $M^{(\Lambda)}/\mathrm{Ker}f$ is simple, $(M + \mathrm{Ker}f)/\mathrm{Ker}f = M^{(\Lambda)}/\mathrm{Ker}f$. This implies that N is M-injective. Hence M is a V-module and then M is simple. So under the assumptions of (3) either M is simple or $\mathcal{T}_{\mathcal{M}} = \sigma[M]$ (compare with [8, Proposition 3.8]).
(4) It is clear by Theorem 5 (7). □

Proposition 7 *Let M be a module. If $M/\tau_{\mathcal{M}}(M)$ is semisimple then $\tau_{\mathcal{M}} = (\mathcal{T}_{\mathcal{M}}, \mathcal{F}_{\mathcal{M}})$ is the same as the torsion theory cogenerated by simple M-injective modules.*

Proof By definitions and Proposition 3. □

Is $\tau_{\mathcal{M}}$ Cohereditary?

$\tau_{\mathcal{M}}$ is not *cohereditary*, i.e. $\mathcal{F}_{\mathcal{M}}$ is not closed under factor modules in general:

Example 8 *There exist a module M which is not semisimple, $N \in \sigma[M]$ and $L \leq N$ such that $Z^*_M(N) = 0$ and $Z^*_M(N/L) \neq 0$.*

Proof Let R be the full ring of linear transformations on a vector space V_F of dimension $\aleph$ over a field F. Suppose that $\aleph$ is infinite and $|F| \leq 2^{\aleph_0}$. Then R is a regular right self-injective ring and any simple injective right R-module is isomorphic to a right ideal of R [10, Theorem 2].

Since R is not semiprime Artinian, there exists a proper essential right ideal E of R. Let L be a maximal right ideal of R such that $E \leq L$. Then R/L is a simple non-injective right R-module [12, Example 2.10]. So $\mathrm{Z}^*(R_R) = \mathrm{Rad}R_R = 0$ but $\mathrm{Z}^*(R/L) = R/L$. □

If M is a V-module then $\tau_{\mathcal{M}}$ is cohereditary. And if $M/\tau_{\mathcal{M}}(M)$ is semisimple for a module M, then $\tau_{\mathcal{M}}$ is cohereditary by Proposition 3.

Let $\mathcal{C}$ be a class of modules in $\sigma[M]$ such that it is closed under direct sums and factor modules. A module $N \in \sigma[M]$ is called *$(M, \mathcal{C})$-injective* if N is injective with respect to every exact sequence $0 \to K \to L$ in $\sigma[M]$ with $L/K \in \mathcal{C}$. If $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory in $\sigma[M]$, then $N \in \sigma[M]$ is $(M, \mathcal{T})$-injective if and only if $\widehat{N}/N \in \mathcal{F}$; [18, 9.11]. The corresponding proposition to the following result in Mod-R is Proposition 4.5 in [8].

Proposition 9 *Let M be a module. The following are equivalent.*
(1) $\tau_{\mathcal{M}}$ is cohereditary,
(2) every $\mathcal{F}_{\mathcal{M}}$-module is $(M, \mathcal{T}_{\mathcal{M}})$-injective,
(3) for every $N \in \mathcal{F}_{\mathcal{M}}$, $\widehat{N}/N \in \mathcal{F}_{\mathcal{M}}$.

If one of the above conditions holds then every $\mathcal{F}_{\mathcal{M}}$-module is a V-module.

Proof $(2 \Leftrightarrow 3)$ By [18, 9.11]. $(1 \Rightarrow 3)$ It is clear.
$(3 \Rightarrow 1)$ Let $N \in \mathcal{F}_{\mathcal{M}}$ and $K \leq N$. Consider the exact sequence

$$0 \to \widehat{K}/K \to \widehat{N}/K \to \widehat{N}/\widehat{K} \to 0.$$

Let T be a submodule of $\widehat{N}$ such that $\widehat{N} = \widehat{K} \oplus T$. Since $\mathrm{Z}^*_M(X) = 0 \Leftrightarrow \mathrm{Rad}\widehat{X} = 0$ for any $X \in \sigma[M]$, $\mathcal{F}_{\mathcal{M}}$ is closed under essential extensions. Then $T \in \mathcal{F}_{\mathcal{M}}$, i.e. $\widehat{N}/\widehat{K} \in \mathcal{F}_{\mathcal{M}}$. On the other hand by (3) $\widehat{K}/K \in \mathcal{F}_{\mathcal{M}}$. Since $\mathcal{F}_{\mathcal{M}}$ is closed under extensions, $\widehat{N}/K \in \mathcal{F}_{\mathcal{M}}$. This implies that $N/K \in \mathcal{F}_{\mathcal{M}}$. □

Let M be a module and consider the torsion theory $\tau_V = (\mathcal{T}_V, \mathcal{F}_V)$ cogenerated by $\mathcal{M}$. This torsion theory is investigated by Talebi and Vanaja [16]. They denoted $\overline{Z}_M(N) :=$ Rej$(N, \mathcal{M})$. Then

$$\mathcal{T}_V = \{A \in \sigma[M] : \ \overline{Z}_M(A) = A\}$$
$$\mathcal{F}_V = \{B \in \sigma[M] : \forall 0 \neq K \leq B, \ \overline{Z}_M(K) \neq K\}.$$

$\mathcal{M} \subseteq \mathcal{F}_V$ and τ_V is not necessarily hereditary [16].

Proposition 10 *$\mathcal{F}_{\mathcal{M}} = \mathcal{T}_V$ if and only if $\tau_{\mathcal{M}}$ is cohereditary and τ_V is hereditary.*

Proof It is clear by definitions, and compare with [8, Lemma 2.2]. □

When Is $\mathcal{T_M}$ Equal To $\{N \in \sigma[M] : \mathbf{Z}^*_M(N) = N\}$?

Let M be a module. A module $N \in \sigma[M]$ is called *hereditary* if every submodule of N is projective in $\sigma[M]$. Then a hereditary module in $\sigma[M]$ is itself projective in $\sigma[M]$.

Proposition 11 *Let M be a module. If M is hereditary, then*

$$\mathcal{T_M} = \{N \in \sigma[M] : Z^*_M(N) = N\}.$$

Proof It is clear that the given class is a subclass of $\mathcal{T_M}$. For the converse, let $N \in \mathcal{T_M}$ and $n \in N \setminus \mathrm{Z}^*_M(N)$. Then nR is not small in $\widehat{nR}$. Let L be a submodule of $\widehat{nR}$ such that $\widehat{nR} = nR + L$. Then $\widehat{nR}/L \cong nR/nR \cap L$ is injective by [17, 39.6]. Let $K/nR \cap L$ be a maximal submodule of $nR/nR \cap L$. Then nR/K is simple injective, i.e. $nR/K \in \mathcal{F_M}$. Since $\mathcal{T_M}$ is closed under submodules and factor modules, $nR/K \in \mathcal{T_M} \cap \mathcal{F_M} = \{0\}$. This contradicts to that $K \neq nR$. □

Let M be a module. Assume that M has a projective cover P in $\sigma[M]$ and consider the torsion theory generated by P, $\tau_P = (\mathcal{T}_P, \mathcal{F}_P)$ where
$\mathcal{F}_P = \{F \in \sigma[M] : \mathrm{Hom}(P, F) = 0\}$
$\mathcal{T}_P = \{T \in \sigma[M] : \forall F \in \mathcal{F}_P,\ \mathrm{Hom}(T, F) = 0\}$.

This is cohereditary and the dual Lambek torsion theory in $\sigma[M]$ (see [1]). Since $P \in \mathcal{T}_P$, $\mathrm{Gen}_M(P) \subseteq \mathcal{T}_P$. And

$$\mathcal{M} \subseteq \mathcal{F}_V, \quad \mathcal{T}_V \subseteq \mathcal{T}_P.$$

Proposition 12 is proved in [7].

Proposition 12 *Assume that M has a projective cover P in $\sigma[M]$. Then*
1) $\mathcal{F}_P \subseteq \mathcal{M}$.
2) If $\overline{Z}_M(M) = M$ then $\overline{Z}_M(P) = P$, $\mathcal{F}_P = \mathcal{M}$ and $\mathcal{M}$ is closed under direct sums.

Theorem 15 gives the relations between torsion theories $\tau_{\mathcal{M}}, \tau_V$ and τ_P. First we give the following lemma.

Lemma 13 *Let $N \in \sigma[M]$ be such that $\overline{Z}_M(N) = N$. Then $Z^*_M(N) = Rad(N)$.*

Proof Let $n \in \mathrm{Z}^*_M(N)$. Then nR is an M-small submodule of N. By [16, Lemma 2.3(1)] $nR \ll N$. Hence $\mathrm{Z}^*_M(N) \leq \mathrm{Rad}(N)$. □

Example 14 The converse of Lemma 13 is not true in general: Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/4\mathbb{Z}$. Then M is M-injective and it can be seen that $\mathrm{Z}^*_M(M) = \mathrm{Rad}(M) = \overline{Z}_M(M) = 2\mathbb{Z}/4\mathbb{Z}$.

Theorem 15 *Let M be a module and assume that P is a projective cover of M in $\sigma[M]$. Then the following are equivalent.*
(1) $\overline{Z}_M(M) = M$,
(2) $\mathcal{F}_P = \mathcal{M}$,
(3) $\mathcal{T}_P = \mathcal{T}_V$,
(4) $Gen_M(P) \subseteq \mathcal{T}_V$.

In this case $\mathcal{M} = \mathcal{F}_V = \mathcal{T}_\mathcal{M} = \{N \in \sigma[M] : Z^*_M(N) = N\} = \{N \in \sigma[M] : \overline{Z}_M(N) = 0\}$.

Proof $(1 \Rightarrow 2)$ By Proposition 12.
$(2 \Rightarrow 3)$ Let $T \in \mathcal{T}_P$ and C be an M-small module. Then $C \in \mathcal{F}_P$ implies that $\mathrm{Hom}(T, C) = 0$, i.e. $T \in \mathcal{T}_V$.
$(3 \Rightarrow 4)$ $\mathrm{Gen}_M(P) \subseteq \mathcal{T}_P = \mathcal{T}_V$.
$(4 \Rightarrow 1)$ Since $M \in \mathrm{Gen}_M(P)$, $M \in \mathcal{T}_V$ and hence $\overline{Z}_M(M) = M$.

For the last part assume that $\mathcal{F}_P = \mathcal{M}$. It is clear that if N is an M-small module in $\sigma[M]$, then $N \in \mathcal{F}_V \cap \mathcal{T}_\mathcal{M}$, $\mathrm{Z}^*_M(N) = N$ and $\overline{Z}_M(N) = 0$.

Now let $N \in \mathcal{F}_V$ and $f : P \to N$ be a homomorphism. Then $P/\mathrm{Ker}f \cong \mathrm{Im}f \leq N \in \mathcal{F}_V$. Since $\overline{Z}_M(P) = P$ by Proposition 12, $P \in \mathcal{T}_V$. This implies that $P/\mathrm{Ker}f \in \mathcal{F}_V \cap \mathcal{T}_V = 0$, i.e. $f = 0$. Hence $\mathcal{F}_V \subseteq \mathcal{M}$.

Let $\mu = \{N \in \sigma[M] : \mathrm{Z}^*_M(N) = N\}$. Since for an R-module L, $\mathrm{Tr}(\mathcal{M}, L) = L$ if and only if L is $\mathcal{M}$-generated [17, 13.5], $\mu = \mathrm{Gen}_M(\mathcal{M}) = \mathrm{Gen}(\mathcal{M}) \cap \sigma[M]$. Let $N \in \mu$. Then there exists an epimorphism from a direct sum of M-small modules to N. Any direct sum of M-small modules is M-small by Proposition 12. It follows that N is M-small.

Let $\beta = \{N \in \sigma[M] : \overline{Z}_M(N) = 0\}$. Since $\beta \subseteq \mathcal{F}_V$, by above $\beta \subseteq \mathcal{M}$.

Let $N \in \mathcal{T}_\mathcal{M}$ and $f : P \to N$ a homomorphism. Let $K := P/\mathrm{Ker}f$. Since $\overline{Z}_M(P) = P$, $\overline{Z}_M(K) = K$ by [16, Proposition 2.4], and by Lemma 13 $\mathrm{Z}^*_M(K) = \mathrm{Rad}(K)$. If $\mathrm{Z}^*_M(K) = K$, we have seen that K is M-small. Since $\overline{Z}_M(K) = K$, $f = 0$. If $\mathrm{Z}^*_M(K) \neq K$, there is a cyclic submodule C that is not small in K. Therefore K has a cyclic factor module and hence a simple factor module, say K/X. Then $\overline{Z}_M(K/X) = K/X$. Again by Lemma 13 $\mathrm{Z}^*_M(K/X) = \mathrm{Rad}(K/X) = 0$. Hence $K/X \in \mathcal{F}_\mathcal{M} \cap \mathcal{T}_\mathcal{M} = 0$, a contradiction. So $N \in \mathcal{F}_P$. □

Let M be a module. A module N in $\sigma[M]$ is called *semiperfect* in $\sigma[M]$ if every factor module of N has a projective cover in $\sigma[M]$ [17]. Then if M is semiperfect in $\sigma[M]$, M has a projective cover in $\sigma[M]$.

Corollary 16 *Let M be a module. If M is hereditary or semiperfect, then the result of Theorem 15 holds.*

Note that if M is a hereditary module then for every injective module N in $\sigma[M]$, $\overline{Z}_M(N) = N$ by [16, Proposition 2.7].

Proposition 17 *Let M be a module and assume that P is a projective cover of M. Then P is a generator $\Leftrightarrow \mathcal{F}_P = \{0\} \Leftrightarrow \mathcal{T}_P = Gen_M(P) = \sigma[M]$.*

Proof Assume that P is a generator. Let $F \in \mathcal{F}_P$. Since F is P-generated there exists an epimorphism $P^{(\Lambda)} \to F$ for some index set Λ. This yields a homomorphism from P to F which is zero. This implies that $F = 0$.

Now assume that $\mathcal{F}_P = \{0\}$. Let E be a simple module in $\sigma[M]$. If $\mathrm{Hom}(P, E) = 0$ then $E \in \mathcal{F}_P$ which is a contradiction. Hence by [17, 18.5] P is a generator. The last part is clear now. □

Corollary 18 *Let M be a module and assume that P is a projective cover of M. If $\overline{Z}_M(M) = M$ and P is a generator, then M is a V-module. In this case $\mathcal{T}_P = \mathcal{F}_\mathcal{M} = \sigma[M]$.*

Proof Let S be a simple module in $\sigma[M]$. Since P generates S by [17, 18.5], we have that $\overline{Z}_M(S) = S$ by [16, Proposition 1.3]. Then S can not be M-small. Hence M is a V-module. Then $\mathcal{M} = \{0\}$. By Theorem 15 $\mathcal{F}_\mathcal{M} = \sigma[M]$. By Proposition 17 $\mathcal{T}_P = \sigma[M]$. □

About $\mathbf{Z}_M^{*\,n}(.)$

Let N be a submodule of a module M. N is called a *weak supplement* of L in M if $N + L = M$ and $N \cap L \ll M$. N is called a *weak supplement* in M if there exists a submodule L such that N is a weak supplement of L in M. M is called *weakly supplemented* if every submodule N of M has a weak supplement (see [19]). If M is weakly supplemented then $M/\mathrm{Rad}M$ is semisimple. For if $\mathrm{Rad}M \leq K \leq M$, by hypothesis $M = K + L$ and $K \cap L \ll M$ for some L. Then $K \cap L \leq \mathrm{Rad}M$ and so $M/\mathrm{Rad}M = K/\mathrm{Rad}M \oplus (L + \mathrm{Rad}M)/\mathrm{Rad}M$.

Lemma 19 *Let $N \in \sigma[M]$. If $\widehat{N}$ is weakly supplemented, then $N/Z_M^*(N)$ is semisimple.*

Proof Let $N \in \sigma[M]$. Then $\widehat{N}/\mathrm{Rad}(\widehat{N}) = \widehat{N}/\mathrm{Z}_M^*(\widehat{N})$ is semisimple. Then $N/\mathrm{Z}_M^*(N) = N/N \cap \mathrm{Z}_M^*(\widehat{N}) \cong N + \mathrm{Z}_M^*(\widehat{N})/\mathrm{Z}_M^*(\widehat{N}) \leq \widehat{N}/\mathrm{Z}_M^*(\widehat{N})$ and hence $N/\mathrm{Z}_M^*(N)$ is semisimple. □

Now we denote the submodules ${\mathrm{Z}_M^*}^n(N)$ of a module $N \in \sigma[M]$ as follows. ${\mathrm{Z}_M^*}^1(N) = \mathrm{Z}_M^*(N)$, $\mathrm{Z}_M^*(N/{\mathrm{Z}_M^*}^{n-1}(N)) = {\mathrm{Z}_M^*}^n(N)/{\mathrm{Z}_M^*}^{n-1}(N) (n = 2, 3, \ldots)$. It is not known whether ${\mathrm{Z}_M^*}^2(N) = {\mathrm{Z}_M^*}^3(N) = \ldots$. But since ${\mathrm{Z}_M^*}^2(N)/\mathrm{Z}_M^*(N) \in \mathcal{T}_\mathcal{M}$ and $\mathrm{Z}_M^*(N) \in \mathcal{T}_\mathcal{M}$, ${\mathrm{Z}_M^*}^2(N) \in \mathcal{T}_\mathcal{M}$. By the same argument we have that ${\mathrm{Z}_M^*}^n(N) \in \mathcal{T}_\mathcal{M}$ for all n. Hence $\mathrm{Z}_M^*(N) \leq {\mathrm{Z}_M^*}^2(N) \leq {\mathrm{Z}_M^*}^3(N) \leq \ldots \leq \tau_\mathcal{M}(N)$.

Lemma 20 *Let $N \in \sigma[M]$. If $N/Z^*_M(N)$ is semisimple then ${Z^*_M}^2(N) = {Z^*_M}^3(N)$ and $N/{Z^*_M}^2(N)$ is N-injective.*

Proof Let $N/\mathrm{Z}^*_M(N) = N_1 \oplus N_2$ where N_1 is a direct sum of simple M-injective modules and N_2 is a direct sum of simple M-small modules. Then $\mathrm{Z}^*_M(N/\mathrm{Z}^*_M(N)) = N_2$. On the other hand $N/{\mathrm{Z}^*_M}^2(N) \cong (N/\mathrm{Z}^*_M(N))/N_2 \cong N_1$. Hence $\mathrm{Z}^*_M(N/{\mathrm{Z}^*_M}^2(N)) = 0$, i.e. ${\mathrm{Z}^*_M}^2(N) = {\mathrm{Z}^*_M}^3(N)$. By Proposition 2, $N/{\mathrm{Z}^*_M}^2(N)$ is N-injective. □

Proposition 21 *If every injective module in $\sigma[M]$ is weakly supplemented, then*
1) $\mathcal{F}_\mathcal{M} = \{N \in \sigma[M] : {Z^*_M}^2(N) = 0\}$
2) $\mathcal{T}_\mathcal{M} = \{N \in \sigma[M] : {Z^*_M}^2(N) = N\}$
3) $\tau_\mathcal{M}(N) = {Z^*_M}^2(N)$.
4) $\tau_\mathcal{M}$ is cohereditary.

Proof 1) Let $\gamma = \{N \in \sigma[M] : {\mathrm{Z}^*_M}^2(N) = 0\}$ and $N \in \mathcal{F}_\mathcal{M}$. Then $\mathrm{Z}^*_M(N) = 0$ and $\mathrm{Z}^*_M(N/\mathrm{Z}^*_M(N)) = {\mathrm{Z}^*_M}^2(N)/\mathrm{Z}^*_M(N) = 0$ implies ${\mathrm{Z}^*_M}^2(N) = 0$. Hence $N \in \gamma$ and so $\mathcal{F}_\mathcal{M} \subseteq \gamma$. Let $N \in \gamma$. Then ${\mathrm{Z}^*_M}^2(N) = 0$. Since $\mathrm{Z}^*_M(N) \leq {\mathrm{Z}^*_M}^2(N)$, $N \in \mathcal{F}_\mathcal{M}$. Hence $\gamma \leq \mathcal{F}_\mathcal{M}$.
2) Let $N \in \sigma[M]$ be such that ${\mathrm{Z}^*_M}^2(N) = N$. Then $\mathrm{Z}^*_M(N/\mathrm{Z}^*_M(N)) = N/\mathrm{Z}^*_M(N) \in \mathcal{T}_\mathcal{M}$ and it follows that $N \in \mathcal{T}_\mathcal{M}$. For the converse let $N \in \mathcal{T}_\mathcal{M}$. $N/\mathrm{Z}^*_M(N)$ is semisimple by Lemma 19. Then $N/\mathrm{Z}^*_M(N)$ is the sum of simple M-small modules. This implies that ${\mathrm{Z}^*_M}^2(N) = N$. Now (3) and (4) are clear. □

Every $\mathcal{T}_\mathcal{M}$-module is M-projective

A module M is called a *GCO-module* if every simple singular module is M-projective or M-injective. M is a GCO-module if and only if every simple M-singular module is M-injective. [4]

Theorem 22 *The following are equivalent for a module M.*
(1) M is a GCO-module,
(2) every M-small module in $\sigma[M]$ is M-projective,
(3) every $\mathcal{T}_\mathcal{M}$-module is M-projective,
(4) every simple $\mathcal{T}_\mathcal{M}$-module is M-projective.

Proof $(1 \Leftrightarrow 2)$ By [13, Theorem 5].
$(1 \Rightarrow 3)$ Let $N \in \mathcal{T}_\mathcal{M}$ and $x \in N$. If K is a maximal submodule of xR, xR/K is M-injective or M-projective. Since $N \in \mathcal{T}_\mathcal{M}$, xR/K can not be M-injective. Then xR/K is M-projective. It follows that K is a direct summand of xR. Hence xR, and then N is semisimple. Again by hypothesis N is M-projective.

(3 $\Rightarrow$ 4) Clear.
(4 $\Rightarrow$ 1) Let N be a simple module in $\sigma[M]$. If N is M-small, then N is M-projective by hypothesis. Hence N is M-injective or M-projective. $\square$

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