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THE Z^* FUNCTOR FOR RINGS WHOSE PRIMITIVE IMAGES ARE ARTINIAN

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ABSTRACT

Given a ring R, we investigate a subfunctor Z^* of the identity functor on the category of all right R-modules which is defined by $Z^*(M) = \{m \in M : mR \text{ is a small module}\}$, for any R-module M. We prove that if the ring R satisfies the descending chain condition for right annihilators and R/P is an Artinian ring for every primitive ideal P then $Z^*(M) = \{m \in M : mS = 0\}$ for every right R-module M, where S is the left socle of R. Moreover the ring R is semiprime Artinian if and only if R is right bounded, R satisfies the descending chain condition for right annihilators and $Z^*(M) = 0$ for some faithful right R-module M.

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1 INTRODUCTION AND NOTATION

Throughout this note all rings are associative with identity and all modules are unital right modules unless specified otherwise. For any module $M, E(M)$ will denote the injective hull of M, Soc M the socle of M and Rad M the radical of M (see[1] for more information). A submodule N of a module M is called *small* (*in M*), written $N \ll M$, if $M \neq N + K$ for every proper submodule K of M. Recall that Rad M is the sum of all small submodules of M (see[1, Proposition 9.13]).

Let R be any ring. An R-module M is called *small* if there exists an R-module M' and a monomorphism $\phi : M \to M'$ such that $\phi(M) \ll M'$. Leonard[2, Theorem 1] proved that the module M is small if and only if $M \ll E(M)$. Recall that an R-module M is singular if and only if there exists a (projective) R-module P and an epimorphism $\phi : P \to M$ such that the kernel of ϕ is an essential submodule of P (see, for example,[3, 4.6]). Thus, small modules can be thought of as a dual to singular modules. Singular modules have been extensively studied, but small modules less so.

For any R-module M, the *singular submodule* $Z(M)$ of M can be defined as follows: $Z(M) = \{m \in M : mR \text{ is a singular module}\}$. In[4, §2] (see also[5]), a subfunctor Z^* of the identity functor on the category of all R-modules is defined as follows: $Z^*(M) = \{m \in M : mR \text{ is a small module}\}$, for any R-module M. There are some interesting correspondences between $Z(M)$ and $Z^*(M)$ for a given module M, as we shall see later. In view of the importance of the functor Z for Module Theory it seems worthwhile to study the functor Z^*.

It is not difficult to establish the following facts (see[5, p.671]).

Lemma 1.1. *For any module M,* $\operatorname{Rad} M \subseteq Z^*(M) = M \cap \operatorname{Rad}(E(M))$.

Corollary 1.2. *Let M be any module. Then*

(i) $Z^*(M) = \operatorname{Rad} M$ *if M is an injective module.*
(ii) $Z^*(M) = M$ *if M is a small module.*
(iii) $Z^*(M) = 0$ *if and only if* $\operatorname{Rad}(E(M)) = 0$.

Proof. By Lemma 1.1. □

In general, the converse of Corollary 1.2(i) is false. Let R be a commutative domain which is not Dedekind. By[6, Theorem 4.25], there exists an R–module M which is divisible (i.e., $M = Mc$ for all $0 \neq c \in R$) but which is not injective. It follows that M does not contain a maximal submodule. Hence $M = \operatorname{Rad} M \subseteq Z^*(M) \subseteq M$, i.e., $Z^*(M) = \operatorname{Rad} M$, by Lemma 1.1. More generally, Özcan[7, Theorem 13] has proved that a prime ring R satisfying a polynomial identity (i.e., a prime *PI*-ring) has the property that every

R-module with $Z^*(M) = \operatorname{Rad} M$ is injective if and only if R is an hereditary Noetherian ring.

Lemma 1.3. *Let M be a module. Then*

(i) $Z^*(M) = \Sigma Z^*(N)$ *where the sum is taken over all finitely generated (cyclic) submodules of M.*
(ii) *If $M = \bigoplus_{i \in I} M_i$ is a direct sum of submodules $M_i (i \in I)$ then $Z^*(M) = \bigoplus_{i \in I} Z^*(M_i)$.*
(iii) $Z^*(N) = N \cap Z^*(M)$ *for any submodule N of M.*

Proof. (i), (iii) By the definition of Z^*. □

(ii) By (i).

Let R be a ring and let M be an R-module. For any non-empty subset X of R we define

$$\mathbf{r}_R(X) = \{r \in R : xr = 0 \text{ for all } x \in X\}, \text{ and}$$
$$\mathbf{l}_M(X) = \{m \in M : mx = 0 \text{ for all } x \in X\}.$$

Note that $\mathbf{r}_R(X)$ is a right ideal of R and such a right ideal is called a *right annihilator*. The *left socle* $\operatorname{Soc}({}_RR)$ of the ring R is the socle of the left R-module R and the *right socle* $\operatorname{Soc}(R_R)$ of R is the socle of the right R-module R. We mentioned above that there are various correspondences between the functors Z and Z^*.

Proposition 1.4. *Let R be a right Artinian ring. Then $Z(M) = \mathbf{l}_M(\operatorname{Soc}(R_R))$ and $Z^*(M) = \mathbf{l}_M(\operatorname{Soc}({}_RR))$ for any right R-module M.*

Proof. Let M be any R-module. Because $\operatorname{Soc}(R_R)$ is an essential right ideal of R,[1, Proposition 9.7] gives that

$$\begin{aligned} Z(M) &= \{m \in M : mA = 0 \text{ for some essential right ideal } A \text{ of } R\} \\ &= \{m \in M : m(\operatorname{Soc}(R_R)) = 0\} = \mathbf{l}_M(\operatorname{Soc}(R_R)). \end{aligned}$$

On the other hand, $Z^*(M) = \mathbf{l}_M(\operatorname{Soc}({}_RR))$ by[8, Theorem 3].

If a ring R is right hereditary then we can describe $Z^*(M)$ for any R-module M, as follows.

Theorem 1.5. *Let R be a right hereditary ring and let M be any right R-module. Then $Z^*(M) = \bigcap\{N : N$ is a maximal submodule of M and M/N is a simple injective R-module$\}$.*

Proof. Let $m \in Z^*(M)$. Let N be a maximal submodule of M such that M/N is a simple injective R-module. If $E = E(M)$ then M/N is a direct summand of E/N, so that $E = K + M$ for some submodule K of E such that $K \cap M = N$. Note that $E/K \cong M/N$ and hence K is a maximal submodule of E. It follows that $m \in K$ and hence $m \in K \cap M = N$. Thus $Z^*(M) \subseteq \bigcap\{N :$ N is a maximal submodule of M and M/N is a simple injective R-module$\}$.

Now suppose that $m \in M \backslash Z^*(M)$. By Lemma 1.1, there exists a maximal submodule L of E such that $m \notin L$. In this case, $E = L + mR$ and $M = (L \cap M) + mR$. Note that $E = L + M$ and $M/(L \cap M) \cong E/L$, so that $L \cap M$ is a maximal submodule of M. Moreover, by[1, p.215 ex. 10] the module $M/(L \cap M)$ is simple injective. Clearly $m \notin L \cap M$. The result follows. □

Next in this section we shall consider some examples. Let k be a field of characteristic 0 and let $A(k)$ be the first Weyl algebra over k. That is, $A(k)$ is the k-algebra with generators x, y subject to the relation $xy - yx = 1$. By[9, Theorems 1.3.5 and 7.5.8], $A(k)$ is a simple hereditary Noetherian domain. Moreover, no simple $A(k)$-module is injective by[10, Lemma 5.4]. By Theorem 1.5, $Z^*(M) = M$ for every $A(k)$-module M.

A ring R is called a *right V-ring* if every simple right R-module is injective. In,[11] Cozzens gives examples of simple principal right and left ideal domains R which are right V-rings, and for such a ring $R, Z^*(M) = 0$ for every R-module M by[12, Theorem 12].

In[13] Osofsky considers twisted polynomial rings $R = F[x; \sigma]$, where F is a field of characteristic $p > 0$ and $\sigma : F \to F$ is the endomorphism given by $\sigma(a) = a^p$ ($a \in F$). The ring R consists of all polynomials

$$a_0 + xa_1 + x^2a_2 + \cdots + x^na_n,$$

where n is a non-negative integer, $a_i \in F (0 \leq i \leq n)$, and multiplication is given by the relation

$$ax = x\sigma(a) \quad (a \in F).$$

Note that R is a principal right ideal domain (see[13, p.597]). Let A denote the ideal xR of R. Clearly A is a maximal right ideal of R and the R-module R/A is not injective because $R/A \neq (R/A)x$ (see[6, Proposition 2.6]). In,[13, Proposition 9] Osofsky gives an example of a field F such that the R-module R/sR is injective for all $s \in R \backslash xR$. Thus some simple R-modules are injective and some are not. In particular, for the principal right ideal domain R, $Z^*(M_1) = M_1$ and $Z^*(M_2) = 0$ for some simple R-modules M_1 and M_2. In this case, $Z^*(M_1 \oplus M_2) = M_1 \oplus 0 \neq 0, M_1 \oplus M_2$, by Lemma 1.3(ii).

Next we wish to note some information about annihilators. In what follows we shall be interested in rings which satisfy the descending chain condition (*dcc*) for right annihilators. It is well known that a ring R satisfies *dcc* for right annihilators if and only if R satisfies the ascending chain condition (*acc*) for left annihilators (see, for example,[14, p.2]). In particular, left Noetherian rings satisfy *dcc* for right annihilators and, more generally, so too do left Goldie rings. If a ring R satisfies *dcc* for right annihilators then it is easy to check that so too does any subring of R. Thus any subring of a left Goldie ring or a right Artinian ring satisfies *dcc* for right annihilators. Note the following result of Faith.[15, Corollary 20.2B]

Lemma 1.6. *A ring R satisfies dcc for right annihilators if and only if, for each left ideal L of R, there exists a finitely generated left ideal L' of R such that $L' \subseteq L$ and $\mathbf{r}_R(L) = \mathbf{r}_R(L')$.*

In this paper, a left ideal L of a ring R will be called *almost finitely generated* if $\mathbf{r}_R(L) = \mathbf{r}_R(L')$ for some finitely generated left ideal $L' \subseteq L$. Clearly any finitely generated left ideal is almost finitely generated. However any left ideal L such that $\mathbf{r}_R(Rc) = 0$ for some element c in L is almost finitely generated because $\mathbf{r}_R(L) = \mathbf{r}_R(Rc)$, but L need not be finitely generated. Note that Lemma 1.6 can be restated thus: a ring R satisfies *dcc* for right annihilators if and only if every left ideal of R is almost finitely generated. The next result is probably known but we do not have a reference.

Lemma 1.7. *Let R be a left nonsingular ring and let L be a left ideal of R such that the left R-module L has finite uniform dimension. Then L is almost finitely generated.*

Proof. By[14, Lemma 1.9], there exists a finitely generated left ideal L' of R such that L' is an essential submodule of the left R-module L. Let $r \in \mathbf{r}_R(L')$. For any $x \in L$, there exists an essential left ideal A of R such that $Ax \subseteq L'$ (see[14, Lemma 1.1]), so that $Axr = 0$ and hence $xr = 0$. It follows that $\mathbf{r}_R(L') \subseteq \mathbf{r}_R(L)$, and hence $\mathbf{r}_R(L) = \mathbf{r}_R(L')$. □

The main result of §2 concerns rings R such that every right primitive ideal is an almost finitely generated left ideal and R/P is an Artinian ring for every right or left primitive ideal P (see Theorem 2.9). Recall that an ideal P of an arbitrary ring R is called *right primitive* if $P = \mathbf{r}_R(U)$ for some simple R-module U. The ring R is *right primitive* if its zero ideal is right primitive. There are analogous definitions for left primitive ideals and left primitive rings. Examples are known of right primitive ideals which are not left primitive (see,[16–18]).

As we mentioned above, we are interested in rings R such that R/P is an Artinian ring for each right (or left) primitive ideal P. For example, if R is a ring with Jacobson radical J such that R/J is an Artinian ring (in particular if R is a semiperfect ring) then every right (or left) primitive image of

R is Artinian. Moreover, Kaplansky's Theorem states that if R is a ring satisfying a polynomial identity (i.e., a *PI*-ring) then R/P is an Artinian ring for every right (or left) primitive ideal P (see, for example,[9, Theorem 13.3.8]).

Following Chatters and Hajarnavis,[14] a ring R is called *right bounded* if every essential right ideal contains an ideal which is essential as a right ideal. A ring R is called *right fully bounded* if every prime factor ring of R is right bounded. It can be shown that every right fully bounded semiprime ring which has only a finite number of minimal prime ideals is right bounded. Clearly right Artinian rings are fully right (and left) bounded and semiprime Artinian rings are right (and left) bounded. Moreover *PI*-rings are fully right (and left) bounded by[9, Corollary 13.6.6]. A ring R is called a *right FBN ring* if R is a right fully bounded right Noetherian ring. If R is a right *FBN* ring then R/P is an Artinian ring for every right primitive ideal P of R (see[19, Proposition 8.4]).

Another interesting class of rings R such that every primitive image is Artinian is provided by group rings. A group G is *polycyclic-by-finite* if there exist a positive integer n and a chain $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = 1$ of subgroups $G_i (0 \leq i \leq n)$ of G such that G_i is a normal subgroup of G_{i-1} and the group G_{i-1}/G_i is cyclic or finite for each $1 \leq i \leq n$. A ring R is called a *Jacobson ring* if every prime factor ring of R has zero Jacobson radical, i.e., every prime ideal of R is an intersection of right primitive ideals. Let S be a commutative Noetherian Jacobson ring such that, for each maximal ideal P of S, the field S/P is an algebraic extension of a finite field. For example, S could be the ring $\mathbb{Z}$ of rational integers or a finite field. Let G be a polycyclic-by-finite group and let R be the group ring $S[G]$. Roseblade[20, Corollary A] proved that for every simple R-module U there exists a maximal ideal P of S such that $UP = 0$ and U is a finite dimensional vector space over the field S/P. In particular, this means that the ring R/Q is Artinian for every right (or left) primitive ideal Q.

If a ring R is von Neumann regular then R/P is an Artinian ring for every right primitive ideal P if and only if R/Q is an Artinian ring for every left primitive ideal Q (see[21, Theorem 6.2]). We do not know an example of a ring R such that R/P is Artinian for every right primitive ideal P but R/Q is not Artinian for some left primitive ideal Q. Note that if such an example exists then there exists a non-Artinian left primitive ring R' such that R'/P is Artinian for every right primitive ideal P.

2 RINGS WHOSE PRIMITIVE IMAGES ARE ARTINIAN

Let R be a ring and let M be a (right) R-module. If R is right hereditary then Theorem 1.5 gives a description of $Z^*(M)$, although it is not clear how one would calculate it in practice. In general, to find $Z^*(M)$ seems a difficult problem because it involves information about the radical of the injective hull of M. In this section we shall show that for a large class of rings, including all left Noetherian rings whose primitive images are Artinian, it is possible to give a very satisfactory description of $Z^*(M)$. The first result of this section illustrates why rings whose primitive images are Artinian feature in the study of the Z^* functor.

Lemma 2.1. *Let R be a ring such that R/P is an Artinian ring for every left primitive ideal P. Then $Z^*(M) \subseteq \mathbf{l}_M(\mathrm{Soc}(_R R))$ for every R-module M.*

Proof. Let M be any R-module and let $E = E(M)$. If $\mathrm{Soc}(_R R) = 0$ then there is nothing to prove. Suppose that $\mathrm{Soc}(_R R) \neq 0$. Let U be a minimal left ideal of R. Let $P = \mathbf{l}_R(U)$. Then P is a left primitive ideal of R and, by hypothesis, R/P is an Artinian ring. Note that the R-module E/EP is semisimple and hence $\mathrm{Rad}\, E \subseteq EP$. Now $Z^*(M)U \subseteq (\mathrm{Rad}\, E)U \subseteq (EP)U = 0$. It follows that $Z^*(M) \subseteq \mathbf{l}_M(\mathrm{Soc}(_R R))$. □

Corollary 2.2. *Let R be a ring such that R/P is an Artinian ring for every left primitive ideal P and let M be an R-module such that $\mathbf{l}_M(\mathrm{Soc}(_R R)) = 0$. Then $Z^*(M) = 0$.*

Proof. By Lemma 2.1. □

Let R be a commutative ring. Then every (left) primitive ideal is maximal and R/P is a field for every primitive ideal P. By Lemma 2.1, $Z^*(M) \subseteq \mathbf{l}_M(\mathrm{Soc}(_R R))$ for every R-module M. We shall show that for many commutative rings R, $Z^*(M) = \mathbf{l}_M(\mathrm{Soc}(_R R))$ for every R-module M (see Theorem 2.9). However, this is not always the case for commutative rings and, in fact, can fail spectacularly. For example, if R is a (commutative) von Neumann regular ring with zero socle then $Z^*(M) = 0$ and $\mathbf{l}_M(\mathrm{Soc}(_R R)) = M$ for every R-module M (see[15, Corollary 19.53] and[12, Theorem 12]). An example of such a ring R can be produced as follows. Let F be any field and let T be the direct product of any infinite number of copies of F. Then T is a commutative ring. Let I be the ideal of T consisting of all elements which have at most a finite number of non-zero components. Then the ring $R = T/I$ is a commutative von Neumann regular ring with zero socle.

For any ring R, R_R will denote the R-module R and $R_R^{(n)}$ will denote the direct sum of n copies of R_R, for any positive integer n. The next lemma is key to our investigation. It is probably known but we do not have a reference.

Lemma 2.3. *Let R be any ring and let E be an injective R-module. Then $EL = \mathbf{l}_E(\mathbf{r}_R(L))$ for every almost finitely generated left ideal L of R.*

Proof. Note first that $(EL)\mathbf{r}_R(L) = 0$, so that $EL \subseteq \mathbf{l}_E(\mathbf{r}_R(L))$. Conversely, let $e \in \mathbf{l}_E(\mathbf{r}_R(L))$. By hypothesis, $\mathbf{r}_R(L) = \mathbf{r}_R(Rx_1 + \cdots + Rx_n)$ for some positive integer n and elements $x_i \in L (1 \leq i \leq n)$. Consider the element $(x_1, \ldots, x_n)$ of the free R-module $R_R^{(n)}$ and define a mapping ϕ: $(x_1, \ldots, x_n)R \to E$ by

$$\phi(x_1 r, \ldots, x_n r) = er \quad (r \in R).$$

Note that if $r \in R$ and $(x_1 r, \ldots, x_n r) = 0$ then $r \in \mathbf{r}_R(L)$ and hence $er = 0$. Thus ϕ is well defined and is clearly a homomorphism. Because E is injective, there exists a homomorphism $\theta : R_R^{(n)} \to E$ such that ϕ is the restriction of θ to $(x_1, \ldots, x_n)R$.

For each $1 \leq i \leq n$, let e_i denote the element $(0, \ldots, 0, 1, 0, \ldots, 0)$ of $R_R^{(n)}$, where 1 is the ith component of e_i. Then

$$\begin{aligned} e &= \phi(x_1, \ldots, x_n) = \theta(e_1 x_1 + \cdots + e_n x_n) \\ &= \theta(e_1)x_1 + \cdots + \theta(e_n)x_n \in EL. \end{aligned}$$

It follows that $\mathbf{l}_E(\mathbf{r}_R(L)) \subseteq EL$. Hence $EL = \mathbf{l}_E(\mathbf{r}_R(L))$. □

For any ring $R, \pi(R)$ will denote the collection of all right primitive ideals of R.

Lemma 2.4. *Let R be a ring such that every right primitive ideal is an almost finitely generated left ideal. Let A denote the ideal $\sum_{P\in\pi(R)} \mathbf{r}_R(P)$ of R. Then $\mathbf{l}_M(A) \subseteq Z^*(M)$ for every R-module M.*

Proof. Let M be any R-module and let $E = E(M)$. If N is a maximal submodule of E then the ideal $Q = \mathbf{r}_R(E/N)$ is right primitive and $EQ \subseteq N$. It follows that $\cap_{P\in\pi(R)} EP \subseteq \operatorname{Rad} E$. By Lemmas 1.1 and 2.3,

$$\begin{aligned} Z^*(M) = M \cap \operatorname{Rad} E &\supseteq \bigcap_{P\in\pi(R)} (M \cap EP) = \bigcap_{P\in\pi(R)} \mathbf{l}_M(\mathbf{r}_R(P)) \\ &= \mathbf{l}_M\left(\sum_{P\in\pi(R)} \mathbf{r}_R(P)\right) = \mathbf{l}_M(A). \end{aligned}$$ □

Corollary 2.5. *Let R be a ring such that for every right primitive ideal P of R there exists a finitely generated left ideal L of R such that $L \subseteq P$ and $\mathbf{r}_R(L) = 0$. Then $Z^*(M) = M$ for every R-module M.*

Proof. Let P be any right primitive ideal of R. By hypothesis there exists a finitely generated left ideal $L \subseteq P$ such that $\mathbf{r}_R(L) = 0$. But this implies that

$\mathbf{r}_R(P) = 0 = \mathbf{r}_R(L)$. Hence every right primitive ideal P is an almost finitely generated left ideal and satisfies $\mathbf{r}_R(P) = 0$. By Lemma 2.4, $Z^*(M) = M$ for every R-module M. □

An element c of a ring R is called *right regular* if $cr \neq 0$ for all $0 \neq r \in R$, i.e., $\mathbf{r}_R(c) = 0$. Note that Corollary 2.5 shows that if R is a ring such that every right primitive ideal contains a right regular element then $Z^*(M) = M$ for every R-module M. The next result gives another situation where every module M satisfies $Z^*(M) = M$. It generalizes[12, Corollary 9] where it is proved for prime right Goldie rings which are not right primitive.

Proposition 2.6. *Let R be a semiprime ring which satisfies dcc for right annihilators such that no minimal prime ideal is right primitive. Then $Z^*(M) = M$ for every R-module M.*

Proof. Let P be any right primitive ideal of R. For any minimal prime ideal Q of R, $P \neq Q$ and $P\mathbf{r}_R(P) = 0 \subseteq Q$ so that $\mathbf{r}_R(P) \subseteq Q$. Because R is semiprime, we deduce that $\mathbf{r}_R(P) = 0$. By Lemma 1.6, P is an almost finitely generated left ideal. The result follows by Corollary 2.5. □

In particular, Proposition 2.6 shows that if R is a prime ring which satisfies *dcc* for right annihilators but which is not right primitive then $Z^*(M) = M$ for every R-module M. Here the fact that R is not right primitive is crucial because of the example in Section 1 of a simple Noetherian ring R for which $Z^*(M) = 0$ for every R-module M. Note further that we showed in Section 1 that there exist simple Noetherian rings R such that $Z^*(M) = M$ for every R-module M. Next we prove an analogue of Lemma 2.4.

Lemma 2.7. *Let R be a ring with Jacobson radical J such that J is an almost finitely generated left ideal. Then $\mathbf{l}_M(\mathbf{r}_R(J)) \subseteq Z^*(M)$ for every R-module M. In particular, $\mathbf{l}_R(\mathbf{r}_R(J)) \subseteq Z^*(R_R)$.*

Proof. Let M be any R-module and let $E = E(M)$. By[1, Corollary 15.18] and Lemma 2.3,

$$\mathbf{l}_E(\mathbf{r}_R(J)) = EJ \subseteq \operatorname{Rad} E,$$

and hence

$$\mathbf{l}_M(\mathbf{r}_R(J)) = M \cap \mathbf{l}_E(\mathbf{r}_R(J)) \subseteq M \cap \operatorname{Rad} E = Z^*(M).$$

The last part is clear. □

Using Lemma 2.7 we can generalize a result of Rayar[8, Theorem 3] (see also[5, Lemma 2.2]) who proved it for right Artinian rings.

Proposition 2.8. *Let R be a ring with Jacobson radical J such that J is an almost finitely generated left ideal and R/J is an Artinian ring. Then $Z^*(M) = \mathbf{l}_M(\mathrm{Soc}(_RR)) = \mathbf{l}_M(\mathbf{r}_R(J))$ for every R-module M. In particular, $Z^*(R_R) = \mathbf{l}_R(\mathrm{Soc}(_RR)) = \mathbf{l}_R(\mathbf{r}_R(J))$.*

Proof. Note that $\mathrm{Soc}(_RR) = \mathbf{r}_R(J)$. Apply Lemmas 2.1 and 2.7. □

Let R be a ring with Jacobson radical J such that J is an almost finitely generated left ideal and R/J is an Artinian ring. Let P be a right primitive ideal of R. Because the ring R/J is semiprime Artinian, the left ideal P/J is principal. It follows that $P = J + Ra$ for some element a of P. Now $\mathbf{r}_R(J) = \mathbf{r}_R(L)$ for some finitely generated left ideal $L \subseteq J$ and hence $\mathbf{r}_R(P) = \mathbf{r}_R(L + Ra)$. It follows that every right primitive ideal of R is an almost finitely generated left ideal. Moreover it is clear that R/Q is an Artinian ring for every right (or left) primitive ideal Q. Our aim now is to generalize Proposition 2.8. We do this in the next result.

Theorem 2.9. *Let R be a ring such that every right primitive ideal is an almost finitely generated left ideal and R/P is an Artinian ring for every right primitive ideal P. Then $\mathbf{l}_M(\mathrm{Soc}(_RR)) \subseteq Z^*(M) = \mathbf{l}_M\left(\sum_{P\in\pi(R)} \mathbf{r}_R(P)\right)$ for every right R-module M. Moreover, if in addition R/Q is an Artinian ring for every left primitive ideal Q then $Z^*(M) = \mathbf{l}_M(\mathrm{Soc}(_RR))$ for every right R-module M.*

Proof. Let M be any R-module and let $E = E(M)$. Set $A = \sum_{P\in\pi(R)} \mathbf{r}_R(P)$. By Lemma 2.4, $\mathbf{l}_M(A) \subseteq Z^*(M)$. For each P in $\pi(R)$, the ring R/P is simple Artinian and hence the left (R/P)-module $\mathbf{r}_R(P)$ and the right (R/P)-module E/EP are both semisimple. It follows that $\mathrm{Rad}\, E \subseteq EP$. By Lemma 1.1, $Z^*(M)A \subseteq (\mathrm{Rad}\, E)A \subseteq \left(\bigcap_{P\in\pi(R)} EP\right)A = 0$, so that $Z^*(M) \subseteq \mathbf{l}_M(A)$ and hence $Z^*(M) = \mathbf{l}_M(A)$. Moreover $A \subseteq \mathrm{Soc}(_RR)$ and we deduce that $\mathbf{l}_M(\mathrm{Soc}(_RR)) \subseteq \mathbf{l}_M(A)$.

Now suppose that, in addition, R/Q is an Artinian ring for every left primitive ideal Q. By Lemma 2.1, $Z^*(M) \subseteq \mathbf{l}_M(\mathrm{Soc}(_RR))$ and hence $Z^*(M) = \mathbf{l}_M(\mathrm{Soc}(_RR))$. □

Any commutative Noetherian ring satisfies the hypotheses of Theorem 2.9 but need not satisfy the hypotheses of Proposition 2.8.

Let M be any module (over an arbitrary ring). Then we define a chain of submodules $0 = Z_0^*(M) \subseteq Z_1^*(M) \subseteq Z_2^*(M) \subseteq \cdots$ of M as follows: for each integer $n \geq 1$, $Z_n^*(M)/Z_{n-1}^*(M) = Z^*(M/Z_{n-1}^*(M))$. Clearly $Z_1^*(M) = Z^*(M)$. Harada[5, Proposition 1.2] proved that if R is a ring with Jacobson radical J such that R/J is an Artinian ring then $Z_2^*(M) = Z_3^*(M)$ for every

R-module M, i.e., $Z_2^* = Z_3^*$. Now we show that $Z_2^* = Z_3^*$ for all rings R satisfying the hypotheses of Theorem 2.9.

Theorem 2.10. *Let R be a ring such that every right primitive ideal is an almost finitely generated left ideal and R/P is an Artinian ring for every right or left primitive ideal P. Then $Z_2^*(M) = Z_3^*(M) = \mathbf{l}_M(S^2)$ for every right R-module M, where $S = \mathrm{Soc}({}_R R)$.*

Proof. Let M be any R-module. By Theorem 2.9, $Z_2^*(M) = \{m \in M : mS \subseteq Z^*(M)\} = \{m \in M : mS^2 = 0\} = \mathbf{l}_M(S^2)$ and, similarly, $Z_3^*(M) = \mathbf{l}_M(S^3)$. If $S = 0$ then there is nothing to prove. Suppose that $S \neq 0$. Let L be a minimal left ideal of R. Then $L^2 = L$ or $L^2 = 0$. Moreover, if $L^2 = 0$ then $LL' = 0$ for every minimal left ideal L' of R. There exist disjoint collections $\{L_\lambda : \lambda \in \Lambda(i)\}$ $(1 \leq i \leq 3)$ of independent minimal left ideals of R such that $S = H_1 \oplus H_2 \oplus H_3$, where $H_i = \oplus_{\lambda \in \Lambda(i)} L_\lambda$ $(1 \leq i \leq 3)$, and moreover,

(a) $L_\lambda^2 = L_\lambda$ for all λ in $\Lambda(1)$,
(b) $L_\lambda^2 = 0$ and $H_1 L_\lambda = L_\lambda$ for all λ in $\Lambda(2)$, and
(c) $L_\lambda^2 = 0$ and $H_1 L_\lambda = 0$ for all λ in $\Lambda(3)$.

By the above remarks $(H_2 \oplus H_3)S = 0$ and hence $S^2 = H_1 \oplus H_2 = S^3$. It follows that $Z_2^*(M) = Z_3^*(M)$. □

Theorem 2.10 can be viewed as an analogue of[22, p.148 Proposition 6.2]. Let R be a commutative von Neumann regular ring with socle $S = 0$. Then, in contrast to Theorem 2.10, R/P is a field for every primitive ideal P but $Z_2^*(M) = Z_3^*(M) = 0$ and $l_M(S^2) = M$ for every R-module M. Like Harada[5, p.671] we do not know if $Z_2^* = Z_3^*$ for any (commutative) ring R. Note that if R is a right hereditary ring then $Z_2^*(M) = Z_3^*(M)(= Z_1^*(M))$ for every R-module M, by Theorem 1.5.

3 MODULES *M* WITH $Z^*(M) = 0$

In this section we shall study rings R with the property $Z^*(R_R) = 0$ and more generally modules M such that $Z^*(M) = 0$. In Corollary 1.2(iii) we observed that a module M has the property $Z^*(M) = 0$ if and only if $\mathrm{Rad}\, E(M) = 0$. Note that every right self-injective von Neumann regular ring satisfies $Z^*(R_R) = 0$ (see Corollary 1.2). In[12], Özcan proved that a ring R is a right V-ring (i.e., every simple right R-module is injective) if and only if $Z^*(M) = 0$ for every (simple) right R-module M. In particular if R is a right V-ring then $Z^*(R_R) = 0$. However, the converse is false. For, by

Corollary 2.2, any commutative ring R with socle S such that $\mathbf{l}_R(S) = 0$ satisfies $Z^*(R) = 0$, and not all such rings are V-rings. We have the following specific example.

Example 3.1. There exists a commutative ring R with socle S such that $\mathbf{l}_R(S) = 0$ but R is not a V-ring.

Proof. Let K be a commutative domain which is not a field and let F be the field of fractions of K. Let T be the direct product of a countable number of copies of F. Then T is the ring consisting of all sequences $a_1, a_2, a_3, \ldots$ of elements $a_i (i \geq 1)$ of F. Let R be the subring of T consisting of all sequences $a_1, a_2, a_3, \ldots$ in T such that there exist b in S and a positive integer n with $a_i = b$ for all $i \geq n$. Then R is a commutative ring whose socle K consists of all sequences $a_1, a_2, a_3, \ldots$ in R such that there exists a positive integer k with $a_i = 0$ for all $i \geq k$. Clearly $\mathbf{l}_R(S) = 0$ and hence $Z^*(R) = 0$ by Corollary 2.2. Let c be any non-zero non-unit in K and let r be the sequence $c, c, c, \ldots$ in R. Clearly $r \notin Rr^2$. Thus the ring R is not von Neumann regular. By[15, Corollary 19.53] R is not a V-ring. □

Next we prove a lemma which will be useful in the sequel.

Lemma 3.2. *Let P be a prime ideal of a ring R such that the ring R/P satisfies dcc for right annihilators and $\mathbf{l}_M(P) \neq 0$ for some R-module M satisfying $Z^*(M) = 0$. Then P is a right primitive ideal of R.*

Proof. Let $E = E(M)$. By Corollary 1.2(iii), $\operatorname{Rad} E = 0$. Let $X = \{e \in E : eP = 0\}$. By[6, Proposition 2.27], X is an injective (R/P)-module and $X \neq 0$ because $0 \neq \mathbf{l}_M(P) \subseteq X$. Note that, as R-modules, $\operatorname{Rad} X \subseteq \operatorname{Rad} E = 0$ by[1, Proposition 9.14]. Hence X is an injective (R/P)-module which contains a maximal submodule. By Proposition 2.6, P is a right primitive ideal. □

The next two results are presumably known but we do not have references.

Lemma 3.3. *Every right bounded semiprime ring is right nonsingular.*

Proof. Let A be any essential right ideal of a right bounded semiprime ring R and let $r \in \mathbf{l}_R(A)$. There exists an ideal I of R such that I is an essential right ideal of R and $I \subseteq A$. If $r \neq 0$ then $rR \cap I \neq 0$. However, $(rR \cap I)^2 \subseteq (rR)I = rI \subseteq rA = 0$, so that $rR \cap I = 0$, a contradiction. Thus $r = 0$. It follows that R is right nonsingular. □

Lemma 3.4. *Let R be a right bounded ring and let P be a prime ideal of R such that P is not an essential right ideal of R. Then R/P is a right bounded ring.*

Proof. Let A be a right ideal of R such that $P \subseteq A$ and A/P is an essential right ideal of the ring R/P. It can easily be checked that A is an essential right ideal of R. By hypothesis, there exists an ideal B of R such that $B \subseteq A$ and B is an essential right ideal of R. By hypothesis, $B \nsubseteq P$ and it is clear that $(B+P)/P$ is an ideal of R/P, $(B+P)/P \subseteq A/P$ and $(B+P)/P$ is an essential right ideal of R/P.

The next result generalizes[14, Theorem 1.24]. Note that if R is a semiprime ring then $\mathrm{Soc}(R_R) = \mathrm{Soc}(_R R)$ (see, for example,[14, p.18]) and in this case we shall call $\mathrm{Soc}(R_R)$ the *socle* of R.

Lemma 3.5. *Let R be a prime ring which satisfies acc or dcc for right annihilators such that R has non-zero socle. Then R is a simple Artinian ring.*

Proof. If R satisfies *dcc* for right annihilators then R satisfies *acc* for left annihilators (see[14, p.2]). Thus it is sufficient to prove the result in case R satisfies *acc* for right annihilators. Let S denote the socle of R. For any $0 \neq a \in R$, $0 \neq aS \subseteq aR \cap S$ and hence S is an essential right ideal of R. Let U_1 be a minimal right ideal of R. It is well known that, because R is a prime ring, there exists an idempotent element e_1 of R such that $U_1 = e_1 R$. Hence $R = U_1 \oplus V_1$, where $V_1 = (1 - e_1)R$. Suppose that $V_1 \neq 0$. Then $S \cap V_1 \neq 0$. Let U_2 be a minimal right ideal of R such that $U_2 \subseteq V_1$. By the above remarks, $R = U_2 \oplus V_2$ for some right ideal V_2 and hence $V_1 = U_2 \oplus (V_1 \cap V_2)$ and $R = U_1 \oplus U_2 \oplus (V_1 \cap V_2)$.

If $V_1 \cap V_2 \neq 0$ then $S \cap V_1 \cap V_2 \neq 0$ and by the above argument there exists a minimal right ideal U_3 of R such that $U_3 \subseteq V_1 \cap V_2$ and $R = U_3 \oplus V_3$ for some right ideal V_3. In this case, $R = U_1 \oplus U_2 \oplus U_3 \oplus (V_1 \cap V_2 \cap V_3)$. This process produces a strictly ascending chain $U_1 \subset U_1 \oplus U_2 \subset U_1 \oplus U_2 \oplus U_3 \subset \dots$ of right annihilators. Since R satisfies *acc* for right annihilators it follows that this process must stop, so that $V_1 \cap \cdots \cap V_n = 0$ for some positive integer n. Hence $R = U_1 \oplus \cdots \oplus U_n$. It follows that the ring R is simple Artinian. □

Note that the proof of Lemma 3.5 shows that if R is a prime ring which does not contain an infinite set of orthogonal idempotents such that R has non-zero socle then R is simple Artinian (see[15, Lemma 22.28]). However Lemma 3.5 is in the form we shall need. The next result is an immediate consequence of Corollary 1.2 and Lemma 1.3.

Lemma 3.6. *Let a module* $M = \oplus_{i \in I} M_i$ *be a direct sum of simple injective submodules* $M_i (i \in I)$. *Then* $Z^*(M) = 0$.

This brings us to the main result of this section. It shows that if R is a right bounded ring then often the only way for a faithful R-module M to satisfy $Z^*(M) = 0$ is for M to be semisimple injective.

Theorem 3.7. *The following statements are equivalent for a ring R.*

(i) *R is a semiprime Artinian ring.*
(ii) *R is a right bounded ring which satisfies dcc for right annihilators such that $Z^*(M) = 0$ for every right R-module M.*
(iii) *R is a right bounded ring which satisfies dcc for right annihilators such that $Z^*(R_R) = 0$.*
(iv) *R is a right bounded ring which satisfies dcc for right annihilators such that $Z^*(M) = 0$ for some faithful right R-module M.*

Proof. (i) $\Rightarrow$ (ii) By Lemma 3.6.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) Clear.

(iv) $\Rightarrow$ (i) Let J be the Jacobson radical of R. By[1, Corollary 15.18] and Lemma 1.1, we have $MJ \subseteq \operatorname{Rad} M \subseteq Z^*(M)$, so that $MJ = 0$ and hence $J = 0$. In particular, the ring R is semiprime. By[14, Lemma 1.16], there exist a positive integer n and prime ideals $P_i (1 \leq i \leq n)$ of R such that $P_1, \ldots, P_n$ are the minimal prime ideals of R. Clearly to complete the proof it is sufficient to prove that R/P_i is an Artinian ring for each $1 \leq i \leq n$.

Choose $1 \leq i \leq n$. Set $P = P_i$ and $A = \prod_{j \neq i} P_j$. Clearly $P = \mathbf{r}_R(A)$. Let $\bar{R}$ denote the prime ring R/P. Note that $\bar{R}$ is right bounded by Lemma 3.4. For any non-empty subset X of R, let $\bar{X} = \{x + P : x \in X\}$ and note that $\mathbf{r}_{\bar{R}}(\bar{X}) = \mathbf{r}_R(AX)/P$. It follows that $\bar{R}$ satisfies *dcc* for right annihilators. By Lemma 3.2 $\bar{R}$ is a right primitive ring. Since $\bar{R}$ is right bounded it follows that $\operatorname{Soc}(\bar{R}_{\bar{R}}) \neq 0$. Finally $\bar{R}$ is Artinian by Lemma 3.5. □

Note that in Theorem 3.7 the condition "R satisfies *dcc* for right annihilators" can be replaced throughout by the condition "R has finite right uniform dimension." This is because of the following result.

Corollary 3.8. *A ring R is semiprime Artinian if and only if R is right bounded, R has finite right uniform dimension and $Z^*(M) = 0$ for some faithful right R-module M.*

Proof. The necessity is clear. Conversely, suppose that R is right bounded with finite right uniform dimension such that $Z^*(M) = 0$ for some faithful right R-module M. As in the proof of Theorem 3.7 (iv) $\Rightarrow$ (i), R is a semiprime ring. By Lemma 3.3 R is right nonsingular and by[14, Lemma 1.14] R satisfies *dcc* for right annihilators. Now the ring R is semiprime Artinian by Theorem 3.7. □

For modules we have the following result.

Corollary 3.9. *Let R be a ring and let M be a right R-module with injective hull $E = E(M)$ such that the ring $R/\mathbf{r}_R(E)$ is right bounded and satisfies dcc for right annihilators. Then the following statements are equivalent.*

(i) *M is semisimple injective.*
(ii) *M is injective and* $\operatorname{Rad} M = 0$.
(iii) $Z^*(M) = 0$

Proof. (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iii) By Corollary 1.2.

(iii) $\Rightarrow$ (i) By Corollary 1.2 $\operatorname{Rad} E = 0$, i.e., $Z^*(E) = 0$. Now the ring $R/\mathbf{r}_R(E)$ is semiprime Artinian by Theorem 3.7. It follows that E is a semisimple R-module so that $M = E$ and hence M is semisimple injective. □

Recall that a module M is Σ–*injective* if every direct sum of copies of M is injective. For a given ring R, an R-module M is Σ–injective if and only if M is injective and R satisfies *acc* on right ideals of the form $\mathbf{r}_R(X)$, where X is a non-empty subset of M (see[15, Proposition 20.3A]). We now show that for a large class of rings R, including all right Noetherian *PI*-rings, an R-module M satisfies $Z^*(M) = 0$ if and only if M is semisimple injective.

Theorem 3.10. *Let R be a ring such that R/S is a right bounded ring which satisfies dcc for right annihilators, for each semiprime ideal S of R. Then the following statements are equivalent for a right R-module M.*

(i) *M is semisimple* Σ–*injective.*
(ii) *M is semisimple injective.*
(iii) $Z^*(M) = 0$.

Proof. (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iii) By Lemma 3.6.

(iii) $\Rightarrow$ (i) Let M' be any R-module such that M' is a direct sum of copies of M. By Lemma 1.3, $Z^*(M') = 0$. Let $E = E(M')$. If J is the Jacobson radical of R then $EJ \subseteq \operatorname{Rad} E = 0$, by[1, Corollary 15.18] and Corollary 1.2. It follows that the ring $R/\mathbf{r}_R(E)$ is semiprime and hence $R/\mathbf{r}_R(E)$ is right bounded and satisfies *dcc* for right annihilators. By Corollary 3.9, E is semisimple. Hence $M' = E$ and M' is semisimple and injective. It follows that M is a semisimple Σ-injective module.

REFERENCES

1. Anderson, F.W.; Fuller, K.R. *Rings and Categories of Modules*; Springer-Verlag: New York, 1974.
2. Leonard, W.W. Small Modules. Proc. Amer. Math. Soc. **1966**, *17*, 527–531.

3. Dung, N.V.; Huynh, D.V.; Smith, P.F.; Wisbauer, R. *Extending Modules*; Longman: Harlow, 1994.
4. Rayar, M. *Small and Cosmall Modules*, Ph.D. dissertation, Indiana Univ., 1971.
5. Harada, M. Non–small Modules and Non-Cosmall Modules. In Ring theory, Proceedings of the 1978 Antwerp Conference, van Oystaeyen, F., Ed.; Dekker: New York, 1979; 669–690.
6. Sharpe, D.W.; Vamos, P. *Injective Modules*; Cambridge Univ. Press: Cambridge, 1972.
7. Özcan, A.Ç. On a Class of Modules. Publ. Math. Debrecen **2000**, *57* (3–4), 269–275.
8. Rayar, M. On Small and Cosmall Modules. Acta Math. Acad. Sci. Hungar. **1982**, *39*, 389–392.
9. McConnell, J.C.; Robson, J.C. *Noncommutative Noetherian Rings*; Wiley-Interscience: Chichester, 1987.
10. McConnell, J.C.; Robson, J.C. Homomorphisms and Extensions of Modules Over Certain Differential Polynomial Rings. J. Algebra **1973**, *26*, 319–342.
11. Cozzens, J.H. Homological Properties of the Ring of Differential Polynomials. Bull. Amer. Math. Soc. **1970**, *76*, 75–79.
12. Özcan, A.Ç. Some Characterizations of *V*-modules and Rings. Vietnam J. Math. **1998**, *26*, 253–258.
13. Osofsky, B.L. On Twisted Polynomial Rings J. Algebra **1971**, *18*, 597–607.
14. Chatters, A.W.; Hajarnavis, C.R. *Rings With Chain Conditions*; Pitman: Boston, 1980.
15. Faith, C. *Algebra II Ring Theory*; Springer-Verlag: Berlin, 1976.
16. Bergman, G.M. A Ring Primitive on the Right But Not On The Left. Proc. Amer. Math. Soc. **1964**, *15*, 473–475.
17. Irving, R.S. On The Primitivity of Certain Ore Extensions. Math. Ann. **1979**, *242*, 177–192.
18. Jategaonkar, A.V. A Counter–Example in Ring Theory and Homological Algebra. J. Algebra **1969**, *12*, 418–440.
19. Goodearl, K.R.; Warfield R.B. Jr, *An Introduction to Noncommutative Noetherian Rings*; Cambridge Univ. Press: Cambridge, 1989.
20. Roseblade, J.E. Group Rings of Polycyclic Groups. J. Pure Appl. Algebra **1973**, *3*, 307–328.
21. Goodearl, K.R. *Von Neumann Regular Rings*; Pitman: London, 1979.
22. Stenström, B. *Rings of Quotients*; Springer-Verlag: Berlin, 1975.

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