# Semi co–Hopfian and Semi Hopfian Modules

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**Abstract**

A module $M$ is called *semi co–Hopfian* (resp. *semi Hopfian*) if any injective (resp. surjective) endomorphism of $M$ has a direct summand image (resp. kernel). In this article, some properties of semi co–Hopfian and semi Hopfian modules are investigated with examples.

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## 1 Introduction

Hopfian and co–Hopfian groups, rings and modules have been studied by many authors since 1960s. Recall that a module $M$ is called *co–Hopfian* (resp. *Hopfian*) if any injective (resp. surjective) endomorphism of $M$ is an isomorphism. Note that any Artinian module is co–Hopfian, and any Noetherian module is Hopfian. In this article, we concerned with semi co–Hopfian and semi Hopfian modules. A module $M$ is called *semi co–Hopfian* (resp. *semi Hopfian*) if any injective (resp. surjective) endomorphism of $M$ has a direct summand image (resp. kernel).

Semi co–Hopfian and semi Hopfian modules are used as a tool by many authors, for example, see [3, 17, 18]. In this paper, we deal with some properties of semi co-Hopfian and semi Hopfian modules and rings, among others direct sums and direct products of them are considered with many examples.

Recall from [13] that a module $M$ has **(C2)** if for any submodule $N$ of $M$ which is isomorphic to a direct summand of $M$, is a direct summand of $M$; and **(D2)** if any submodule $N$ such that $M/N$ is isomorphic to a direct summand of $M$ is a direct summand of $M$. If $M$ has $(C2)$, then $M$ is semi co–Hopfian. If $M$ has $(D2)$, then $M$ is semi Hopfian. Hence any self–injective module is semi co–Hopfian, and any self–projective module is semi Hopfian by [13]. But their converses are not true in general, examples are given in the article.

In the last section, we consider the ring of continuous functions. We prove that for a compact Hausdorff space $X$, if $X$ is a semi co–Hopfian (resp. semi Hopfian) topological space, then $C(X)$ is a semi Hopfian (resp. semi co–Hopfian) $\mathbb{R}$–algebra.

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A module $M$ is called *Dedekind finite* if $M \cong M \oplus N$ for some module $N$, $N = 0$. For a ring $R$, ${}_RR$ is Dedekind finite if and only if $ab = 1$ implies that $ba = 1$ for any $a, b \in R$. It is well known that any co–Hopfian or Hopfian module is Dedekind finite.

Throughout this paper, $R$ denotes an associative ring with identity and modules $M$ are unitary left $R$–modules. For a module $M$, $Rad(M)$, $Soc(M)$ and $Z(M)$ are the Jacobson radical, the socle and the singular submodule of $M$, respectively. In the ring case we use the abbreviations $Z_r = Z(R_R)$ and $Z_l = Z({}_RR)$. For any $m \in M$, $l_R(m)$ will denote the left annihilator of $m$ over $R$. $N \leq^{\oplus} M$ means that $N$ is a direct summand of $M$.

## 2 Semi co–Hopfian Modules

A module $M$ is called *semi co–Hopfian* if any injective endomorphism of $M$ has a direct summand image, i.e. any injective endomorphism of $M$ splits. A ring $R$ is called *left semi co–Hopfian* if ${}_RR$ is a semi co–Hopfian module. In [17] and [18] semi co–Hopfian modules are named $GC2$. They generalized some results about injectivity via modules with GC2.

Clearly, any co–Hopfian module is semi co–Hopfian. The converse is not true in general, for example, let ${}_{\mathbb{Z}}M = \mathbb{Q}^{(\mathbb{N})}$. Since $M$ is quasi–injective, it is semi co–Hopfian (by Lemma 2.1 and [13, Proposition 2.1]). But since $M \cong M \oplus \mathbb{Q}$, $M$ is not Dedekind finite, hence not co–Hopfian. Also it is clear that if $p$ is a prime and $n$ is a positive integer , then any direct sum of copies of $\mathbb{Z}/\mathbb{Z}p^n$ is not a co–Hopfian $\mathbb{Z}$–module, but it is semi co–Hopfian because it is quasi–injective.

**Lemma 2.1** *The following are equivalent for a module $M$.*
*1) $M$ is semi co–Hopfian.*
*2) Any submodule $N$ of $M$ which is isomorphic to $M$, is a direct summand of $M$.*

**Proof** (2 $\Rightarrow$ 1) It is obvious. (1 $\Rightarrow$ 2) Let $N \leq M$ be such that $N \cong M$. Then we have an injective endomorphism $\alpha$ of $M$ where $Im\alpha = N$. By (1), $N$ is a direct summand of $M$. □

By Lemma 2.1, if $M$ has $C2$, then $M$ is semi co–Hopfian. In particular any quasi–injective module is semi co–Hopfian by [13, Proposition 2.1]. Therefore, the concept of semi co–Hopfian modules is a generalization of co–Hopfian modules and modules with $C2$.

**Example 2.2** *There exists a semi co–Hopfian module which has not $C2$.*

**Proof** Let $R$ be the ring of $2 \times 2$ lower triangular matrices over a field $F$. Then $R$ is Artinian and so co–Hopfian, but ${}_RR$ has not $C2$ (see [13, Example 2.9]). □

Any semi co–Hopfian quasi continuous module has $C2$ (see [13, Lemma 3.14]).

For a ring $R$ we have the following characterization.

**Proposition 2.3** *The following are equivalent for a ring $R$.*
*1) $R$ is left semi co–Hopfian.*
*2) If $l_R(a) = 0$, $a \in R$, then $aR$ is a direct summand of $R$.*
*3) If $l_R(a) = 0$, $a \in R$, then $aR = R$.*
*4) Every $R$–isomorphism $Ra \to R$, $a \in R$, extends to $R$.*

**Proof** (1) $\Rightarrow$ (4) If $Ra \cong R$, $a \in R$, then $Ra$ is a direct summand of $R$ by Lemma 2.1. So (4) holds.
(4) $\Rightarrow$ (3) Let $l_R(a) = 0$, $a \in R$. Then the isomorphism $f : Ra \to R$ defined by $f(ra) = r$, $r \in R$, extends to $R$ by $g$. Then $1 = f(a) = g(a) = ag(1) \in aR$.
(3) $\Rightarrow$ (2) It is obvious.
(2) $\Rightarrow$ (1) Let $f : R \to R$ be a left $R$–monomorphism. Since $l_R(f(1)) = 0$, $f(1)R$ is a direct summand of $R$. Hence $Imf = Rf(1)$ is a direct summand of $R$. □

It is well–known that if $R$ is an integral domain, then ${}_RR$ has $C2$ if and only if $R$ is a division ring. The following result is obvious by definitions.

**Proposition 2.4** *If $R$ is a ring with only idempotents 0 and 1, then the following are equivalent.*

*1) ${}_RR$ has $C2$.*

*2) ${}_RR$ is co–Hopfian.*

*3) ${}_RR$ is semi co–Hopfian.*

*In particular, if $R$ is an integral domain, $(1)-(3)$ are equivalent to*

*4) $R$ is a division ring.*

Another example of a semi co–Hopfian module is related with the summand sum property. A module $M$ has the *summand sum property (SSP)* if the sum of any two direct summands of $M$ is a direct summand. Note that $M$ has SSP if and only if for every decomposition $M = A \oplus B$ and every $R$–homomorphism $f$ from $A$ to $B$, the image of $f$ is a direct summand [1].

Hence if $M \oplus M$ has SSP, then $M$ is semi co–Hopfian. But the converse is not true: Let the $\mathbb{Z}$–module $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$. Since $M$ is Artinian, it is co–Hopfian. Denote the $\mathbb{Z}$–homomorphism $f : M \to M$ by $f(\overline{a}, \overline{b}) = (0, \overline{pb})$. Then $Imf = 0 \oplus p\mathbb{Z}_{p^2}$ is not a direct summand of $M$. Hence $M \oplus M$ has not SSP.

If $M$ is co–Hopfian, then it is Dedekind finite. The converse is true if $M$ is self–injective by [7, Proposition 1.4]. Also in [15, Proposition 3.7], it is proved that if $M$ has finite uniform dimension and $C2$, then $M$ is co–Hopfian. Since any self–injective module has $C2$ (see [13, Proposition 2.1]) and a module with a finite uniform dimension is Dedekind finite (see [11, Exercises 6(1)]), as a generalization of [7, Proposition 1.4] and [15, Proposition 3.7] we have the following proposition.

A module $M$ is called *weakly co–Hopfian* (in [7]) if any injective endomorphism of $M$ has an essential image.

**Proposition 2.5** *The following are equivalent for a module $M$.*

*1) $M$ is co–Hopfian.*

*2) $M$ is Dedekind finite and semi co–Hopfian.*

*3) $M$ is weakly co–Hopfian and semi co–Hopfian.*

**Proof** (3)$\Leftrightarrow$ (1) $\Rightarrow$ (2) are obvious.
(2) $\Rightarrow$ (1) Let $f$ be an injective endomorphism of $M$. Then $M = f(M) \oplus K$ for some $K \leq M$. Define a homomorphism $\varphi : M \oplus K \longrightarrow M$ by $\varphi(m, k) = f(m) + k$. Then $\varphi$ is an isomorphism. Since $M$ is Dedekind finite, $K = 0$. Hence $f(M) = M$ and so $f$ is an isomorphism. □

Recall that a ring is *I–finite* if it contains no infinite set of orthogonal idempotents. $R$ is $I$–finite if and only if $R$ has ACC on right direct summands if and only if $R$ has DCC on left direct summands (see [11, 6.59]). If a ring $R$ is $I$–finite and has left $C2$, then $_RR$ is co–Hopfian by [14, Example 7.5]. By the same proof it can be seen that any $I$–finite and left semi co–Hopfian ring is left co–Hopfian.

From now on we investigate some properties of semi co–Hopfian modules.

**Lemma 2.6** *Any direct summand of a semi co–Hopfian module is semi co–Hopfian.*

**Proof** Let $N$ be a direct summand of $M$ and $f : N \longrightarrow N$ a monomorphism. Write $M = N \oplus N'$. Then $g : M \to M$, $g(n + n') = f(n) + n'$ where $n \in N$, $n' \in N'$, is a monomorphism. Since $Img = Imf \oplus N'$ is a direct summand of $M$, we get that $Imf$ is a direct summand of $N$. □

We say that a submodule $N$ of $M$ is a *non–summand* of $M$ if $N$ is not a direct summand of $M$.

**Lemma 2.7** *If for any non–summand submodule $N$ of $M$, $N$ is semi co–Hopfian, then $M$ is semi co–Hopfian.*

**Proof** If $M$ is not semi co–Hopfian, then there exists a non–summand submodule $N$ of $M$ such that $N \cong M$. By hypothesis, we have a contradiction. □

Any finite direct sum of semi co–Hopfian modules need not be semi co–Hopfian.

**Example 2.8** *There exists a simple module $U$ and an injective module $V$ such that $U \oplus V$ is not semi co–Hopfian.*

**Proof** Let $R$ be a right Noetherian ring which is not a right $V$–ring (see [4]). Let $U$ be a simple right $R$–module which is not injective and $E$ denote the injective envelope of $U$. For each positive integer $n$, let $E_n = E$ and let $V = \oplus_{n\geq 1} E_n$. Then $V$ is injective.

Let $M = U \oplus V$. Define $f : M \to M$ by $f((u, e_1, e_2, \ldots)) = (0, u, e_1, e_2, \ldots)$ where $u \in U$, $e_i \in E_i$. Then $f$ is clearly a monomorphism. If $f(M)$ was a direct summand of $M$, then $U$ would have to be a direct summand of $E$, a contradiction. □

**Proposition 2.9** *Let $M = \oplus_{i\in I} M_i$, where $M_i$ is invariant under any injection of $M$ for all $i \in I$. Then $M$ is semi co–Hopfian if and only if $M_i$ is semi co–Hopfian for all $i \in I$.*

**Proof** The necessity is by Lemma 2.6. For the sufficiency, let $f : M \to M$ be a monomorphism. Then restriction of $f$ to $M_i$ ($i \in I$), is an injective endomorphism of $M_i$. By hypothesis, $f(M_i) \leq^{\oplus} M_i$ ($i \in I$). This implies that $M = (\oplus_{i\in I} f(M_i)) \oplus X = f(M) \oplus X$ for some $X \leq M$. Hence $M$ is semi co–Hopfian. □

Note that $U$ is not invariant under the monomorphism $f$ in Example 2.8.

**Proposition 2.10** *A direct product $R = \prod_{i\in I} R_i$ of rings $R_i$ is semi co–Hopfian left $R$–module if and only if each $R_i$ is semi co–Hopfian left $R_i$–module.*

**Proof** Clear by Proposition 2.3(3). □

But any direct product of semi co–Hopfian modules need not be semi co–Hopfian.

**Example 2.11** Let $p$ be prime and the $\mathbb{Z}$–module $M = \prod_{n=1}^{\infty} \mathbb{Z}_{p^n}$. $M$ is not semi co–Hopfian. For define $f : M \to M$ by $f(a_1 + p\mathbb{Z}, a_2 + p^2\mathbb{Z}, a_3 + p^3\mathbb{Z}, \ldots) = (0, pa_1 + p^2\mathbb{Z}, pa_2 + p^3\mathbb{Z}, \ldots)$. Then $f$ is a $\mathbb{Z}$–monomorphism and $Imf = pM$. If $pM$ is a direct summand of $M$, then $M = pM \oplus L$ for some submodule $L$. Since $pL = 0$, $L \subseteq \mathbb{Z}_p \times p\mathbb{Z}_{p^2} \times p^2\mathbb{Z}_{p^3} \times \cdots$. But $(0, 1 + p^2\mathbb{Z}, 0, 0, \ldots) \notin pM \oplus L$. This gives that $pM$ is not a direct summand of $M$.

**Proposition 2.12** *Let a $\mathbb{Z}$–module $M = M' \oplus M''$ be a direct sum of a semisimple module $M'$ and an injective module $M''$ such that $M''$ has finite uniform dimension. Then $M$ is semi co–Hopfian.*

**Proof** Let $f : M \to M$ be any monomorphism. $f(M'')$ is injective and hence $f(M'')$ is contained in $M''$ (since $M$ is a $\mathbb{Z}$–module). Then $M'' = f(M'') \oplus N$ for some submodule $N$. But $M''$ and $f(M'')$ have the same uniform dimension. Therefore $N = 0$. Thus $f(M'') = M''$ and $f(M) = M'' \oplus (f(M) \cap M')$ which is a direct summand of $M$ because $M'$ is semisimple. □

A module $M$ is called *torsion free* if $rm = 0$ then $r = 0$ or $m = 0$ for any $r \in R$ and $m \in M$. Torsion free modules need not be semi co–Hopfian, for example $\mathbb{Z}_{\mathbb{Z}}$.

**Proposition 2.13** *Let $R$ be a commutative domain and let $M$ be a torsion free semi co–Hopfian $R$–module. Then $M$ is injective.*

**Proof** Let $c$ be any non–zero element of $R$. Define $f : M \to M$ by $f(m) = cm$, $m \in M$. Then $f$ is a monomorphism. By assumption there exists a submodule $N \leq M$ such that $M = f(M) \oplus N = cM \oplus N$. Then $cN = 0$ so that $N = 0$. Thus $M = cM$ for all non–zero $c \in R$. Since $M$ is torsion–free it follows that $M$ is injective. □

Now we will consider the descending chain condition (DCC) on non–summands.

**Proposition 2.14** *If $M$ has DCC on non–summand submodules, then $M$ is semi co–Hopfian.*

**Proof** If $M$ is not semi co–Hopfian, then there exists a non–summand submodule $M_1$ of $M$ such that $M_1 \cong M$. Since $M_1$ is not semi co–Hopfian there exists a non–summand submodule $M_2$ of $M_1$ such that $M_2 \cong M_1$. Repeating this argument we have a strictly descending chain of non–summand submodules of $M$. □

Moreover,

**Proposition 2.15** *Let $\mathcal{P}$ be a property of modules preserved under isomorphism. If a module $M$ has the property $\mathcal{P}$ and satisfies DCC on non–summand submodules with property $\mathcal{P}$, then $M$ is semi co–Hopfian.*

**Proof** By a proof similar to Proposition 2.14. □

**Corollary 2.16** *If $M$ has DCC on its non–semi co–Hopfian submodules, then $M$ is semi co–Hopfian.*

As for the endomorphism ring of a semi co–Hopfian module, note that by [18, Lemma 1.1] if $M$ has a finite uniform dimension and is semi co–Hopfian, then the endomorphism ring $End_R(M)$ is semilocal.

**Proposition 2.17** *Let $M$ be a module. If the ring ${}_SS = End_R(M)$ is semi co–Hopfian, then ${}_RM$ is semi co–Hopfian. The converse is true if $Ker(\alpha)$ is generated by $M$ whenever $\alpha \in S$ is such that $l_S(\alpha) = 0$.*

**Proof** (Since $M$ is a right $S$–module, we will consider $Imf$ as $(M)f$.) Let $f : M \to M$ be a monomorphism. Then $S \cong Sf$. Since ${}_SS$ is semi co–Hopfian there exists an idempotent $e \in S$ such that $Sf = Se$ by Lemma 2.1. Then $Imf = Ime$ is a direct summand of $M$.

For the converse; let $\alpha \in S$ be such that $l_S(\alpha) = 0$. If we prove that $\alpha S = S$, then $S$ will be semi co–Hopfian by Proposition 2.3. By hypothesis, $Ker(\alpha) = \sum\{Imh \,|\, h \in S, Imh \subseteq Ker(\alpha)\}$. If $Imh \subseteq Ker(\alpha)$, we have that $h = 0$. Then $Ker(\alpha) = 0$ and hence $\varphi : Im\alpha \to M$, defined by $((m)\alpha)\varphi = m$, is an isomorphism. Since $M$ is semi co–Hopfian, $Im\alpha$ is a direct summand of $M$. Let $\pi : M \to Im\alpha$ be the projection. Then $\alpha(\pi\varphi) = 1 \in \alpha S$. So $S$ is semi co–Hopfian. □

The converse of Proposition 2.17 is not true in general. For example, let ${}_{\mathbb{Z}}M = \mathbb{Z}_{p^\infty}$ for a prime $p$. Then the endomorphism ring $S$ of $M$ is isomorphic to the ring of $p$–adic integers. ${}_{\mathbb{Z}}M$ is co–Hopfian but ${}_SS$ is not (see [12]). Since only idempotents in $S$ are 0 and 1, ${}_SS$ is not semi co–Hopfian by Proposition 2.4.

Since a free module generates all its submodules, we have;

**Corollary 2.18** *If $M$ is free, then $M$ is semi co–Hopfian if and only if $End_R(M)$ is semi co–Hopfian. In particular, $R^n$ is semi co–Hopfian left $R$–module if and only if $M_n(R)$ is semi co–Hopfian left $M_n(R)$–module.*

Let $M$ be a module. The elements of $M[X]$ are formal sums of the form $a_0 + a_1X + \cdots + a_kX^k = \sum_{i=1}^{k} a_iX^i$ with $k$ an integer greater than or equal to 0 and $a_i \in M$. Addition is defined by adding the corresponding coefficients. The $R[X]$–module structure is given by

$$(\sum_{i=0}^{k} \lambda_iX^i) \cdot (\sum_{j=0}^{z} a_jX^j) = \sum_{\mu=0}^{k+z} c_\mu X^\mu$$

where $c_\mu = \sum_{i+j=\mu} \lambda_i a_j$, for any $\lambda_i \in R$, $a_j \in M$.

**Theorem 2.19** *Let $M$ be an $R$– module. If $M[X]$ is semi co–Hopfian $R[X]$–module, then $M$ is semi co–Hopfian $R$–module.*

**Proof** Let $f : M \to M$ be an injective endomorphism of $M$. Then $f[X] : M[X] \to M[X]$ with $f[X](\sum m_iX^i) = \sum f(m_i)X^i$ is an injective endomorphism of $M[X]$. Since $M[X]$ is semi co–Hopfian, $Im(f[X]) = (Imf)[X] \leq^{\oplus} M[X]$. Now we claim that $Imf \leq^{\oplus} M$. Let $M[X] = (Imf)[X] \oplus K$ for some submodule $K$ of $M[X]$ and $K'$ denote the submodule of $M$ which is generated by the constant polynomials of $K$. Note that $K' \neq 0$ if $M \neq Imf$. We will show that $M = Imf \oplus K'$. Let $m \in M$. Then $m \in M[X]$ and so $m = g(X) + k(X)$ where $g(X) \in (Imf)[X]$, $k(X) \in K$. Since $m$ is a constant polynomial in $M[X]$, we have $m = g(0) + k(0)$ where $g(0) \in Imf$ and $k(0) \in K'$. Next, take $k' \in Imf \cap K'$. But $k' \in (Imf)[X] \cap K = 0$. □

There exists a left semi co–Hopfian ring that is not right semi co–Hopfian.

**Example 2.20** (Faith-Menal) The left $C2$ ring $R$ which is not right $C2$ is the example: Let $D$ be any countable, existentially closed division ring over a field $F$, and let $R = D \bigotimes_F F(x)$. Then the trivial extension of $D$ by $R$, $T(R, D)$ is a non–Artinian left $P$–injective, right finite dimensional ring (see [14, Example 7.11 and 8.16] ). Since $R$ is left $P$–injective, it is left $C2$ and so left semi co–Hopfian. If $R$ is right semi co–Hopfian, then $R$ is right co–Hopfian by Proposition 2.5. By Camps–Dicks Theorem (see [14, Theorem C.2]), $R$ is semilocal. Since J(R) is nilpotent, $R$ is right Artinian by Hopkins–Levitzki Theorem. But this is a contradiction.

It is well–known that if ${}_RR$ has $C2$, then $Z_l \subseteq J(R)$. By a proof similar to Lemma 2.3 in [15] we have the following generalization. But we give the proof for completeness.

**Proposition 2.21** *If ${}_RR$ is semi co–Hopfian, then $Z_l \subseteq J(R)$.*

**Proof** Let $a \in Z_l$. Then $l_R(1-a) = 0$ and so $R(1-a) \cong R$. This isomorphism gives an injective endomorphism of $R$ such that $Imf = R(1-a)$. By hypothesis, $R(1-a)$ is a direct summand of $R$. Then $(1-a)R$ is a direct summand. Let $e^2 = e \in R$ be such that $(1-a)R = eR$. Since $(1-e)(1-a) = 0$, we have $1-e = 0$. Hence $1-a$ is right invertible. Since this holds for all $a \in Z_l$, $Z_l \subseteq J(R)$. □

The converse is not true in general. The localization $\mathbb{Z}_{(p)}$ of the ring of integers at the prime $p$ is commutative domain with $Z_l = 0$ but not a division ring. By Proposition 2.4, ${}_RR$ is not semi co–Hopfian.

# 3 Semi Hopfian Modules

In this section we consider the dual version of semi co–Hopfian modules. A module $M$ is called *semi Hopfian* if any surjective endomorphism of $M$ has a direct summand kernel, i.e. any surjective endomorphism of $M$ splits. Then any Hopfian module is semi Hopfian.

**Example 3.1** If $R$ is semisimple Artinian, then a module ${}_RM$ is Hopfian if and only if ${}_RM$ has finite length (see [9]). Also a vector space over a field is Hopfian if and only if it is finite dimensional. Hence an infinite dimensional vector space over a field is semi Hopfian (it is semisimple) but not Hopfian.

**Example 3.2** Let $p$ be prime and $M$ be any direct sum of copies of $\mathbb{Z}_{p^2}$. Then we claim that $M$ is a semi Hopfian $\mathbb{Z}$–module. Let $f : M \to M$ be an epimorphism. Since $p^2M = 0$, $f$ is an $\mathbb{Z}_{p^2}$–epimorphism. Since $M$ is a free $\mathbb{Z}_{p^2}$–module, $f$ splits. This implies that $M$ is a semi Hopfian $\mathbb{Z}$–module. But it is well known that $M$ is not a Hopfian $\mathbb{Z}$–module.

Note that for any ring $R$, $R_R$ is Hopfian if and only if ${}_RR$ is Hopfian if and only if it is Dedekind finite [16, Proposition 1.2]. Hence any ring which is not Dedekind finite is an example of a module which is semi Hopfian but not Hopfian. For example, the ring of linear endomorphisms of an infinite dimensional left vector space over a division ring is not Dedekind finite.

The following characterization can be seen easily.

**Lemma 3.3** *The following are equivalent for a module $M$.*
*1) $M$ is semi Hopfian.*
*2) Any submodule $N$ of $M$ which satisfies $M/N \cong M$ is a direct summand of $M$.*

Hence any module with $D2$ is semi Hopfian. In particular, any quasi–projective module is semi Hopfian by [13, Proposition 4.38]. Therefore, the concept of semi Hopfian modules is a generalization of Hopfian modules and modules with $D2$.

**Example 3.4** *There exists a semi Hopfian module which has not $D2$.*

**Proof** Let ${}_{\mathbb{Z}}M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$, $p$ is a prime number. Then $M$ is not relatively projective but has $D1$ by [10, Example 4]. Hence $M$ has not $D2$ by [13, Lemma 4.23]. Since ${}_{\mathbb{Z}}M$ is Noetherian, it is Hopfian and hence semi Hopfian. □

Note that any semi Hopfian quasi–discrete module has $D2$ (see [13, Lemma 5.1]).

Dual of the summand sum property is the summand intersection property. A module $M$ has the *summand intersection property (SIP)* if the intersection of any two direct summands of $M$ is a direct summand. Note that $M$ has SIP if and only if for every decomposition $M = A \oplus B$ and every $R$–homomorphism $f$ from $A$ to $B$, the kernel of $f$ is a direct summand [8].

Hence if $M \oplus M$ has SIP, then $M$ is semi Hopfian. But the converse is not true: Let the $\mathbb{Z}$–module $M = \mathbb{Z} \oplus \mathbb{Z}_2$. Since $M$ is Noetherian, it is semi Hopfian. Denote an $R$-homomorphism $f : M \to M$ by $f(a, \overline{b}) = (0, \overline{a})$ where $a, b \in \mathbb{Z}$. Then $Ker f = 2\mathbb{Z} \oplus \mathbb{Z}_2$ is not a direct summand. Hence $M \oplus M$ has not SIP.

Now consider torsion free modules. Torsion free modules need not be semi Hopfian.

**Example 3.5** *There exists a torsion free module which is not semi Hopfian.*

**Proof** Let $F$ be the direct sum of copies of $\mathbb{Z}$ and the $\mathbb{Z}$–module $M = \mathbb{Q} \oplus F \oplus F \oplus \cdots$. Then there exists an epimorphism $\alpha : F \to \mathbb{Q}$. Define $\theta : M \to M$ be such that $\theta(q, f_1, f_2, \ldots) = (\alpha(f_1), f_2, \ldots)$ where $q \in \mathbb{Q}, f_i \in F$ for all $i$. Then $\theta$ is a $\mathbb{Z}$–epimorphism and $Ker\theta = \mathbb{Q} \oplus Ker\alpha \oplus 0 \oplus 0 \cdots$. If $Ker\theta$ was a direct summand of $M$, then $Ker\alpha$ would have to be a direct summand of $F$, this is a contradiction. So $M$ is not semi Hopfian. □

Unlike Proposition 2.13, there exists a commutative domain $R$ and a torsion free semi Hopfian $R$–module $M$ which is not injective or not projective. For example, $M_{\mathbb{Z}} = \mathbb{Z}$ is not injective and $M_{\mathbb{Z}} = \mathbb{Q}$ is not projective.

Recall that $M$ has *hollow dimension* $n \in \mathbb{N}$ if there exists an epimorphism from $M$ to a direct sum of $n$ nonzero modules but no epimorphism from $M$ to a direct sum of more than $n$ nonzero modules (see [3, 5.2]). In [3, 5.4(3)], it is proved that a semi Hopfian module with finite hollow dimension (i.e. dual Goldie dimension) is Hopfian. Any module with finite hollow dimension is Dedekind finite, for let $M$ be a module with finite hollow dimension and $M \oplus K \cong M$. Consider the isomorphism $\varphi : M \longrightarrow M \oplus K$ and the projection $\pi : M \oplus K \longrightarrow M$. Clearly, $\pi\varphi$ is a surjection. Since $M$ has finite hollow dimension, $Ker(\pi\varphi) \ll M$ by [3, 5.4(3)]. Then $Ker(\pi\varphi) = \varphi^{-1}(Ker\pi) = \varphi^{-1}(K) \ll M$ implies that $\varphi\varphi^{-1}(K) = K \ll M \oplus K$. But $K \leq^{\oplus} M \oplus K$. Hence $K = 0$, i.e. $M$ is Dedekind finite.

So the following result which is known in the literature generalizes [3, 5.4(3)]. A module $M$ is called *generalized Hopfian* (in [5]) if any surjective endomorphism of $M$ has a small kernel.

**Proposition 3.6** *The following are equivalent for a module $M$.*

*1) $M$ is Hopfian.*

*2) $M$ is Dedekind finite and semi Hopfian.*

*3) $M$ is generalized Hopfian and semi Hopfian.*

**Proof** (3) $\Leftrightarrow$ (1) $\Rightarrow$ (2) are obvious.
(2) $\Rightarrow$ (1) (see also [11, Exc. 1.8]) Let $f : M \longrightarrow M$ be a surjection. Since $M$ is semi Hopfian $f$ splits. Then there exists an endomorphism $g : M \longrightarrow M$ such that $fg = 1$. But Dedekind finiteness of $End_R(M)$ implies $gf = 1$. Hence, $f$ is an injection. □

It is also known that a semi Hopfian module with finite hollow dimension has a semilocal endomorphism ring [3, 19.2].

Now we investigate some properties of semi Hopfian modules.

**Proposition 3.7** *Any direct summand of a semi Hopfian module is semi Hopfian.*

**Proof** Let $K$ be a direct summand of a semi Hopfian module $M$ and $f : K \to K$ be a surjection. Then $M = K \oplus K'$ for some $K'$, and $f \oplus 1_{K'} : M \to M$ is also a surjection. Thus $Ker(f \oplus 1_{K'}) = Kerf \leq^{\oplus} M$ and hence $Kerf \leq^{\oplus} K$. □

**Proposition 3.8** *If $M/N$ is semi Hopfian for every non–summand submodule $N$ of a module $M$, then $M$ is semi Hopfian.*

**Proof** Suppose that $M$ is not semi Hopfian. Then there exists a surjective endomorphism $f : M \to M$ such that $Kerf$ is not a direct summand of $M$. But by assumption $M/Kerf \cong M$ is semi Hopfian, a contradiction. □

Any direct sum of semi Hopfian modules need not be semi Hopfian.

**Example 3.9** Let $p$ be prime and $M_1 = \mathbb{Z}_p$ and $M_2$ an infinite direct sum of copies of $\mathbb{Z}_{p^2}$. Then $M_1$ is simple and $M_2$ is semi Hopfian by Example 3.2. But $M = M_1 \oplus M_2$ is not semi Hopfian $\mathbb{Z}$–module. For, define $f : M \to M$ by $f(a_1+p\mathbb{Z}, a_2+p^2\mathbb{Z}, a_3+p^2\mathbb{Z}, \ldots) = (a_2+p\mathbb{Z}, a_3+p^2\mathbb{Z}, \ldots)$. Then $f$ is a $\mathbb{Z}$–epimorphism and $Kerf = \mathbb{Z}_p \oplus p\mathbb{Z}_{p^2} \oplus 0 \oplus 0 \cdots$ is not a direct summand of $M$ since $p\mathbb{Z}_{p^2}$ is not a direct summand of $\mathbb{Z}_{p^2}$.

Any direct product of semi Hopfian modules need not be semi Hopfian: If we let the $\mathbb{Z}$–module $M = \mathbb{Z}_p \times \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \cdots$, then $M$ is not semi Hopfian by a proof similar to Example 3.9.

**Proposition 3.10** *Let $M = \oplus_{i\in I} M_i$, where $M_i$ is invariant under any surjection of $M$ for all $i \in I$. Then $M$ is semi Hopfian if and only if $M_i$ is semi Hopfian for all $i \in I$.*

**Proof** The necessity is clear from Proposition 3.7. For the sufficiency, let $f : M \to M$ be a surjection. Then $f|_{M_i} : M_i \to M_i$ is a surjection for all $i \in I$. Since $Ker(f|_{M_i}) \leq^{\oplus} M_i$ for all $i$ we have that $Kerf = \oplus_{i\in I} Ker(f|_{M_i}) \leq^{\oplus} M$. □

Note that $M_2$ is not invariant under the surjection $f$ of $M$ in Example 3.9.

As a dual of DCC on non–summands we consider the ascending chain condition (ACC) on non–summands.

**Theorem 3.11** *If $M$ has ACC on non–summand submodules, then $M$ is semi Hopfian.*

**Proof** Assume that $f : M \to M$ is a surjection and $Kerf$ is a non–summand of $M$. Then $Kerf \subseteq Kerf^2 \subseteq Kerf^3 \subseteq \cdots$ is an ascending chain of non–summand submodules of $M$. By hypothesis there exists an integer $n$ such that $Ker(f^n) = Ker(f^{n+1})$. Now we claim that $Kerf = 0$. Let $x \in M$ be such that $f(x) = 0$. Since $f$ is surjective, $f(a_1) = x$ for some $a_1 \in M$. Again, $f(a_2) = a_1$ for some $a_2 \in M$. By repeating this argument, we have $f(a_n) = a_{n-1}$ for some $a_n \in M$. Then $f(a_1) = f^2(a_2) = \cdots = f^n(a_n) = x$. Hence $f(x) = f^{n+1}(a_n) = 0$ implies that $a_n \in Ker(f^{n+1}) = Ker(f^n)$. As a result, $x = 0$. So we have a contradiction. □

**Proposition 3.12** *Let $\mathcal{P}$ be a property of modules preserved under isomorphism. If a module $M$ has the property $\mathcal{P}$ and satisfies ACC on non–summand submodules $N$ such that $M/N$ has the property $\mathcal{P}$, then $M$ is semi Hopfian.*

**Proof** Suppose that $M$ is not semi Hopfian. Then there exists a non–summand submodule $N_1$ of $M$ such that $M/N_1 \cong M$. Since $M/N_1$ has the property $\mathcal{P}$ but is not semi Hopfian, there exists a non–summand submodule $N_2/N_1$ of $M/N_1$ such that $M/N_2 \cong M/N_1$. $N_2$ is also a non–summand of $M$. Continuing in this way we get an ascending chain $0 \subset N_1 \subset N_2 \subset \cdots$ of non–summand submodules of $M$. But this is a contradiction. □

**Corollary 3.13** *If $M$ is semi co–Hopfian and satisfies ACC on non–summand submodules $N$ such that $M/N$ is semi co–Hopfian, then $M$ is semi Hopfian.*

**Proof** Take the property $\mathcal{P}$ as being semi co–Hopfian and apply Proposition 3.12. □

**Corollary 3.14** *If $M$ is semi Hopfian and satisfies DCC on non–summand semi Hopfian submodules, then $M$ is semi co–Hopfian.*

**Proof** Clear by Proposition 2.15. □

**Corollary 3.15** *If $M$ satisfies ACC on non–summand submodules $N$ such that $M/N$ is not semi Hopfian, then $M$ is semi Hopfian.*

**Proof** It follows from Proposition 3.12 by letting $\mathcal{P}$ be the property of not being semi Hopfian. □

The following result can be seen by a proof similar to Theorem 2.19.

**Theorem 3.16** *Let $M$ be an $R$–module. If $M[X]$ is semi Hopfian $R[X]$–module, then $M$ is semi Hopfian $R$–module.*

# 4 Algebra of Continuous functions

**Definition 4.1** A topological space $X$ is said to be *semi Hopfian* (resp. *semi co–Hopfian*) in the category of topological spaces **Top** if every surjective (resp. injective) continuous map $\alpha : X \to X$, there exists a continuous map $\beta : X \to X$ such that $\alpha \circ \beta = id_X$ (resp. $\beta \circ \alpha = id_X$).

**Definition 4.2** Let $K$ be a commutative ring and $K$–**alg** denote the category of $K$–algebras. A $K$–algebra $A$ is said to be *semi Hopfian* (resp. *semi co–Hopfian*) as an $K$–algebra if for any surjective (resp. injective) $K$–algebra homomorphism $\alpha : A \to A$ there exists a $K$–algebra homomorphism $\beta : A \to A$ such that $\alpha \circ \beta = 1_A$ (resp. $\beta \circ \alpha = 1_A$)

For any compact Hausdorff space $X$, $C(X)$ denote the $\mathbb{R}$–algebra of continuous functions from $X$ to $\mathbb{R}$. Varadarajan [16, Theorem 5.3] prove that if $X$ is a compact Hausdorff space, then $C(X)$ is Hopfian (resp. co–Hopfian) as an $\mathbb{R}$–algebra if and only if $X$ is co–Hopfian (resp. Hopfian) as a topological space. Here we prove that if $X$ is semi co–Hopfian (resp. semi Hopfian), then $C(X)$ is semi Hopfian (resp. semi co–Hopfian) in the category of $\mathbb{R}$–algebras. The converse of this result is open.

If $\varphi : X \to Y$ is a continuous map of compact Hausdorff spaces, there is an induced homomorphism $\varphi^* : C(Y) \to C(X)$ in the $\mathbb{R}$–algebra given by $\varphi^*(g) = g \circ \varphi$ for every $g \in C(Y)$. Also given any $\mathbb{R}$–algebra homomorphism $\alpha : C(Y) \to C(X)$, there is a unique continuous map $\varphi : X \to Y$ such that $\alpha = \varphi^*$ (see [16]).

**Proposition 4.3** [16, Proposition 5.2] *Let $\varphi : X \to Y$ be a continuous map of a compact Hausdorff spaces. Then $\varphi^* : C(Y) \to C(X)$ is injective (resp. surjective) if and only if $\varphi : X \to Y$ is surjective (resp. injective).*

**Theorem 4.4** *Let $X$ be a compact Hausdorff space. If $X$ is a semi co–Hopfian (resp. semi Hopfian) topological space, then $C(X)$ is a semi Hopfian (resp. semi co–Hopfian) $\mathbb{R}$–algebra.*

**Proof** Assume that $X$ is a semi co–Hopfian topological space. Let $\alpha : C(X) \to C(X)$ be a surjective $\mathbb{R}$–algebra homomorphism. Then there exists a unique continuous map $\varphi : X \to X$ such that $\alpha = \varphi^*$. By Proposition 4.3, $\varphi$ is injective. By assumption, there exists a continuous map $\gamma : X \to X$ such that $\gamma \circ \varphi = id_X$. Then $\gamma$ is surjective and again $\gamma^* : C(X) \to C(X)$ is an injective $\mathbb{R}$–algebra homomorphism. Let $f \in C(X)$. Then $(\alpha \circ \gamma^*)(f) = \alpha(\gamma^*(f)) = \alpha(f \circ \gamma) = \varphi^*(f \circ \gamma) = f \circ \gamma \circ \varphi = f \circ id_X = f$. So we have that $\alpha \circ \gamma^* = 1_{C(X)}$. Hence $\alpha$ splits.

The result in parenthesis can be seen similarly. □

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