A Generalization of Semiregular and Almost Principally Injective Rings

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Dedicated to Professor Abdullah Harmancı, on his 65th birthday.

Abstract. In this article, we call a ring R *right almost I–semiregular* if, for any $a \in R$, there exists a left R–module decomposition $l_R r_R(a) = P \oplus Q$ such that $P \subseteq Ra$ and $Q \cap Ra \subseteq I$, where I is an ideal of R, l and r are the left and right annihilators, respectively. This definition generalizes the right almost principally injective rings defined by Page and Zhou [10], I–semiregular rings defined by Nicholson and Yousif [7], and right generalized semiregular rings defined by Xiao and Tong [11]. We prove that R is I–semiregular if and only if, for any $a \in R$, there exists a decomposition $l_R r_R(a) = P \oplus Q$, where $P = Re \subseteq Ra$ for some $e^2 = e \in R$ and $Q \cap Ra \subseteq I$. Among the results for right almost I–semiregular rings, we are able to show that if I is the left socle $Soc(_R R)$ or the right singular ideal $Z(R_R)$ or the ideal $Z(_R R) \cap \delta(_R R)$, where $\delta(_R R)$ is the intersection of essential maximal left ideals of R, then R being right almost I–semiregular implies that R is right almost J–semiregular, where J is the Jacobson radical of R. We show that $\delta_l(eRe) = e\delta(_R R)e$ for any idempotent e of R satisfying $ReR = R$ and, for such an idempotent, R being right almost $\delta(_R R)$–semiregular implies that eRe is right almost $\delta_l(eRe)$–semiregular.

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1 Introduction

Throughout this paper, R denotes an associative ring with identity and all modules are unitary right R–modules.

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Let M be an R–module and F a submodule of M_R. Following Alkan and Özcan [1], M is called *F–semiregular* if, for any $m \in M$, there exists a decomposition $M = P \oplus Q$ such that P is projective, $P \subseteq mR$ and $Q \cap mR \subseteq F$. If F is a fully–invariant submodule of M_R, then M is F–semiregular if and only if, for any $m \in M$, there exists a decomposition $mR = P \oplus S$ such that P is a projective (direct) summand of M and $S \subseteq F$. A ring R is called *I–semiregular* for an ideal I of R if R_R is an I–semiregular module. Such rings are studied in [7] and [9]. Note that being I–semiregular for an ideal I of a ring R is left-right symmetric by [9, Lemma 27 and Theorem 28].

A module M is said to be *principally injective* (or *P–injective* for short) if $l_M r_R(a) = Ma$ for all $a \in R$, where l and r are the left and right annihilators, respectively. As a generalization of P–injective modules, Page and Zhou [10] call a module M *almost principally injective* (or *AP–injective* for short) if, for any $a \in R$, there exists an S–submodule X_a of M such that $l_M r_R(a) = Ma \oplus X_a$ as S–modules, where $S = End_R(M)$. A ring R is called *right AP–injective* if R_R is AP–injective.

In [13], M is called *almost principally quasi–injective* (or *APQ–injective* for short) if, for any $m \in M$, there exists an S–submodule X_m of M such that $l_M r_R(m) = Sm \oplus X_m$, where $S = End_R(M)$. Then R_R is APQ–injective if and only if R_R is AP–injective.

In this article, we call a right R–module M *almost F–semiregular* if, for any $m \in M$, there exists an S–module decomposition $l_M r_R(m) = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \subseteq F$, where $S = End_R(M)$ and F is a submodule of ${}_SM$. A ring R is called *right almost I–semiregular* for an ideal I of R if R_R is almost I–semiregular. If ${}_SM$ is F–semiregular, then M_R is almost F–semiregular. An APQ–injective module M_R is almost F–semiregular for any S–submodule F of M. Moreover,

$$M_R \text{ is } APQ\text{–injective} \Leftrightarrow M_R \text{ is almost } 0\text{–semiregular}.$$

Right almost J-semiregular rings are examined in [11] and named as right generalized semiregular rings.

In Section 2, firstly we give a new characterization of F–semiregular modules by modifying the definition of almost F–semiregular modules. Next, we give conditions under which a right almost I–semiregular ring is I–semiregular. Some of the results in [11] are extended. We also prove that if R is a right almost I–semiregular ring, then eRe is a right almost eIe–semiregular ring for a right semicentral idempotent e of R (i.e., $eR = eRe$) or an idempotent e of R satisfying $ReR = R$. If the matrix ring $M_n(R)$ is right almost $M_n(I)$–semiregular for an ideal I of R, then R is right almost I–semiregular.

In [1, Corollary 4.6], it is shown that if M_R is projective and $Soc(M)$–semiregular, then M is semiregular (i.e., for any $m \in M$, there exists a decomposition $M = A \oplus B$ such that A is projective, $A \subseteq mR$ and $B \cap mR \ll M$).

In the last section, we prove that if M_R is almost $Soc(_SM)$–semiregular, then M_R is almost semiregular, i.e., for any $m \in M$, there exists an S–module decomposition $l_M r_R(m) = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \ll {_S}M$. We also consider right almost I–semiregular rings for some ideals such as the socle, the singular ideal and the ideal δ. If R is right almost Z_r–semiregular, then R_R satisfies $(C2)$ and is almost semiregular.

The following implications hold for a ring R.

S_l–semiregular $\Rightarrow$ right almost S_l–semiregular $\overset{3.3}{\Rightarrow}$ right almost semiregular $\Rightarrow$ right almost δ_r–semiregular and right almost δ_l-semiregular.

Z_r–semiregular $\Rightarrow$ right almost Z_r–semiregular $\overset{3.17}{\Rightarrow}$ right almost semiregular $\Rightarrow$ right almost δ_r–semiregular and right almost δ_l-semiregular.

Counterexamples to each of the inverse implications are given.

It is well known that $J(eRe) = eJe$ for any idempotent $e \in R$. But $\delta_r(eRe) \neq e\delta_r(R)e$ even for a right semicentral idempotent e (see Example 3.13). However if $e \in R$ is an idempotent with $ReR = R$, then $\delta_r(eRe) = e\delta_r(R)e$. Consequently, if R is right almost $\delta(_RR)$–semiregular and $ReR = R$, then eRe is right almost $\delta_l(eRe)$–semiregular.

The symbols $Rad(M)$, $Soc(M)$ and $Z(M)$ will stand for the Jacobson radical, the socle and the singular submodule of a module M, respectively. In the ring case we use the abbreviations: $S_r = Soc(R_R)$, $S_l = Soc(_RR)$, $Z_r = Z(R_R)$ and $Z_l = Z(_RR)$. We write $J = J(R)$ for the Jacobson radical of R. For a small (resp. an essential) submodule K of M, we write $K \ll M$ (resp. $K \leq_e M$). For any non-empty subset X of R, $l_M(X)$ (resp. $r_M(X)$) is used for the left (resp. right) annihilator of X in M. For any subset N of M, $l_R(N)$ (resp. $r_R(N)$) will denote the left (resp. right) annihilator of N in R.

Following [12], a submodule N of a module M is called *δ–small* in M, denoted by $N \ll_\delta M$, if $N + K \neq M$ for any submodule K of M with M/K singular. Let

$$\delta(M) = \cap\{N \subseteq M : M/N \text{ is singular simple}\}.$$

Then $\delta(M)$ is the sum of all δ–small submodules of M and is a fully invariant submodule of M [12, Lemma 1.5]. Clearly $Rad(M) \leq \delta(M)$. If M is a projective module, then $Soc(M) \subseteq \delta(M)$ [12, Lemma 1.9]. We use δ_r for $\delta(R_R)$ and δ_l for $\delta(_RR)$. Note that δ_r need not be equal to δ_l. For example, if R is the ring of 2×2 upper triangular matrices over a field F, then $\delta_r = S_r$ and $\delta_l = S_l$.

2 Almost F–semiregular Modules

Definition 2.1. Let M be a right R–module, $S = End_R(M)$ and F a submodule of $_SM$. The module M_R is called *almost F–semiregular* if, for any $m \in M$, there exists an S–module decomposition $l_M r_R(m) = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \subseteq F$. A ring R is called *right almost I–semiregular* for an ideal I of R if R_R is almost I–semiregular.

If M_R is APQ–injective, then M_R is almost F–semiregular for any submodule F of ${}_SM$. Moreover, M_R is almost 0–semiregular if and only if M_R is APQ–injective.

Proposition 2.2. *Let M be a right R–module, $S = End_R(M)$ and F any submodule of ${}_SM$. If ${}_SM$ is F–semiregular, then M_R is almost F–semiregular.*

Proof. Let $m \in M$. Then there exists a decomposition ${}_SM = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \subseteq F$. Since $l_M r_R(m) = l_M r_R(m) \cap M$, by the modular law, we have $l_M r_R(m) = P \oplus (l_M r_R(m) \cap Q)$ and $(l_M r_R(m) \cap Q) \cap Sm = Q \cap Sm \subseteq F$. Hence, M_R is almost F–semiregular. □

In particular, if ${}_SM$ is semiregular, then M_R is almost $Rad({}_SM)$–semiregular. If R is an I–semiregular ring for an ideal I, then it is right and left almost I–semiregular, because the notion of I–semiregular rings is left–right symmetric.

When we take the summand P of $l_M r_R(m)$ as a summand of M in Definition 2.1, we have the following result.

Theorem 2.3 *Let M be a right R–module and $S = End_R(M)$. If ${}_SM$ is projective and ${}_SF$ is a fully–invariant submodule of ${}_SM$, then the following are equivalent:*

(1) *${}_SM$ is F–semiregular.*

(2) *For any $m \in M$, there exists an S–module decomposition $l_M r_R(m) = P \oplus Q$, where $P \subseteq Sm$, P is a summand of M and $Q \cap Sm \subseteq F$.*

Proof. (1) $\Rightarrow$ (2) Follows from the proof of Proposition 2.2.

(2) $\Rightarrow$ (1) Let $m \in M$ and $l_M r_R(m) = P \oplus Q$, where $P \subseteq Sm$, P is a summand of M and $Q \cap Sm \subseteq F$. Then $Sm = P \oplus (Q \cap Sm)$, where P is a projective summand of M and $Q \cap Sm \subseteq F$. Hence, ${}_SM$ is F–semiregular. □

By Theorem 2.3, we obtain the following characterization of I–semiregular rings for an ideal I.

Corollary 2.4 *Let I be an ideal of a ring R. The following are equivalent:*

(1) *R is I–semiregular.*

(2) *For any $a \in R$, there exists a decomposition $l_R r_R(a) = P \oplus Q$, where $P = Re \subseteq Ra$ for some $e^2 = e \in R$ and $Q \cap Ra \subseteq I$.*

(3) *For any $a \in R$, there exists a decomposition $r_R l_R(a) = P \oplus Q$, where $P = eR \subseteq aR$ for some $e^2 = e \in R$ and $Q \cap aR \subseteq I$.*

Now we consider the module-theoretic version of right generalized semiregular rings defined by Xiao and Tong [11].

Definition 2.5 Let M be a right R–module and $S = End_R(M)$. M is called *almost semiregular* if, for any $m \in M$, there exists an S–module decomposition $l_M r_R(m) = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \ll M$. A ring R is called a *right almost semiregular* if R_R is almost semiregular.

Clearly, R is right almost J–semiregular if and only if R is right almost semiregular. Semiregular or right AP–injective rings are right almost semiregular by [11, Proposition 1.2]. Example 1.3 in [11] shows that right almost semiregular rings need not be right AP–injective or semiregular.

Let M be a right R–module and $S = End_R(M)$. If ${}_SM$ is semiregular, then M_R is almost semiregular by a proof similar to that of Proposition 2.2. Moreover, if M_R is almost semiregular, then it is almost $Rad({}_SM)$–semiregular. The converse is true if $Rad({}_SM) \ll {}_SM$.

The following result generalizes [11, Lemma 1.4].

Proposition 2.6 *Let I be an ideal of a ring R. If R is right almost I–semiregular and there exists $e^2 = e \in R$ such that $r_R(a) = r_R(e)$ for any $a \in R$, then R is I–semiregular.*

Proof. Let $a \in R$. Then there exists a decomposition $l_R r_R(a) = P \oplus Q$ such that $P \subseteq Ra$ and $Q \cap Ra \subseteq I$ as left ideals. Since $r_R(a) = r_R(e)$ for some $e^2 = e \in R$, $Re = P \oplus Q$ and $a = ae$. Let $e = p + q$, where $p = ra \in P$ and $q \in Q$. Then $a = ae = ara + aq$ and $ra = rara + raq$. Since $ra - rara = raq \in P \cap Q = 0$, ra is an idempotent. Also, we have $a(1 - ra) = a - ara = aq \in Q \cap Ra \subseteq I$. Hence, R is I–semiregular. □

Corollary 2.7 *If $l_R r_R(a)$ is a summand of R for any $a \in R$ and R is right almost I–semiregular for an ideal I, then R is I–semiregular.*

Proof. Let $a \in R$. By hypothesis $l_R r_R(a) = Re$ for some idempotent e. Then $r_R(a) = r_R(e)$ and the claim holds by Proposition 2.6. □

A ring R is called a *right PP–ring* if every principal right ideal of R is projective ([2]), or equivalently, for any $a \in R$, $r_R(a) = eR$ for some idempotent $e \in R$. Hence, we have the following result.

Corollary 2.8 *Let R be a right PP–ring. If R is a right almost I–semiregular ring for an ideal I, then R is I–semiregular.*

Nicholson and Zhou [9, Proposition 41] prove that if R is I–semiregular for an ideal I, then eRe is eIe–semiregular for any idempotent e of R. We consider this property for almost I–semiregular rings.

An idempotent $e \in R$ is called *right semicentral* if $eR = eRe$ [3].

Theorem 2.9 *If R is a right almost I–semiregular ring for an ideal I and e is a right semicentral idempotent of R, then eRe is a right almost eIe–semiregular ring.*

Proof. Let $a \in eRe$. Then there is a decomposition $l_R r_R(a) = P \oplus Q$ such that $P \subseteq Ra$ and $Q \cap Ra \subseteq I$. Since e is right semicentral, by the proof of [11, Proposition 1.11], $l_{eRe} r_{eRe}(a) = eP \oplus eQ$. Then $eP \subseteq eRa = eRea$ and $eQ \cap eRea \subseteq e(eQ \cap eRea)e$. Hence, $eQ \cap eRea \subseteq Q \cap Ra \subseteq I$ implies that $eQ \cap eRea \subseteq eIe$. □

Theorem 2.10 *Let e be an idempotent of R such that $ReR = R$. If R is a right almost I–semiregular ring for an ideal I, then eRe is a right almost eIe–semiregular ring.*

Proof. Follows from the proof of [11, Theorem 1.15]. □

Proposition 2.11 *Let S be a right almost I–semiregular ring for an ideal I of S. If $\varphi : S \to R$ is a ring isomorphism, then R is a right almost $\varphi(I)$–semiregular ring.*

Proof. Let $a \in R$. Then there is a decomposition $l_S r_S(\varphi^{-1}(a)) = P \oplus Q$ such that $P \subseteq S\varphi^{-1}(a)$ and $Q \cap S\varphi^{-1}(a) \subseteq I$. If $x \in l_R r_R(a)$, then $\varphi^{-1}(x) \in l_S r_S(\varphi^{-1}(a))$. Then we obtain a decomposition $l_R r_R(a) = \varphi(P) \oplus \varphi(Q)$, where $\varphi(P) \subseteq Ra$ and $\varphi(Q) \cap Ra \subseteq \varphi(I)$. Hence, R is a right almost $\varphi(I)$–semiregular ring. □

The following result generalizes [11, Corollary 1.16].

Corollary 2.12 *Let I be an ideal of a ring R and let $n \geq 1$. If $M_n(R)$ is right almost $M_n(I)$–semiregular, then R is right almost I–semiregular.*

Proof. Let $S = M_n(R)$. Then $Se_{11}S = S$ and $R \cong e_{11}Se_{11}$, where e_{11} is the $n \times n$ matrix whose $(1, 1)$-entry is 1, others are 0. By Theorem 2.10, $e_{11}Se_{11}$ is right almost $e_{11}M_n(I)e_{11}$–semiregular. Let $\varphi : e_{11}Se_{11} \to R$ be the isomorphism. Since $\varphi(e_{11}M_n(I)e_{11}) = I$, R is right almost I–semiregular by Proposition 2.11. □

3 Special cases: Soc, δ, Z

In this section, we consider a few fully invariant submodules. We begin with some examples.

Recall that if R is a ring and V is an R–R bimodule, *the trivial extension* $R \propto V$ of R by V is the ring with additive group $R \oplus V$ and multiplication $(a, v)(b, w) = (ab, aw + vb)$.

Example 3.1 There exists a right AP-injective ring R that is not semiregular. Hence, there exists a right almost I–semiregular ring R that is not I–semiregular for ideals $I = J$ or $Z(R)$ or $Soc(R)$.

Proof. Let $R = \mathbb{Z} \propto (\mathbb{Q}/\mathbb{Z})$ be the trivial extension. So R is a commutative AP-injective ring that is not semiregular by [7, Examples (8), p. 2435]. R is almost I–semiregular for any ideal I, because R is AP-injective. But R is neither $Z(R)$–semiregular nor $Soc(R)$–semiregular by [7, Theorem 2.4] and [1, Corollary 4.6]. □

Example 3.2 There exists a right almost $Soc(R)$–semiregular ring R that is not $Soc(R)$–semiregular.

Proof. Let $R = \mathbb{Z}_8$. Since R is a self–injective ring, it is almost I–semiregular for any ideal I of R. But since $2R = J \nsubseteq Soc(R) = 4R$, R is not $Soc(R)$–semiregular (see [1, Example 4.21]). □

Example 3.1 also shows that the class of right almost semiregular rings is not closed under homomorphic images, because $R/J \cong \mathbb{Z}$ is not right almost semiregular by [11, Example 4.8].

In [1], it is proved that if M_R is a projective $Soc(M_R)$–semiregular module, then M_R is semiregular.

Proposition 3.3 *Let M be a right R–module and $S = End_R(M)$. If M_R is almost $Soc(_SM)$–semiregular, then M_R is almost semiregular.*

Proof. Let $m \in M$. Then there exists a decomposition $l_M r_R(m) = A \oplus B$ such that $A \subseteq Sm$ and $B \cap Sm \subseteq Soc(_SM)$. By the modular law, $Sm = A \oplus (B \cap Sm)$. Then $B \cap Sm$ is a finite direct sum of simple S–submodules. If every simple submodule of $B \cap Sm$ is in $Rad(_SM)$, then $B \cap Sm \ll M$ and hence M_R is almost semiregular. Assume that there exists a simple submodule S_1 of $B \cap Sm$ such that $S_1 \nsubseteq Rad(_SM)$. Then S_1 is a summand of M and hence a summand of B. Let L_1 be such that $B = S_1 \oplus L_1$. Then $l_M r_R(m) = A \oplus S_1 \oplus L_1$.

Similarly, $L_1 \cap Sm$ is a finite direct sum of simple submodules. If every simple submodule of $L_1 \cap Sm$ is in $Rad(_SM)$, then M_R is almost semiregular. Assume that there exists a simple submodule S_2 of $L_1 \cap Sm$ such that $S_2 \nsubseteq Rad(_SM)$. Then S_2 is a summand of M and so there exists a submodule L_2 such that $L_1 = S_2 \oplus L_2$. It follows that $l_M r_R(m) = A \oplus S_1 \oplus S_2 \oplus L_2$. This process produces a strictly descending chain $B \cap Sm \supset L_1 \cap Sm \supset L_2 \cap Sm \ldots$. Since $B \cap Sm$ is semisimple and finitely generated, it is Artinian. Hence, this process must stop so that $L_n \cap Sm \subseteq Rad(_SM)$ for some positive integer n. Hence, $l_M r_R(m) = (A \oplus S_1 \oplus \ldots \oplus S_n) \oplus L_n$, where $A \oplus S_1 \oplus \ldots \oplus S_n \leq Sm$ and $L_n \cap Sm \ll M$. Thus, M_R is almost semiregular. □

Corollary 3.4 *If R is right almost S_l–semiregular, then R is right almost semiregular.*

The next example shows that the converse of Corollary 3.4 is not true in general.

Example 3.5 There exists a right almost semiregular ring that is not right almost S_l (S_r)–semiregular.

Proof. (Camillo Example) (see [8, p. 39 and p. 114]) Let $R = \mathbb{Z}_2[x_1, x_2, \ldots]$, where the x_i are commuting indeterminants satisfying the relations $x_i^3 = 0$ for all i, $x_i x_j = 0$ for all $i \neq j$ and $x_i^2 = x_j^2$ for all i and j. Let $m = x_1^2 = x_2^2 = \ldots$. Then R is a commutative local uniform (i.e., every nonzero right ideal is essential) ring. Then R is semiregular with $J = \mathrm{Span}_{\mathbb{Z}_2}\{m, x_1, x_2, \ldots\}$ and $S_l = S_r = J^2 = \mathbb{Z}_2 m$. We claim that R is not (right) almost S_l–semiregular. Let $a = x_1 + x_2$. If R is almost S_l–semiregular, then there exists a decomposition $l_R r_R(a) = P \oplus Q$ such that $P \subseteq Ra$ and $Q \cap Ra \subseteq S_l$. Since $l_R r_R(a)$ is uniform, either $P = 0$ or $Q = 0$. If $P = 0$, then we have that $l_R r_R(a) \cap Ra = Ra \subseteq S_l$, a contradiction. If $Q = 0$, then $l_R r_R(a) = Ra$. But since $r_R(a) = \mathrm{Span}_{\mathbb{Z}_2}\{m, x_3, x_4, \ldots\}$, $x_1 \in l_R r_R(a)$ and $x_1 \notin Ra$. This gives a contradiction. Hence, R is not almost S_l–semiregular. □

If R is right almost S_l–semiregular, then R need not be semiregular, because right AP–injective rings need not be semiregular (see Example 3.1).

We know from [9, Corollary 30] that R is S_l–semiregular if and only if R/S_l is (von Neumann) regular. If R is right almost S_l–semiregular, then $(Ra+S_l)/S_l$ is a summand of $(l_R r_R(a) + S_l)/S_l$ for any $a \in R$ by [4, Lemma 18.4].

Note also that if R is S_l–semiregular, then R is semiregular, $J \subseteq S_l$ and $Z_r \subseteq S_l$ by [7, Theorem 1.2], [1, Theorem 2.3] and by the proof of [1, Theorem 4.5]. On the other hand, J or Z_r need not be contained in S_l if R is right almost S_l–semiregular (see Example 3.2).

According to [11], we know that if R is right almost semiregular, then $Z_r \subseteq J$. Hence, if R is right almost S_l–semiregular, then $Z_r \subseteq J$.

Because of the fact that $S_l \subseteq \delta_l$, R being right almost S_l–semiregular implies that R is right almost δ_l–semiregular. Also if R is δ_l–semiregular, then $Z_r \subseteq \delta_l$ by [7, Theorem 1.2]. We have the following result for right almost δ_l–semiregular rings.

Proposition 3.6 *If R is right almost δ_l–semiregular and R/S_l is a projective right R–module, then $Z_r \subseteq \delta_l$.*

Proof. Let $a \in Z_r$. If $a \notin \delta_l$, then there exists an essential maximal left ideal N of R such that $a \notin N$. Then $R = Ra + N$. Write $1 = ya + n$, where $y \in R$ and $n \in N$. Since Z_r is an ideal and $R \neq Z_r$, we have $n \neq 0$. Since $r_R(ya) \cap r_R(n) = 0$ and $ya \in Z_r$, we obtain that $r_R(n) = 0$. By hypothesis, $R = l_R r_R(n) = P \oplus Q$, where $P = Re \subseteq Rn$, $Q \cap Rn \subseteq \delta_l$ and $e^2 = e \in R$.

Let $\overline{R} = R/S_l$. If $\overline{R} = 0$, then R is semisimple and $Z_r = 0 \subseteq \delta_l = R$. Assume that $\overline{R} \neq 0$. If $\overline{e} = \overline{1}$, then $\overline{R}\overline{n} = \overline{N} = \overline{R}$. Since $S_l \subseteq N$, $N = R$, which is a contradiction. So $\overline{e} \neq \overline{1}$. Since $r_R(ya) \leq_e R$, $\overline{R}/\overline{r_R(ya)} \cong R/(r_R(ya) + S_l)$ is a singular right R–module. This implies that $\overline{r_R(ya)} \leq_e \overline{R}$, because $\overline{R}$ is a projective right R–module. Since $\overline{r_R(ya)} \subseteq r_{\overline{R}}(\overline{ya})$, we have that $r_{\overline{R}}(\overline{ya}) \leq_e \overline{R}$.

Now $(\overline{1} - \overline{e})\overline{R} \cap r_{\overline{R}}(\overline{ya}) \neq 0$. Let $0 \neq (\overline{1} - \overline{e})\overline{r} \in (\overline{1} - \overline{e})\overline{R} \cap r_{\overline{R}}(\overline{ya})$. Let $n = se + t$, where $s \in R$ and $t \in Q$. Then $t = n - se \in Q \cap Rn \subseteq \delta_l$ and $\overline{t} \in \delta_l/S_l = J(R/S_l)$ by [12, Corollary 1.7]. So $\overline{1} - \overline{t}$ is unit in $\overline{R}$. Also, we have $\overline{n}(\overline{1}-\overline{e})\overline{r} = (\overline{1}-\overline{ya})(\overline{1}-\overline{e})\overline{r} = (\overline{1}-\overline{e})\overline{r}$ and $\overline{n}(\overline{1}-\overline{e})\overline{r} = (\overline{se}+\overline{t})(\overline{1}-\overline{e})\overline{r} = \overline{t}(\overline{1}-\overline{e})\overline{r}$. Then $(\overline{1} - \overline{t})(\overline{1} - \overline{e})\overline{r} = \overline{0}$. Hence, $(\overline{1} - \overline{e})\overline{r} = \overline{0}$, a contradiction. □

Proposition 3.7 *If R is right almost δ_l–semiregular, R/S_l is a projective right R–module and $S_l \subseteq Z_l$, then $Z_r \subseteq J$.*

Proof. By a proof similar to that of Proposition 3.6. □

Example 3.8 There exists a right almost δ_l (or δ_r)-semiregular ring that is not right almost semiregular.

Proof. [12, Example 4.3] Let F be a field and $I = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, and

$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) \mid n \in \mathbb{N},\, x_i \in M_2(F),\, x \in I\}.$$

Then R is δ_r (δ_l)-semiregular but not semiregular by [12]. Since every nonzero one–sided ideal contains a nonzero idempotent, $Z_r = Z_l = J = 0$. If R was right almost semiregular, then R would be regular by [11, Lemma 3.1], which is a contradiction. Hence, R is not right almost semiregular. □

It is well known that $J(eRe) = eJe$ for any idempotent e of R. We consider this property for δ which will be used in the forthcoming corollary. Recall by [12, Theorem 1.6] that

$$\begin{aligned}\delta_r &= \{x \in R : \forall y \in R, \exists \text{a semisimple right ideal}\, Y \text{ of}\, R \ni R_R = (1-xy)R \oplus Y\} \\ &= \bigcap \{\text{ideals}\, P \text{ of}\, R \,:\, R/P \,\text{has a faithful singular simple module}\}\end{aligned}$$

Theorem 3.9 *Let e be an idempotent of R such that $ReR = R$. Then $\delta_l(eRe) = e\delta_l e$.*

Proof. We know that if e is an idempotent such that $ReR = R$, then the category of left R–modules, R–Mod, and the category of left eRe–modules, eRe–Mod, are Morita equivalent (see [6]) under the functors given by

$$\begin{array}{llll}\mathcal{F}: & R\text{-}Mod \longrightarrow eRe\text{-}Mod, & \mathcal{G}: & eRe\text{-}Mod \longrightarrow R\text{-}Mod \\ & M \longmapsto eM & & T \longmapsto Re \otimes_{eRe} T.\end{array}$$

By [12], $\delta_l = R$ if and only if R is semisimple. Therefore if $\delta_l = R$, then R is semisimple and so is eRe. This gives that $\delta_l(eRe) = eRe = e\delta_l e$.

Now assume that $\delta_l \neq R$. Let P be an ideal of R such that R/P has a faithful singular simple module N. Denote $\overline{R} = R/P$. Since $\overline{R}\overline{e}\overline{R} = \overline{R}$, the categories $\overline{R}$-Mod and $\overline{e}\overline{R}\overline{e}$-Mod are Morita equivalent. So $\overline{e}N$ is a faithful $\overline{e}\overline{R}\overline{e}$–module by [6, 18.47 and 18.30], a singular $\overline{e}\overline{R}\overline{e}$–module by [5, p. 34] and a simple $\overline{e}\overline{R}\overline{e}$–module. Since $\overline{e}\overline{R}\overline{e} \cong eRe/ePe$, we have that $\delta_l(eRe) \subseteq ePe \subseteq P$. This holds for any ideal P such that R/P has a faithful singular simple module. Thus, $\delta_l(eRe) \subseteq e\delta_l e$.

For the reverse inclusion, let $a \in \delta_l$. Then $Reae \ll_\delta R$. Now we claim that $eRe(eae) \ll_\delta eRe$. Let K be a left ideal of eRe such that $eRe = eRe(eae) + K$. Write $e = ereae + k$, where $r \in R$ and $k \in K$. This implies that $1 = e + (1-e) = ereae + k + (1-e) \in Reae + RK + R(1-e)$ and so $R = Reae + RK + R(1-e)$. Since $Reae \ll_\delta R$, there exists a semisimple projective left ideal Y of R such that $Y \subseteq Reae$ and $R = Y \oplus [RK + R(1-e)]$ by [12, Lemma 1.2]. Hence, we obtain that $eRe = eYe + (eRe)K = eY + K$. Since $Y \cap RK = 0$, we have that $eY \cap K = 0$. On the other hand, since $ReR = R$, eY is a semisimple projective left eRe–module. So $eRe = eY \oplus K$, $eY \subseteq eRe(eae)$ and eY is a semisimple projective eRe–module. By [12, Lemma 1.2], $eRe(eae) \ll_\delta eRe$. Thus, $e\delta_l e \subseteq \delta_l(eRe)$. □

Corollary 3.10 *Let e be an idempotent of R such that $ReR = R$. If R is right almost δ_l–semiregular, then eRe is right almost $\delta_l(eRe)$–semiregular.*

Proof. Follows from Theorems 3.9 and 2.10. □

Now we consider the ring eRe, where e is a right semicentral idempotent.

Theorem 3.11 *If e is a right semicentral idempotent of R, then $e\delta_l e \subseteq \delta_l(eRe)$ and $\delta_r(eRe) \subseteq e\delta_r e$.*

Proof. Let $a \in \delta_l$. Since δ_l is an ideal, $eae \in \delta_l$. By [12, Theorem 1.6], there exists a semisimple left ideal Y of R such that ${}_RR = R(1-eae) \oplus Y$. Let $1 = x(1-eae)+y$, where $x \in R$ and $y \in Y$. Then $e = ex(1-eae)e + eye = exe(e-eae)+eye$ and so $eRe = eRe(e-eae)+eYe$. Since e is right semicentral, this sum is direct. Now we claim that eYe is semisimple. Let $Y = \oplus_{i=1}^n S_i$, where S_i is a simple left R-module, for $i = 1, 2, \ldots, n$. Since e is right semicentral, $eYe = \oplus_{i=1}^n eS_ie$. Let $S_1 = Rs$ for some $s \in R$. Then $eS_1e = eRse = eRe(ese) \cong eRe/l_{eRe}(ese)$. Let K be a left ideal of eRe such that $l_{eRe}(ese) \subset K$. Then there exists $k \in K$ such that $k \notin l_{eRe}(ese)$. Since $l_{eRe}(ese) = l_{eRe}(es) = l_R(es) \cap eRe$, $k \notin l_R(es)$. Then $kes \neq 0$. But since $l_R(s)$ is maximal in R, we have that $l_R(s) + Rke = R$. Let $1 = x + yke$, where $x \in l_R(s)$ and $y \in R$. Then $e = ex + eyek$. Since $xs = 0$, we have $exese = 0$. Then $ex \in l_{eRe}(ese) \subset K$, so $ex \in K$. It follows that $e \in K$. Hence, we show that $l_{eRe}(ese)$ is a maximal left ideal of eRe. So eS_1e is simple. This proves that eYe is semisimple. Now $eRe = eRe(e-eae) \oplus eYe$ with eYe semisimple. Since a is any element in δ_l, we have that $e\delta_l e \subseteq \delta_l(eRe)$.

For the other inclusion, let P be an ideal of R and V be a faithful singular simple right R/P–module. Then Ve is an eRe–module. If $Ve = 0$, then $\delta_r(eRe) \subseteq eRe \subseteq P$.

Assume that $Ve \neq 0$. Since V is a simple R–module, Ve is a simple eRe–module. We claim that Ve is a singular eRe–module. Let ve be the generator of Ve. To show that $r_{eRe}(ve) = r_R(v) \cap eRe$ is an essential right ideal of eRe, let $0 \neq exe \in eRe$. Since $ex \neq 0$ and $r_R(v)$ is essential in R, there exists $t \in R$ such that $0 \neq ext \in r_R(v)$. Then $0 \neq ext = exte \in r_{eRe}(ve)$ (e is right semicentral). Hence, Ve is a singular simple eRe–module. Now, $V\delta_r(eRe) = Ve\delta_r(eRe) = 0$ by the definition of δ. Since V is a faithful R/P–module, we have that $\delta_r(eRe) \subseteq P$. Therefore $\delta_r(eRe) \subseteq P$ for each ideal P of R such that R/P has a faithful singular simple module. So $\delta_r(eRe) \subseteq \delta_r$ and hence $\delta_r(eRe) \subseteq e\delta_r e$. □

Corollary 3.12 *Let e be a right semicentral idempotent of R. If R is right almost δ_l–semiregular, then eRe is right almost $\delta_l(eRe)$–semiregular.*

Proof. Follows from Theorems 3.11 and 2.9. □

The following example shows that the equality $e\delta_l e = \delta_l(eRe)$ does not hold even for a right semicentral idempotent.

Example 3.13 There exists a right semicentral idempotent $e \in R$ such that $e\delta_l e \subset \delta_l(eRe)$.

Proof. Let R be the ring of 2×2 upper triangular matrices over a field F and $e = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Then $eR = eRe$ and $e\delta_l e = 0$, where δ_l is the first row of R. Since eRe is a semisimple projective left eRe–module, $\delta_l(eRe) = eRe$. □

Recall that R_R is said to satisfy $(C2)$ if any right ideal of R isomorphic to a summand of R_R is itself a summand of R. We have the following results about right almost Z_r (Z_l)–semiregular rings.

Theorem 3.14 *Let I be an ideal of R. If R is right almost I–semiregular and $I \subseteq Z_r$, then R_R satisfies $(C2)$.*

Proof. Let $a \in R$ such that $aR \cong eR$, where $e^2 = e \in R$. By [10, Lemma 2.12], there exists an idempotent $f \in R$ such that $a = af$ and $r_R(a) = r_R(f)$. By the proof of Proposition 2.6, there exists an idempotent $h \in R$ such that $h \in Ra$ and $a(1-h) \in I$. By [9, Lemma 27], there exists an idempotent $g \in R$ such that $g \in aR$ and $(1-g)a \in I$. Then $aR = gR \oplus S$, where $S = (1-g)aR \subseteq I$. By assumption, S is a singular right R–module. Since aR is projective, we have that $S = 0$. Thus, $aR = gR$. □

Corollary 3.15 *Let R be a right PP–ring and I an ideal of R. If R is right almost I–semiregular and $I \subseteq Z_r$, then R is regular.*

Proof. Let $a \in R$ and $r_R(a) = eR$, where e is an idempotent of R. Then $aR \cong (1-e)R$. By Theorem 3.14, there exists an idempotent $g \in R$ such that $aR = gR$. Hence, R is regular. □

Corollary 3.16 *If R is right almost Z_r–semiregular, then R_R satisfies $(C2)$.*

We know from [7, Lemma 2.3] that if R_R satisfies $(C2)$, then $Z_r \subseteq J$. Hence, we have the following result.

Corollary 3.17 *If R is right almost Z_r–semiregular, then R is right almost semiregular.*

The following two examples show that the converse of Corollary 3.17 is not true in general.

Example 3.18 There is an Artinian ring R such that R is Z_l–semiregular but not right almost Z_r–semiregular.

Proof. Let $R = \begin{bmatrix} \mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Then

$$S_r = \begin{bmatrix} 2\mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}, S_l = \begin{bmatrix} 2\mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix},$$

$$Z_r = l_R(S_r) = \begin{bmatrix} 2\mathbb{Z}_4 & 0 \\ 0 & 0 \end{bmatrix}, Z_l = r_R(S_l) = \begin{bmatrix} 2\mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}.$$

By [9, Example 40], R is Z_l–semiregular but not Z_r–semiregular. Now we claim that R is not right almost Z_r–semiregular. Let $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in R. Then $Ra = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ and $l_R r_R(a) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. If R is right almost Z_r–semiregular, then there is a decomposition $l_R r_R(a) = P \oplus Q$, where $P \subseteq Ra$ and $Q \cap Ra \subseteq Z_r$. Since $Ra \cap Z_r = 0$, $Q \cap Ra = 0$. This implies that $Ra = P$ is a summand of $l_R r_R(a)$ which is a contradiction. Hence, R is not right almost Z_r–semiregular. □

Example 3.19 Let R be the ring of 2×2 upper triangular matrices over a field F. Then R is an Artinian ring which does not satisfy $(C2)$ ([8, Example 1.20]). Hence, R is right almost semiregular but not right almost Z_r–semiregular.

Recall that R_R is said to satisfy $(C1)$ if every right ideal of R is essential in a summand of R. A ring R satisfying $(C1)$ and $(C2)$ as a right R-module is called *right continuous*. The following result generalizes [1, Corollary 3.5] in the ring case.

Proposition 3.20 *A ring R is right almost Z_r–semiregular and R_R satisfies $(C1)$ if and only if R is right continuous.*

Proof. It is well known that if R_R is right continuous, then it is semiregular and $Z_r = J$. Now the proof follows from Corollary 3.16. □

The ring R in Example 3.19 is right almost semiregular but not right almost Z_l–semiregular, because $Z_l = 0$ and R is not right AP–injective.

Proposition 3.21 *If R is a right almost Z_l–semiregular and left PP–ring, then R is right AP–injective.*

Proof. Let $a \in R$. By hypothesis, $Ra = P \oplus Q$, where P is a summand of $l_R r_R(a)$ and $Q \subseteq Z_l$. Since Ra is a projective left ideal, Q is projective, and so $Q = 0$. Hence, Ra is a summand of $l_R r_R(a)$. □

Proposition 3.22 *If R is right almost $Z_l \cap \delta_l$–semiregular, then it is right almost semiregular.*

Proof. Let $a \in R$. Then there exists a decomposition $l_R r_R(a) = P \oplus Q$ such that $P \subseteq Ra$ and $Q \cap Ra \subseteq Z_l \cap \delta_l$. We claim that $Q \cap Ra \subseteq J$. Let $x \in Q \cap Ra$. To see that $x \in J$, we must show that $1 - yx$ is left invertible in R for any $y \in R$. Let $u = 1 - yx$, where $y \in R$. Since $x \in \delta_l$, there exists a semisimple left ideal Y of R such that $R(1 - yx) \oplus Y = R$ by [12, Theorem 1.6]. Let $\varphi : R \to Y$ be the projection. Then $\varphi(Q \cap Ra) \subseteq \varphi(Z_l) \subseteq Z(Y) = 0$, and so $Ryx \subseteq Q \cap Ra \subseteq Ker\varphi = R(1 - yx)$. Since $R = Ryx + R(1 - yx)$, we have that $R = R(1 - yx)$. Hence, $x \in J$ and $Q \cap Ra \ll R$. □

Proposition 3.23 *If R is right almost I–semiregular for an ideal I such that $J \cap I = 0$, then $J \subseteq Z_r$.*

Proof. Let $a \in J$ and assume that $a \notin Z_r$. Then there exists a nonzero right ideal K of R such that $r_R(a) \cap K = 0$. Take $s \in K$ such that $as \neq 0$. Let $0 \neq u \in asR$. By hypothesis, $l_R r_R(u) = P \oplus Q$, where $P \subseteq Ru$, $Q \cap Ru \subseteq I$. Without loss of generality we can assume that $u = as$. Then it can be seen that $r_R(as) = r_R(s)$. Then $l_R r_R(as) = l_R r_R(s) = P \oplus Q$. Write $s = das + x$, where $d \in R$ and $x \in Q$. Then $(1 - da)s = x$ and so $u = as = a(1 - da)^{-1}x \in J \cap (Q \cap Ru) \subseteq J \cap I = 0$, a contradiction. Hence, $a \in Z_r$. □

Corollary 3.24 *If R is right almost S_l–semiregular and R/S_l is a projective right R–module, then $J = Z_r$ and R is right almost Z_r–semiregular.*

Proof. Since S_l is a summand of R, $J \cap S_l = Rad(S_l) = 0$. By Proposition 3.23, $J \subseteq Z_r$. By Corollary 3.4, R is right almost semiregular. Then $Z_r \subseteq J$ and hence $J = Z_r$ and R is right almost Z_r–semiregular. □

The following example shows that the assumption "$J \cap I = 0$" in Proposition 3.23 is not removable in case $I = Z_l$.

Example 3.25 Let R be the ring in Example 3.18. R is a right almost Z_l–semiregular ring. Since $J = \begin{bmatrix} 2\mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$, $J \cap Z_l \neq 0$ and $J \nsubseteq Z_r$.

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