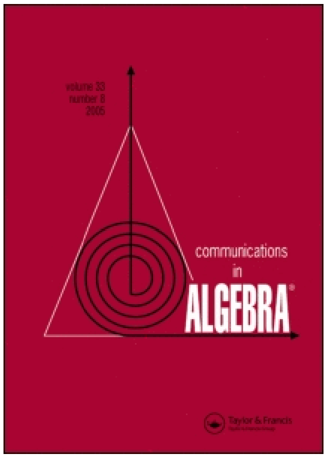

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δ-M-SMALL AND δ-HARADA MODULES

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Let M be a right R-module and $N \in \sigma[M]$. A submodule K of N is called δ-M-small if, whenever $N = K + X$ with N/X M-singular, we have $N = X$. N is called a δ-M-small module if $N \cong K$, K is δ-M-small in L for some $K, L \in \sigma[M]$. In this article, we prove that if M is a finitely generated self-projective generator in $\sigma[M]$, then M is a Noetherian QF-module if and only if every module in $\sigma[M]$ is a direct sum of a projective module in $\sigma[M]$ and a δ-M-small module. As a generalization of a Harada module, a module M is called a δ-Harada module if every injective module in $\sigma[M]$ is δ_M-lifting. Some properties of δ-Harada modules are investigated and a characterization of a Harada module is also obtained.

Key Words: Harada module and ring; Injective module; Lifting module; Noetherian QF-module; Small module.

2000 Mathematics Subject Classification: 16L30; 16E50.

1. INTRODUCTION

Let R denote an associative ring with unit, Mod-R the category of unital right R-modules, and M a unitary right R-module.

We write $Soc(M)$ and $Rad(M)$ for the socle and the Jacobson radical of a module M, respectively. $\widehat{N}$ and $Z_M(N)$ is the M-injective hull and the M-singular submodule of N in $\sigma[M]$, respectively. Recall that $Z_M^2(N)$ is defined as $Z_M(N/Z_M(N)) = Z_M^2(N)/Z_M(N)$ for a module $N \in \sigma[M]$. The notions $K \leq^{\oplus} M$ and $K \leq_e M$ are reserved for a direct summand K and essential submodule K of M, respectively.

A submodule K of a module M is called *small*, (notation $K \ll M$) if $M = K + L$ for some submodule L of M, then we have $L = M$. A module N is called an *M-small module* if $N \cong K \ll L \in \sigma[M]$. In case $M = R$, N is called a *small module*. A module M is called *lifting* (or (D1)) if, for all $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is small in M (Mohamed and Müller, 1990).

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Now we consider some generalizations of the notions "small" and "lifting." Zhou (2000) generalized the notion of a *small submodule* to a *δ-small submodule*. More general, a submodule K of a module N in $\sigma[M]$ is called a *δ-M-small submodule* (notation $K \ll_{\delta_M} N$) if, whenever $N = K + X$ with N/X M-singular, we have $N = X$ (Özcan, 2002).

A module N is called a *δ-M-small module* if $N \cong K \ll_{\delta_M} L \in \sigma[M]$ (Özcan, 2002). We call a module N in $\sigma[M]$ *δ_M-lifting* if, for all $K \leq N$, there exists a decomposition $N = A \oplus B$ such that $A \leq K$ and $K \cap B$ is δ-M-small in N. Clearly, lifting modules in $\sigma[M]$ are δ_M-lifting for any module M.

Recall that a ring R is called a *right Harada ring* (or a right H-ring) if every injective right R-module is lifting (see for example Harada, 1979; Oshiro, 1984a,b). As a module theoretic version of Harada rings, Harada modules are defined in Jayaraman and Vanaja (2000) as modules M such that every injective module in $\sigma[M]$ is lifting. Equivalently, every module in $\sigma[M]$ is a direct sum of an injective module in $\sigma[M]$ and an M-small module.

In this article, δ-Harada modules are defined as an analog of Harada modules. We call a module M a *δ-Harada module* if every injective module in $\sigma[M]$ is δ_M-lifting.

In Chapter 2, we study δ-M-small submodules, and modules with some chain conditions on δ-M-small submodules. We also prove the following theorem.

Theorem. Let M be a module such that finitely generated self-projective and a generator in $\sigma[M]$. Then M is a Noetherian QF-module if and only if every module in $\sigma[M]$ is a direct sum of a projective module in $\sigma[M]$ and a δ-M-small module in $\sigma[M]$.

In Chapter 3, after giving some properties of δ_M-lifting modules, we investigate δ-Harada modules. We prove the following theorem.

Theorem. M is a δ-Harada module if and only if every module in $\sigma[M]$ is a direct sum of an injective module in $\sigma[M]$ and a δ-M-small module.

Corollary. If M is a δ-Harada module, then $M/Soc(M)$ is locally noetherian.

Also we have a characterization of Harada modules.

Theorem. The following are equivalent for a module M:

1. M is a Harada module;
2. M is locally Noetherian and every non-δ-M-small module in $\sigma[M]$ contains a nonzero injective submodule;
3. There exists a subgenerator N in $\sigma[M]$ such that N is $\sum$-lifting and M-injective, and for any exact sequence $P \xrightarrow{f} N \longrightarrow 0$ in $\sigma[M]$ where N is injective in $\sigma[M]$ and $Ker(J) \ll_{\delta_M} P$, P is a direct sum of an injective module in $\sigma[M]$ and a semisimple projective module in $\sigma[M]$.

For the other definitions in this note we refer to Anderson and Fuller (1974) and Wisbauer (1991).

2. δ-M-SMALL MODULES

A submodule K of a module N in $\sigma[M]$ is called a *δ-M-small submodule* (notation $K \ll_{\delta_M} N$) if, whenever $N = K + X$ with N/X M-singular, we have $N = X$ (Özcan, 2002). Define

$$\delta_M(N) = \cap\{K \leq N : N/K \text{ is } M\text{-singular simple}\}$$

and it is the sum of all δ-M-small submodules of N (see Zhou, 2000, Lemma 1.5). Note that every finitely generated submodule of $\delta_M(N)$ is δ-M-small submodule of N. If N is finitely generated module in $\sigma[M]$, then $\delta_M(N) \ll_{\delta_M} N$ (see Zhou, 2000, Lemma 1.5). For any projective module P, $Soc(P) \leq \delta(P)$ (Zhou, 2000, Lemma 1.9), and $J(R/Soc(R_R)) = \delta(R_R)/Soc(R_R)$ (Zhou, 2000, Corollary 1.7) for a ring R.

We begin by stating a lemma which can be seen by a proof similar to Zhou (2000, Lemmas 1.2 and 1.3).

Lemma 2.1. *Let N be a module in $\sigma[M]$.*

1. *If $K \ll_{\delta_M} N$ and $N = K + X$, then $N = Y \oplus X$ for a semisimple projective submodule Y in $\sigma[M]$ with $Y \leq K$.*
2. *If $K \ll_{\delta_M} N$ and $f : N \to L$ is a homomorphism, then $f(K) \ll_{\delta_M} L$. In particular, if $K \ll_{\delta_M} N \subseteq L$, then $K \ll_{\delta_M} L$.*
3. *$K \ll_{\delta_M} N$ and $L \ll_{\delta_M} N$ if and only if $K + L \ll_{\delta_M} N$.*
4. *Let $K_1 \leq M_1 \leq N$, $K_2 \leq M_2 \leq N$ and $N = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_{\delta_M} M_1 \oplus M_2$ if and only if $K_1 \ll_{\delta_M} M_1$ and $K_2 \ll_{\delta_M} M_2$.*

Corollary 2.2. *Let N be a module in $\sigma[M]$. If $K \ll_{\delta_M} N$ and $K \not\ll N$, then K contains a projective simple direct summand of N.*

Proof. By Lemma 2.1(1), K contains a nonzero projective semisimple direct summand of N. □

Corollary 2.3. *Let N be a module in $\sigma[M]$. If $N \ll_{\delta_M} N$, then N is semisimple projective module.*

Proof. As $N = N + 0$, the Corollary follows from Lemma 2.1(1). □

Corollary 2.4. *Let A and B be modules in $\sigma[M]$. Suppose $f : A \to B$ is an epimorphism with $Ker\, f \ll_{\delta_M} A$ and $L \subseteq A$. Then $L \ll_{\delta_M} B$ if and only if $f^{-1}(L) \ll_{\delta_M} A$.*

Proof. By Lemma 2.1(2), if $f^{-1}(L) \ll_{\delta_M} A$, then $L \ll_{\delta_M} B$. Conversely, let $L \ll_{\delta_M} B$. Suppose $A = f^{-1}(L) + K$ where A/K is M-singular. Then $B = L + f(K)$ and $B/f(K)$ is M-singular. As $L \ll_{\delta_M} B$ we have $f(K) = B$. Hence $A = Ker\, f + K$. Now $Ker\, f \ll_{\delta_M} A$ and A/K is M-singular imply $K = A$. □

Al-Khazzi and Smith (1991) investigated the ascending chain condition (ACC) and the descending chain condition (DCC) on $Rad(M)$ for a module M. Now we shall consider the similar results for $\delta_M(N)$.

Clearly if $\delta_M(N)$ is Artinian (Noetherian) for a module $N \in \sigma[M]$, then $Rad(N)$ is Artinian (Noetherian) because $Rad(N) \subseteq \delta_M(N)$. But the converse is not true in general. For example, let $Q = \prod_{i=1}^{\infty} F_i$ where each $F_i = \mathbb{Z}_2$. Let R be the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. Then $J(R) = 0$ and $\bigoplus_{i=1}^{\infty} F_i = Soc(R) = \delta(R)$ (see Zhou, 2000, Example 4.1). Hence $J(R)$ is Artinian (Noetherian) but $\delta(R)$ is not.

The following two propositions can be seen by a proof similar to Proposition 2 and Theorem 5 in Al-Khazzi and Smith (1991). But we give the proofs for completeness.

Proposition 2.5. *The following are equivalent for a module $N \in \sigma[M]$:*

1. *$\delta_M(N)$ is Noetherian;*
2. *N has the ACC on δ-M-small submodules.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Let $X_1 < X_2 < \cdots$ be a strictly ascending chain of submodules of $\delta_M(N)$. Let $x_1 \in X_1$ and $x_i \in X_i - X_{i-1}$ for $i \geq 2$. Clearly, $x_1R < x_1R + x_2R < \cdots$ and, from the definition of $\delta_M(N)$, each x_iR is δ-M-small. Hence, for each n, $\sum_{i=1}^{n} x_iR$ is δ-M-small submodule of N. This follows that N does not have ACC on δ-M-small submodules, a contradiction. □

A module M is called *locally Artinian* if every finitely generated submodule of M is Artinian.

Proposition 2.6. *The following are equivalent for a module $N \in \sigma[M]$:*

1. *$\delta_M(N)$ is Artinian;*
2. *Every δ-M-small submodule of N is Artinian;*
3. *N has the DCC on δ-M-small submodules.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) First we claim that $\delta_M(N)$ is locally Artinian. Let L be a finitely generated submodule of $\delta_M(N)$. Then $L \ll_{\delta_M} N$, and by (3), it is Artinian. Now let K be a proper submodule of $\delta_M(N)$. Let $x \in \delta_M(N) - K$. Then xR is Artinian and $(xR + K)/K$ is a nonzero Artinian module. It follows that $\delta_M(N)/K$ has essential socle.

Suppose that $\delta_M(N)$ is not Artinian. Then there exists a submodule X of $\delta_M(N)$ such that $\delta_M(N)/X$ is not finitely cogenerated. There exists a minimal submodule P of $\delta_M(N)$ with respect to $\delta_M(N)/P$ not finitely cogenerated by Zorn's lemma. Let $Soc(\delta_M(N)/P) = S/P$ where $S \leq \delta_M(N)$. We have seen that S/P is an essential submodule of $\delta_M(N)/P$. Therefore, S/P is not finitely generated (Anderson and Fuller, 1974, Proposition 10.7).

We claim that $P \ll N$. Let $N = P + Q$ for some $Q \leq N$. Then $S = P + (S \cap Q)$. Suppose $P \cap Q \neq P$. Then $\delta_M(N)/(P \cap Q)$ is finitely cogenerated by the choice of P. But $S/P = [P + (S \cap Q)]/P \cong (S \cap Q)/(P \cap Q) \leq Soc(\delta_M(N)/(P \cap Q))$ and hence S/P is finitely generated, a contradiction. Thus $P \ll N$.

Now we claim that $S \ll_{\delta_M} N$. Let $N = S + V$ where N/V is M-singular. Then $N/(P + V) = (S + V)/(P + V) \cong S/[P + (S \cap V)]$. Thus $N/(P + V)$ is semisimple.

If $N \neq P + V$, then there exists a maximal submodule W of N such that $P + V \leq W$. Since N/V is M-singular, $\delta_M(N) \leq W$. But now $S \leq W$ gives the contradiction $N = W$. Then $N = P + V$. Since $P \ll N$, $N = V$. Thus $S \ll_{\delta_M} N$ and by hypothesis S is Artinian. Since S/P is semisimple Artinian, S/P is finitely generated, a contradiction. Thus $\delta_M(N)$ is Artinian. □

Definition 2.7. A module N is called a *δ-M-small module* if $N \cong K \ll_{\delta_M} L \in \sigma[M]$. A module is called *non-δ-M-small* if it is not a δ-M-small module.

Lemma 2.8. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of modules in $\sigma[M]$. If B is δ-M-small, then A and C are δ-M-small.*

Clearly, every M-small module is a δ-M-small module, and any nonzero semisimple non-M-singular injective module in $\sigma[M]$ is a δ-M-small module, but not an M-small module. The following result is by definitions, see also Özcan (2002).

Proposition 2.9. *A module $N \in \sigma[M]$ is a δ-M-small module if and only if $N \ll_{\delta_M} \widehat{N}$.*

If M is a Noetherian injective cogenerator in $\sigma[M]$, then it is called a *Noetherian quasi-Frobenius (QF)-module* (Wisbauer, 1991). For a finitely generated quasi-projective module M, M is a Noetherian QF-module if and only if every injective module in $\sigma[M]$ is projective in $\sigma[M]$ if and only if M is a self-generator and every projective module in $\sigma[M]$ is injective in $\sigma[M]$ by Wisbauer (1991, 48.14). A ring R is called a *quasi-Frobenius ring*, in short QF-ring, if R is Noetherian and injective as a right (or left) R-module. Rayar proved that a ring R is a QF-ring if and only if every right R-module is a direct sum of a projective module and a small module (Rayar, 1982, Theorem 7). Now we generalize this result as follows.

Theorem 2.10. *Let M be a module such that finitely generated self-projective and a generator in $\sigma[M]$. Then the following are equivalent:*

1. *M is a Noetherian QF-module;*
2. *Every module in $\sigma[M]$ is a direct sum of a projective module in $\sigma[M]$ and a δ-M-small module in $\sigma[M]$.*

Proof. (1 ⇒ 2) It follows from Jayaraman and Vanaja (2000, Proposition 3.7).

(2 ⇒ 1) Let N be an injective module in $\sigma[M]$. By the assumption, $N = P \oplus Q$ for a projective module P in $\sigma[M]$ and a δ-M-small module Q. Then Q is injective in $\sigma[M]$. By Proposition 2.9, Q is δ-M-small in Q. By Corollary 2.3, Q is projective in $\sigma[M]$. Hence N is projective in $\sigma[M]$. □

Corollary 2.11. *The following are equivalent for a ring R:*

1. *R is a QF-ring;*
2. *Every right R-module is a direct sum of a projective module and a δ-small module.*

3. δ-HARADA MODULES

In this chapter, we study some properties of δ_M-lifting modules in $\sigma[M]$ for a module M and δ-Harada modules.

A module $N \in \sigma[M]$ is called *δ_M-lifting* if, for all $K \leq N$, there exists a decomposition $N = A \oplus B$ such that $A \leq K$ and $K \cap B \ll_{\delta_M} N$. In case $M = R$, we use $\delta =$ lifting instead of $\delta_R =$ lifting.

Remark 3.1. Clearly any lifting module in $\sigma[M]$ is δ_M-lifting. Suppose $N \in \sigma[M]$ does not contain a projective simple direct summand. By Corollary 2.2, N is δ_M-lifting if and only if it is lifting. Hence if M is indecomposable (for example uniform) or M-singular, then M is δ_M-lifting if and only if it is lifting.

The following lemma can be seen by a proof similar to Koşan (2007, Lemma 2.3).

Lemma 3.2.

1. *The following are equivalent for a module $N \in \sigma[M]$:*
   (a) *N is δ_M-lifting;*
   (b) *For all $K \leq N$, there exists a decomposition $K = A \oplus B$ such that $A \leq^{\oplus} N$ and $B \ll_{\delta_M} N$;*
   (c) *For all $K \leq N$, there exists $A \leq^{\oplus} N$ such that $A \leq K$ and $K/A \ll_{\delta_M} N/A$.*
2. *Any direct summand of a δ_M-lifting module is δ_M-lifting.*

Now we give an example of a δ_M-lifting module.

Example 3.3. $Q = \prod_{i=1}^{\infty} F_i$ where $F_i = \mathbb{Z}_2$. Let R be the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. Then R is δ-semiperfect (i.e., δ_M-lifting) but not semiperfect (i.e., not lifting) by Zhou (2000, Example 4.1).

Theorem 3.4. *Let $N \in \sigma[M]$ be a δ_M-lifting module. If N satisfies DCC (ACC) on δ-M-small submodules, then so also does N/A for any submodule A of N.*

Proof. Suppose N is a δ_M-lifting module and let $A \leq N$. Then $N = X \oplus Y$ with $A = X \oplus (A \cap Y)$ and $A \cap Y \ll_{\delta_M} N$. Consider the natural map $f: Y \to Y/(A \cap Y)$. Then $Ker\, f \ll_{\delta_M} Y$. N has DCC (ACC) on δ-M-small submodules implies that Y has also DCC (ACC) on δ-M-small submodules. From Corollary 2.4 it is easy to conclude $N/A \cong Y/(A \cap Y)$ has DCC (ACC) on δ-M-small submodules. □

By Proposition 2.5 and 2.6, we have the following corollary.

Corollary 3.5. *Let $N \in \sigma[M]$ be a δ_M-lifting module. Then $\delta_M(N)$ is Artinian (Noetherian) if and only if $\delta_M(N/A)$ is Artinian (Noetherian) for every $A \leq N$.*

A family $\{X_i : i \in I\}$ of submodules of a module $N \in \sigma[M]$ is called a *local direct summand of N* if $\sum_{i \in I} X_i$ is direct and $\sum_{i \in F} X_i$ is a direct summand of N for any finite subset F of I. If N is an injective lifting module in $\sigma[M]$, then every local direct summand of N is a direct summand (see Oshiro, 1984b, Lemma 2.5). For an injective δ_M-lifting module in $\sigma[M]$ we have the following result.

Proposition 3.6. *If N is an injective δ_M-lifting module in $\sigma[M]$, then every local direct summand of N is a direct sum of an injective module in $\sigma[M]$ and a semisimple projective module in $\sigma[M]$.*

Proof. Let $N \in \sigma[M]$ be an injective δ_M-lifting module in $\sigma[M]$. Let $X = \Sigma_{i\in I} X_i$ be a local direct summand of N. We have a decomposition $N = M_1 \oplus M_2$ such that $M_1 \leq X$ and $X \cap M_2 \ll_{\delta_M} N$. Thus, $X = M_1 \oplus (X \cap M_2)$. For any $x \in X \cap M_2$, we have $xR \leq X_1 + \cdots + X_n$ for some n. Since $X_1 \oplus \cdots \oplus X_n$ is self-injective, by Mohamed and Müller (1990, Proposition 2.1) $xR \leq_e Z \leq^{\oplus} X_1 \oplus \cdots \oplus X_n$ for some Z. This shows that $Z \cap M_1 = 0$ and so $Z \oplus M_1 \leq^{\oplus} X$. Let $X = Z \oplus M_1 \oplus U$ for some $U \leq X$. Then $Z \oplus U \cong X \cap M_2$. So there exists $Y \leq X \cap M_2$ such that $Z \cong Y$. It follows that $Y \leq^{\oplus} N$. By Lemma 2.1, Y is semisimple projective in $\sigma[M]$ and so is Z. Thus $xR = Z$ is semisimple projective in $\sigma[M]$. Since x can be any element of $X \cap M_2$, $X \cap M_2$ is semisimple projective in $\sigma[M]$. □

Definition 3.7. A module M is called a *δ-Harada module* if every injective module in $\sigma[M]$ is δ-lifting. A ring R is called a *right δ-Harada ring* if every injective right R-module is δ_M-lifting.

Any Harada module is a δ-Harada module.

Theorem 3.8. *The following are equivalent for a module M:*

1. *M is a δ-Harada module;*
2. *Every module in $\sigma[M]$ is a direct sum of an injective module in $\sigma[M]$ and a δ-M-small module.*

Proof. $(1 \Rightarrow 2)$ It is obvious by Lemma 3.2.

$(2 \Rightarrow 1)$ Suppose that every module in $\sigma[M]$ is a direct sum of an injective module in $\sigma[M]$ and a δ-M-small module. Let K be a submodule of an injective module N in $\sigma[M]$. Then K has a decomposition $K = A \oplus B$ such that A is an injective module in $\sigma[M]$ and $B \ll_{\delta_M} \widehat{B}$. Since N is injective in $\sigma[M]$, $B \ll_{\delta_M} N$. Since A is injective, $A \leq^{\oplus} N$. Hence N is δ_M-lifting by Lemma 3.2. □

Consider the following:

$(*)_M$ Every non-M-small module in $\sigma[M]$ contains a nonzero injective submodule;

$(*)_{\delta_M}$ Every non-δ-M-small module in $\sigma[M]$ contains a nonzero injective submodule;

(ICC)$_M$ For any exact sequence $P \xrightarrow{f} N \longrightarrow 0$ in $\sigma[M]$ where N is injective in $\sigma[M]$ and $Ker(f) \ll P$, P is injective in $\sigma[M]$;

(ICC)$_{\delta_M}$ For any exact sequence $P \xrightarrow{f} N \longrightarrow 0$ in $\sigma[M]$ where N is injective in $\sigma[M]$ and $Ker(f) \ll_{\delta_M} P$, P is a direct sum of an injective module in $\sigma[M]$ and a semisimple projective module in $\sigma[M]$.

By Jayaraman and Vanaja (2000, Theorem 2.8) a module M is a Harada module if and only if M is locally Noetherian with $(*)_M$ if and only if there exists a subgenerator N in $\sigma[M]$ such that N is $\sum$-lifting and M-injective, and $(\mathrm{ICC})_M$ holds.

By Oshiro (1984b, Theorem 2.11) and Harada (1979, Proposition 2.1), a ring R is a right Harada ring if and only if R is right Noetherian with $(*)_R$ if and only if R is right Artinian with $(*)_R$ if and only if R is right perfect ring with $(\mathrm{ICC})_R$.

Proposition 3.9. $(*)_M$ *if and only if* $(*)_{\delta_M}$.

Proof. $(\Rightarrow)$ This is clear.

$(\Leftarrow)$ Let $N \in \sigma[M]$ be a non-M-small module. Then there exists a proper submodule X of $\widehat{N}$ such that $\widehat{N} = N + X$. If $N \ll_{\delta_M} \widehat{N}$, then $\widehat{N} = Y \oplus X$ for some semisimple projective submodule Y of N in $\sigma[M]$ by Lemma 2.1. Then Y is a nonzero injective submodule of N in $\sigma[M]$. If N is non-δ-M-small, by hypothesis, N contains a nonzero injective submodule. □

Let M be a module and $N \in \sigma[M]$. P is called a *δ-M-small cover of N* if there exists an epimorphism $f: P \to N$ such that $Ker(f) \ll_{\delta_M} P$.

Theorem 3.10. *Let M be a δ-Harada module. Then the following hold*:

1. *M satisfies* $(*)_M$;
2. *M satisfies* $(ICC)_{\delta_M}$;
3. *Every factor module of an injective module in $\sigma[M]$ has a δ-M-small cover in $\sigma[M]$ which is injective in $\sigma[M]$.*

Proof. (1) By Theorem 3.8, M has $(*)_{\delta_M}$. Then (1) follows from Proposition 3.9.

(2) Let $f: P \longrightarrow N$ be an epimorphism in $\sigma[M]$ where N is injective in $\sigma[M]$ and $Ker(f) \ll_{\delta_M} P$. By Theorem 3.8, $P = X \oplus Y$ where X is injective in $\sigma[M]$ and Y is δ-M-small. We claim that $Y \ll_{\delta_M} P$. Then $N = f(X) + f(Y)$. $f(Y)$ is δ-M-small (see Lemma 2.8). N is injective implies $f(Y) \ll_{\delta_M} N$ (see Proposition 2.9). Since $Ker\, f \ll_{\delta_M} P$, $f^{-1}f(Y) \ll_{\delta_M} P$ (see Corollary 2.4) and hence $Y \ll_{\delta_M} P$.

(3) Let N be injective in $\sigma[M]$ and $K \leq N$. Then N has a decomposition $N = M_1 \oplus M_2$ such that $M_1 \leq K$ and $K \cap M_2 \ll_{\delta_M} N$. Let $f: M_2 \to N/K$ be the canonical epimorphism. Then $Ker(f) = K \cap M_2 \ll_{\delta_M} N$. Hence $K \cap M_2 \ll_{\delta_M} M_2$. So M_2 is an injective δ-M-small cover of N/K. □

Corollary 3.11. $(ICC)_{\delta_M} \Rightarrow (ICC)_M$.

Proof. Let $P \xrightarrow{f} N \longrightarrow 0$ be an exact sequence in $\sigma[M]$ where N is injective in $\sigma[M]$ and $Ker(f) \ll P$. Then $P = X \oplus Y$ where X is injective in $\sigma[M]$ and Y is semisimple projective in $\sigma[M]$. Put $T = \{x \in P : f(x) \in f(X)\}$. Since Y is semisimple,

$P = T \oplus K$ for some submodule K of Y. $f(X) + f(K) = N$ and $Ker\, f \ll P$ imply that $P = X \oplus K$. Hence $T = X$. This implies that $N = f(X) \oplus f(Y)$. Since Y is semisimple and $Ker\, f \ll P$, f is one-to-one on Y. Hence $Y \cong f(Y) \leq N$ implies P is injective. □

Corollary 3.12. *If M is a δ-Harada module, then M has $(ICC)_M$.*

Corollary 3.13. *The following are equivalent for a module M:*

1. *M is a Harada module;*
2. *M is locally Noetherian with $(*)_{\delta_M}$;*
3. *There exists a subgenerator N in $\sigma[M]$ such that N is $\sum$-lifting and M-injective, and $(ICC)_{\delta_M}$ holds.*

Proof. By Jayaraman and Vanaja (2000, Theorem 2.8), Proposition 3.9, and Corollary 3.11. □

Corollary 3.14. *The following are equivalent for a ring R:*

1. *R is a right Harada ring;*
2. *R is right Noetherian with $(*)_\delta$;*
3. *R is right Artinian with $(*)_\delta$;*
4. *R is right perfect with $(ICC)_\delta$.*

Proof. $(1 \Leftrightarrow 2 \Leftrightarrow 3)$ By Oshiro (1984b, Theorem 2.11), Harada (1979, Proposition 2.1) and Proposition 3.9.

$(1 \Leftrightarrow 4)$ By Oshiro (1984b, Theorem 2.11), Theorem 3.10, and Corollary 3.11. □

Corollary 3.15. (a) *Any right Noetherian (or right perfect) right δ-Harada ring is a right Harada ring.*

(b) *A right δ-Harada ring R is a right Harada ring if R has no simple projective module.*

Remark. We couldn't find an example of a right δ-Harada ring which is not right Harada. Such a ring should not be right Noetherian and right perfect. So we have the following open question.

Question. Is there a right δ-Harada ring which is not a right Harada ring?

Corollary 3.16. *If M is a δ-Harada module, then every finitely generated submodule L of M has an ACC on $\{K \leq L : Z^2_M(L/K) = L/K\}$. In particular, $M/Soc(M)$ is locally Noetherian.*

Proof. Let $N \in \sigma[M]$ and assume that $N = \bigoplus_{i \in I} N_i$ where each N_i is M-injective and $Z^2_M(N_i) = N_i$, then $\bigoplus_{i \in I} N_i$ is a local summand of $\widehat{N}$. Since $Z^2_M(N) = N$, it is injective in $\sigma[M]$ by Theorem 3.10(4). By Page and Zhou (1994, Proposition 9 and Lemma 7), every finitely generated submodule L of M has ACC on $\{K \leq L :$

$Z^2_M(L/K) = L/K\}$. Now let L be a finitely generated submodule of M. By Dung et al. (1994, 5.15), $L/Soc(L)$ is Noetherian and hence $M/Soc(M)$ is locally Noetherian. □

So if R is a right δ-Harada ring, then $R/Soc(R_R)$, and hence $R/\delta(R_R)$, is right Noetherian.

Proposition 3.17. *If M is a δ-Harada module with $\delta_M(M) \ll_{\delta_M} M$ and $\delta_M(M) = M \cap \delta_M(\widehat{M})$, then M is δ_M-lifting.*

Proof. Let N be a submodule of M. Since $\widehat{M}$ is δ_M-lifting, there is a decomposition $N = A \oplus B$ such that A is a direct summand of $\widehat{M}$ and B is a δ_M-small module. This implies that A is a direct summand of M and $B \leq M \cap \delta_M(\widehat{M}) = \delta_M(M) \ll_{\delta_M} M$. Hence M is δ_M-lifting. □

Zhou (2000) calls a ring R *δ-semiperfect* if every simple module has a projective δ-cover. R is δ-semiperfect if and only if R_R is δ_M-lifting (Zhou, 2000, Theorem 3.6).

Hence by Proposition 3.17 we have that if R is a right δ-Harada ring with $\delta(R_R) = R \cap \delta(E(R)_R)$, then R is a δ-semiperfect ring.

The following is an example of a ring which is not perfect and not δ-Harada.

Example 3.18. Let $Q = \prod_{i=1}^{\infty} F_i$ where each $F_i = \mathbb{Z}_2$. Let R be the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. Then R is a commutative regular (i.e., cosemisimple; see Anderson and Fuller, 1974), δ-semiperfect ring, and $Soc(R) = \delta(R)$ but not semiperfect (Zhou, 2000). The injective hull of R_R is $E(R_R) = Q_R$. Now we claim that $E(R_R)$ is not δ-lifting. Assume that $E(R_R)$ is δ_M-lifting. Then R has a decomposition $R = A \oplus B$ such that $A \leq^{\oplus} E(R_R)$ and $B \ll_{\delta} E(R_R)$ by Lemma 3.2. Since $R/\delta(R)$ is semisimple, $M\delta(R) = \delta(M)$ for any R-module M (Zhou, 2000, Theorem 1.8). Then we have that $B \leq \delta(E(R)) = E(R)\delta(R) = E(R)Soc(R) \leq Soc(E(R))$. Hence B is semisimple finitely generated submodule of R. Since every simple R-module is injective, we get that B is injective. Consequently, R is self-injective. This gives a contradiction. Hence R is not a δ-Harada ring.

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