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CHARACTERIZATION OF SOME RINGS BY FUNCTOR $Z^*(.)$

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Abstract

Let $\underline{X} = \{M : Z^*(M) = 0\}$ and $\underline{X}^* = \{M : Q \leq P \leq M, P/Q \in \underline{X}$ implies $P/Q = 0\}$ be classes of R-modules. In this note we study the structure of rings R over which every module M has a decomposition $M = M_1 \oplus M_2$ with $M_1 \in \underline{X}$ and $M_2 \in \underline{X}^*$.

Let R be a ring with identity. Throughout all modules will be unital right R-modules and $RadM, E(M), Z(M)$ will denote the radical, injective hull and singular submodule of a module $M.J(R)$ is the Jacobson radical of R.

A module N is called a small *submodule* in a module M if whenever $N+L = M$ for some submodule L of M we have $M = L$. A module M is said to be *small* if M is small in $E(M)$. Let M be an R-module. We set $Z^*(M) = \{m \in M : mR$ is small$\}$ and we define inductively $Z_n^*(M) : Z_1^*(M) = Z^*(M), Z^*(M/Z_{n-1}^*(M)) = Z_n^*(M)/Z_{n-1}^*(M)(n = 2, 3, \ldots)$. It is well-known that $Z_2(M) = Z_3(M) = \cdots$ for $Z(M)$. But it is not known in $Z_2^*(M) = Z_3^*(M) = \cdots$ In this note we consider the classes $\underline{X} = \{M : MR-$ module and $Z^*(M) = 0\}, \underline{X}^* = \{M : MR-$ module and whenever $Q \leq P \leq M, P/Q \in \underline{X}$ implies $P/Q = 0\}$, following [5]. Since $RadM$ is the sum of small submodules of M, then $RadM \leq Z^*(M)$.

A class Ω of modules is called *s-closed* if Ω is closed under submodules and *q-closed* if Ω is closed under homomorphic images, and $\{s, q\}$*-closed* if Ω is s-closed and q-closed. It is known that $\underline{X}^*$ is {s,q}-closed. Let $H_{\underline{X}}(M)$ denote the sum of $\underline{X}^*$-submodules of M. Then $H_{\underline{X}}(M) \in \underline{X}^*, H_{\underline{X}}(M/H_{\underline{X}}(M)) = 0$, and $H_{\underline{X}}$ is fully invariant [5], and $\underline{X} \cap \underline{X}^* = 0$. It is known that the class $\underline{X}$ is closed under submodules, direct products, direct sums, essential extensions and module extensions.

In [9] it is proved that if R is a quasi-Frobenius ring then every module is a direct sum of an injective module and a small module. In this note we show that every module M over a quasi-Frobenius ring has a decomposition $M = M_1 \oplus M_2$ with $M_1 \in \underline{X}$ and $M_2 \in \underline{X}^*$. We also deal with the question: Let R be a ring such that every module M has a decomposition $M = M_1 \oplus M_2$ with $M_1 \in \underline{X}$ and $M_2 \in \underline{X}^*$, then R is quasi-Frobenius?

Lemma 1. *Let M be an R-module. Then*
(i) *If M is small then $Z^*(M) = M$,*
(ii) *If $Z^*(M) = M$ then $M \in \underline{X}^*$,*
(iii) *If M is semisimple injective then $M \in \underline{X}$.*
Proof. **(i)** Clear from definitions.
(ii) Let M be a module such that $Z^*(M) = M$. Assume $Q \leq P \leq M$ and $P/Q \in \underline{X}$. Then $Z^*(P/Q) = 0$. Since $Z^*(M) = M$ and any homomorphic image of a small module is small, then $P/Q = Z^*(P/Q)$. Hence $P/Q = 0$, and so $M \in \underline{X}^*$.
(iii) Assume first M is simple injective. Let $0 \neq m \in M$ be such that mR is small in $E(mR) = M$. This is a contradiction for $mR = M$. Hence $Z^*(M) = 0$ and $M \in \underline{X}$. Assume M is semisimple injective. Since $\underline{X}$ is closed under direct sums, then $M \in \underline{X}$. □

Lemma 2. *Let R be a right perfect ring. Then a module M is small if and only if $Z^*(M) = M$.*
Proof. Let R be a right perfect ring. Assume M is small module. Let $0 \neq m \in M$. Then mR is small in $E(M)$ and so in $E(mR)$. Hence $m \in Z^*(M)$ and then $Z^*(M) = M$. Conversely suppose that $Z^*(M) = M$. Since R is right perfect and $Z^*(M) = M$, then $Z^*(M) \leq RadE(M)$ and $RadE(M)$ is small in $E(M)$ [1]. Hence M is small. □

Theorem 3. *Let R be a right hereditary ring. Then $\underline{X}^* = \{M : Z^*(M) = M\}$.*
Proof. Let R be a right hereditary ring and M a module with $Z^*(M) = M$. By Lemma 1(ii), $M \in \underline{X}^*$. Assume M is a module with $M \in \underline{X}^*$. Let $m \in M$ be such that $m \notin Z^*(M)$. Then mR is not small in $E(mR)$. Hence there is a submodule L of $E(mR)$ such that $mR + L = E(mR)$. Since R is right hereditary, then $E(mR)/L$ is injective and so the cyclic module $mR/(mR \cap L)$ is injective. Let $K/(mR \cap L)$ be a maximal submodule in $mR/(mR \cap L)$. Then mR/K is simple, and injective as a quotient of injective module. By Lemma 1(iii), $mR/K \in \underline{X}$. Since $M \in \underline{X}^*$ and $\underline{X}^*$ is $\{$s-q$\}$-closed, then $mR/K \in \underline{X}^*$. Hence mR/K is a zero module. This is a contradiction. Thus $Z^*(M) = M$. □

Theorem 4. *Let R be a ring such that $R/J(R)$ is a semisimple ring. Then $\underline{X} = \{M : M$ is semisimple injective R-module$\}$.*
Proof. Let M be an $\underline{X}$-module. Since $R/J(R)$ is a semisimple ring, then $RadM = MJ(R)$ [1]. Since $RadM$ is contained in $Z^*(M)$ and $Z^*(M) = 0$, then $RadM = 0$. It follows that M is semisimple. Since $M \in \underline{X}$, implies $E(M) \in \underline{X}$ then $E(M)$ is semisimple. Hence $M = E(M)$ and so M is injective. Conversely suppose M is a semisimple injective module. By Lemma 1(iii), $M \in \underline{X}$. It completes the proof. □

Lemma 5. *Let M be a semisimple module with $M = \oplus_{i \in I} M_i, M_i$ simple for each $i \in I$. Then M has a decomposition $M = M_1 \oplus M_2$ where $M_1 \in \underline{X}$ and $M_2 \in \underline{X}^*$.*

Proof. Let $M_i (i \in I)$ be a simple module. Then M_i is either small or injective. Let $M = \oplus_{i \in I} M_i, M_i$ simple, write $I = I_1 \cup I_2, I_1 \cap I_2, = \emptyset$ with $i \in I_1$ implies M_i is injective and $i \in I_2$ implies M_i is small. Then $M = M_1 \oplus M_2$ with $M_1 = \oplus_{i \in I_1} M_i$ and $M_2 = \oplus_{i \in I_2} M_i$. It is clear that $M_1 \in \underline{X}$ and $M_2 \in \underline{X}^*$ □

Lemma 6. *Let R be a quasi-Frobenius ring. Then every R-module M has a decomposition $M = M_1 \oplus M_2$ with $M_1 \in \underline{X}$ and $M_2 \in \underline{X}^*$.*

Proof. Assume R quasi-Frobenius ring. Let M be an R-module. Then $M = N_1 \oplus N_2$ with N_1 is injective and N_2 is small by [9]. $N_2 \in \underline{X}^*$ by Lemma 1(ii). Since R is Noetherian ring and N_1 is injective R-module, then $N_1 = \oplus_{i \in I} L_i$ with L_i indecomposable injective [1]. Now if $H_{\underline{X}}(L_i) = 0$ then $L_i \in \underline{X}$. If not, $L_i / H_{\underline{X}}(L_i) = K_1 \oplus K_2$ where K_1 is injective and K_2 is small. Then $K_2 \in \underline{X}^*$, and since $H_{\underline{X}}(L_i / H_{\underline{X}}(L_i)) = 0$, then $K_2 = 0$. Hence $L_i / H_{\underline{X}}(L_i)$ is injective, and so $L_i / H_{\underline{X}}(L_i)$ is projective since R is a quasi-Frobenius ring. Thus $L_i / H_{\underline{X}}(L_i) = L_i$ and then $L_i \in \underline{X}^*$. Hence for $i \in I$, either $L_i \in \underline{X}$ or $L_i \in \underline{X}^*$. Thus $N_1 = L \oplus K$ with $L \in \underline{X}$ and $K \in \underline{X}^*$ as in the proof of Lemma 5. Therefore $M = L \oplus K \oplus N_2$ with $L \in \underline{X}, K \oplus N_2 \in \underline{X}^*$. This completes the proof. □

Corollary 7. *Every module M over a semisimple ring has a decomposition $M = M_1 \oplus M_2$ with $M_1 \in \underline{X}, M_2 \in \underline{X}^*$.*

Proof. Let M be a module over a semisimple ring R. Then M is semisimple. Corollary is now clear from Lemma 5. □

In this note we investigate the converse statements of Lemma 5, Lemma 6 and Corollary 7. For this we set $^{(*)}$.

Lemma 8. *We assume R satisfies$^{(*)}$. Then $\underline{X}^*$ is closed under essential extensions.*

Proof. Let M be an $\underline{X}^*$-module. It is enough to show $E(M) \in \underline{X}^*$. By hypothesis, $E(M) = M_1 \oplus M_2, M_1 \in \underline{X}, M_2 \in \underline{X}^*$. Since M is essential in $E(M), M_1 \in \underline{X}$ and $M_1 \cap M \in \underline{X} \cap \underline{X}^* = 0$, then $M_1 = 0$. Hence $E(M) \in \underline{X}^*$. □

Lemma 9. *Assume $\underline{X}^*$ closed under essential extensions. Then every injective module M has a decomposition $M = M_1 \oplus M_2$ with $M_1 \in \underline{X}$ and $M_2 \in \underline{X}^*$.*

$^{(*)}$ Every module M has a decomposition $M = M_1 \oplus M_2$ with $M_1 \in \underline{X}, M_2 \in \underline{X}^*$.

Proof. Let M be an injective R-module. We note that $H_{\underline{X}}(M) \in \underline{X}^*$ and then by assumption, $E(H_{\underline{X}}(M)) \in \underline{X}^*$. Since $E(M) = E(H_{\underline{X}}(M)) \oplus K$ for some submodule K of $E(M)$, then $E(M) = H_{\underline{X}}(E(M)) + K$. Let $x \in K$ be such that $xR \in \underline{X}^*$. Since $xR \cap M \in \underline{X}^*$, then $xR \cap M \leq H_{\underline{X}}(M) \leq E(H_{\underline{X}}(M))$. Since $K \cap E(H_{\underline{X}}(M)) = 0$ then $xR \cap M = 0$ and so $xR = 0$ for all $x \in K$ with $xR \in \underline{X}^*$. Hence $H_{\underline{X}}(K) = 0$. Then $0 = H_{\underline{X}}(K) = K \cap H_{\underline{X}}(E(M))$ implies $E(M) = H_{\underline{X}}(E(M)) \oplus K$. Since $H_{\underline{X}}(E(M))$ is the largest submodule of $E(M)$ belonging to $\underline{X}^*$, then $K \in \underline{X}$. This completes the proof. □

Let M be an R-module and A, L submodules of $M.L$ is called a *supplement* of A in M if it is minimal with the property $A + L = M$. A submodule K of M is called a *supplement* (in M) if K is a supplement of some submodule of M. It is easy to check that L is a supplement of A in M if and only if $M = A + L$ and $A \cap L$ is small in $L.M$ is called a *supplemented module* if every submodule has a supplement in M. The following lemma is in [6]. We prove for the sake of completeness.

Lemma 10. *Let M be a supplemented module. Then M has a decomposition $M = M_1 \oplus M_2$ with M_1 semisimple and $RadM_2$ is essential in M_2.*

Proof. Let M_1 be a submodule of M such that $RadM \oplus M_1$ is essential in M. Since M is supplemented, then there exists a submodule M_2 of M such that $M = M_1 + M_2$ and $M_1 \cap M_2$ is small in M_2. Hence $M_1 \cap M_2$ is submodule of both $RadM$ and M_1. It follows that $M = M_1 \oplus M_2$, and then $RadM = RadM_2$ is essential in M_2. M_1 is semisimple because $M_1 \cap RadM = 0$ and M is supplemented. □

Lemma 11. *We assume $\underline{X}^*$ is closed under essential extensions. Then every supplemented module M has a decomposition $M = M_1 \oplus M_2, M_1 \in \underline{X}, M_2 \in \underline{X}^*$.*

Proof. Let M be a supplemented module. Then $M = M_1 \oplus M_2$ with M_1 semisimple and $RadM_2$ is essential in M_2 by Lemma 10. By Lemma 5, $M_1 = N \oplus K, N \in \underline{X}, K \in \underline{X}^*$. Also $RadM_2 \in \underline{X}^*$. By hypothesis, $M_2 \in \underline{X}^*$. Then $M = N \oplus K \oplus M_2, N \in \underline{X}, K \oplus M_2 \in \underline{X}^*$. □

Proposition 12. *Let R be a ring such that every module has a projective cover (i.e: right perfect ring). Then the following are equivalent.*

(1) *$\underline{X}^*$ is closed under essential extensions.*

(2) *R satisfies $^{(*)}$.*

Proof. We combine Lemma 8, Lemma 4.40 of [7] and Lemma 11 to prove the equivalence of (1) and (2). □

Theorem 13. *Let R be a right hereditary ring. Then R is a right perfect and R satisfies* $(^1)$ *if and only if R is a right H-ring and $\underline{X}^*$ is closed under essential extensions.*

Proof. Oshiro [8] *class* a ring R a right H-ring if every right R-module is a direct sum of an injective module and a small module. Now, if R is a right hereditary, right perfect ring satisfying $(^1)$ then by Lemma 2 and Theorem 3, $\underline{X}^* = \{M : Z^*(M) = M\} = \{M : M \text{ is small}\}$, and by Theorem 4 $\underline{X} = \{M : M \text{ is semisimple injective}\}$. Let M be an R-module. By hypothesis, $M = M_1 \oplus M_2$ with $M_1 \in \underline{X}$ is injective and $M_2 \in \underline{X}^*$ is small. Then R is a right H-ring. By Lemma 8, we have $\underline{X}^*$ is closed under essential extensions. Assume R is a right H-ring and $\underline{X}^*$ is closed under essential extensions. Then R is a right perfect. By Proposition 12, every module M has a decomposition $M = M_1 \oplus M_2, M_1 \in \underline{X}$ and $M_2 \in \underline{X}^*$. This completes the proof. □

Example 14. The ring of integers is a (right) hereditary ring. It is not a quasi-Frobenius ring. Let K be a field and G a finite group such that the characteristic of K divides the order of G. Then by Mascke's Theorem [10] the group ring KG is not semisimple but a quasi-Frobenius ring [2, Proposition 9.6]. Then the following lemma shows that the quasi-Frobenius ring KG is not right hereditary.

Lemma 15. *Let R be a quasi-Frobenius ring. Then R is a right hereditary if and only if R is semisimple.*

Proof. Let R be a quasi-Frobenius ring. Assume R is semisimple. Then every R-module, in particular, every right ideal of R is projective [1]. Hence R is right hereditary. Conversely, suppose that R is right hereditary. Let $x \in Z(R)$. Then xR is a projective R-module. Since $xR \cong R/r(x)$ then the essential right ideal $r(x)$ is a direct summand of R. Hence $xR = 0$. It follows that $Z(R) = 0$. Since R is a quasi-Frobenius ring, then $J(R) = Z(R)$ by [8, Theorem 4.3] and R is artinian. Hence R is semisimple. □

Remark. Let R be a ring. Then every direct sum of small modules is small if and only if for every injective module $M, RadM$ is small in M [9]. In this case $Z^*(M) = M$ if and only if M is small. To prove this only note that $Z^*(M) = M \cap Rad(E(M))$ for any module M.

Theorem 16. *Let R be a right hereditary ring. Then the following are equivalent.*

(1) *R is a right perfect ring and satisfies* $(^1)$,

(2) *$R/J(R)$ is semisimple and direct sum of small modules is small and R satisfies* $(^1)$,

(3) *R is a quasi-Frobenius ring,*

(4) *R is a semisimple ring,*

(5) *R is a right self-injective ring.*

Proof. **(1) $\Longrightarrow$ (2)** Then $R/J(R)$ is semisimple and for every injective R-module $M, RadM$ is small in M [1]. Hence direct sum of small modules is small by remark.
(2) $\Longrightarrow$ (3) Let M be an R-module. Then by (2) $M = M_1 \oplus M_2, M_1 \in \underline{X}, M_2 \in \underline{X}^*$. Since $R/J(R)$ is semisimple, then M_1 is semisimple injective by Theorem 4. Since R is a right hereditary and direct sum of small modules is small, then M_2 is small by remark. Thus R is a right H-ring. We write $R = I_1 \oplus I_2$ where $I_1 \in \underline{X}, I_2 \in \underline{X}^*$. Then I_1 is injective by Theorem 4, and $E(R) = I_1 \in E(I_2)$. Since $I_2 \in \underline{X}^*$ then $E(I_2) \in \underline{X}^*$ by Lemma 8, and so $E(I_2)$ is small. Hence $E(I_2) = 0$. It follows that $H_{\underline{X}}(R) = 0$ and R is a right self-injective. Since $J(R)$ is $\underline{X}^*$-submodule of M, then $J(R) = 0$. It is clear that every non-zero right ideal of R is injective and then direct summand of R. Hence $Z(R) = 0$. By [8, Theorem 4.3] R is quasi-Frobenius ring.
(3) $\Longrightarrow$ (4) By Lemma 15.
(4) $\Longrightarrow$ (5) Clear.
(5) $\Longrightarrow$ (4) Let R be a right hereditary ring. Assume R right self-injective. Let M be a non-zero R-module and $0 \neq m \in M$. Then $mR \cong R/r(m)$ is injective. Hence M contains a non-zero injective R-module. By [3, Lemma 15.10] R is a semisimple ring.
(4) $\Longrightarrow$ (1) Clear. □

References

[1] Anderson, F.W. and Fuller, K.R.: Rings and Categories of Modules, Berlin-Heidelberg-New York, 1974.

[2] Curtis, C., Reiner, I.: Methods of Representation Theory, Wiley, New York, 1981.

[3] Dung, N.V., Huynh, D.V., Smith, P.F., Wisbauer, R.: Extending Modules, New York, 1994.

[4] Harada, M.: Non-small modules and non-cosmall modules, In Ring Theory: Proceedings of the 1978 Antwerp Conference, F. Van Oystaeyen, ed. New York: Marcel Dekker.

[5] Harmancı, A. and Smith, P.F.: Relative injectivity and module classes, *Comm. in Alg.,* 20(9), 2471-2501 (1992).

[6] Keskin, D., Harmancı, A., Smith, P.F.: On $\oplus$-supplemented modules, preprint.

[7] Mohamed, S.H. and Muller, B.J.: Continuous and Discrete Modules, No. 147, Cambridge University Press, 1990.

[8] Oshiro, K.: Lifting modules, extending modules and their applications to QF-rings, *Hokkaido Math. Journal,* 13, 310-338 (1984).

[9] Rayar, M.: On small and cosmall modules, *Acta Math. Acad. Sci. Hungar,* 39(4), 389-392 (1982).

[10] Rotman, J.J.: An Introduction to Homological Algebra, Academic Press, New York, 1979.

$Z^*(.)$ **Yardımıyla Bazı Halkaların Karakterizasyonu**

Özet

$\underline{X} = \{M : Z^*(M) = 0\}$ ve $\underline{X}^* = \{M : Q \geq M, P/Q \in \underline{X}$ ise $P/Q = 0$ dır.$\}$ modüllerin, sınıfları olsun. Bu çalışmada bir R halkası için her R-modül M, $M = M_1 \oplus M_2$, $M_1 \in \underline{X}$ $M_2 \in \underline{X}^*$ olacak şekilde bir ayrışıma sahipse R'nin yapısı belirleniyor.

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