REFERENCES

1. D. S. Dummit, R. M. Foote, Abstract Algebra, 3rd edition, Wiley, 2003.
2. I. Kaplansky, Commutative rings, Allyn and Becon, 1970.
3. M. D. Larsen, P. J. McCarthy, Multiplicative Theory of Ideals, Academic Press, 1971.
4. H. Matsumura, Commutative Ring Theory, Cambridge University Press, 1986.

All rings considered, unless stated otherwise, are assumed to have a unity and all modules are unital.

CH.1 PROJECTIVE AND FLAT MODULES

1.1. Projective Modules

Let R be a (not necessarily commutative) ring and let M be a left R-module. M is said to be a **free** R-module in case there is a nonempty subset $S \subseteq M$ such that every element of M can be uniquely written in the form

$$\sum_{u \in S} a_u u,$$

where $a_u \in R$ and $a_u \neq 0$ for only a finite number of elements $u \in S$. We note that, in this case,

$$M = \bigoplus_{u \in S} Ru.$$

Let M be a free module over a ring R and let N be another R-module such that there is an R-epimorphism

$$N \xrightarrow{g} M \longrightarrow 0.$$

Assume that a subset $\{u_\alpha\}$ freely generates M over R. Then there exists $n_\alpha \in N$ with $g(n_\alpha) = u_\alpha$ for each α. Thus the mapping from $\{u_\alpha\}$ into N defined by $u_\alpha \mapsto n_\alpha$ can be extended to an R-homomorphism h from M into N. Note that $gh = 1_M$. In this case we say that the homomorphism g splits. There is a certain kind of modules having this property, called projective modules. It follows that once we define projective modules, all free modules turn out to be included in this new class. Now we define projectivity.

Let P be a left module over a ring R. P is said to be **projective** if, for any epimorphism of left R-modules $g : B \to C$, and any R-homomorphism $h : P \to C$, there exists an R-homomorphism $h' : P \to B$ such that $h = g \circ h'$. (See the diagram below.)

$$\begin{tikzcd}
& P \arrow[dl, dashed, "h'"'] \arrow[d, "h"] & \\
B \arrow[r, "g"'] \arrow[ur, phantom, "\circlearrowright"] & C \arrow[r] & 0
\end{tikzcd}$$

We refer to this property by saying that any $h : P \to C$

can be lifted (along g) to a homomorphism $h': P \to B$. Note that in general such a lifting may not be possible. To see this consider the abelian groups $B = \mathbb{Z}/4\mathbb{Z}$ and $C = \mathbb{Z}/2\mathbb{Z}$, which are modules over the ring of integers $\mathbb{Z}$, and the unique epimorphism $g: B \to C$. Then the identity map h from $P = \mathbb{Z}/2\mathbb{Z}$ to C clearly cannot be lifted (along g) to a homomorphism $P \to B$. Therefore, $\mathbb{Z}/2\mathbb{Z}$ is not $\mathbb{Z}$-projective.

For any given R-module ${}_RP$, the functor $\mathrm{Hom}_R(P,-)$ from the category of left R-modules to the category of abelian groups is **left exact**, in the sense that, for any short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \qquad (1)$$

of left R-modules, we have a corresponding exact sequence of abelian groups:

$$0 \to \mathrm{Hom}_R(P,A) \xrightarrow{f_*} \mathrm{Hom}_R(P,B) \xrightarrow{g_*} \mathrm{Hom}_R(P,C).$$

It turns out that ${}_RP$ is projective if and only if $\mathrm{Hom}_R(P,-)$ is exact, which means that, for any short exact sequence (1), we have a short exact sequence

$$0 \to \mathrm{Hom}_R(P, A) \xrightarrow{f_*} \mathrm{Hom}_R(P, B) \xrightarrow{g_*} \mathrm{Hom}_R(P, C) \to 0.$$

<u>1. Proposition.</u> A direct sum $P = \bigoplus_i P_i$ of left R-modules is projective if and only if each P_i is projective. $\square$

<u>2. Theorem.</u> ${}_R P$ is projective if and only if it is a direct summand of a free module, if and only if any epimorphism $N \to P$ splits. $\square$

<u>3. Theorem.</u> (Dual basis lemma) A left R-module P is projective if and only if there exist a family of element $\{a_i : i \in I\} \subseteq P$ and linear functionals $\{f_i : i \in I\} \subseteq P^* = \mathrm{Hom}_R(P, R)$ such that for any $a \in P$, $f_i(a) = 0$ for almost all i, and $a = \sum a_i f_i(a)$. $\square$

<u>4. Example.</u> As we noted earlier, every free module is projective.

<u>5. Example.</u> The $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z} \times \cdots$ is not projective.

<u>proof</u>. Suppose that M is a projective $\mathbb{Z}$-module. Then there exists a free $\mathbb{Z}$-module F and a submodule N of F such that $M \oplus N = F$. We can choose a $\mathbb{Z}$-basis I of F in such a way that I can be written as a disjoint union $I_1 \cup I_2$ of two proper subsets I_1 and

I_2 such that I_1 is countable and the $\mathbb{Z}$-submodule $P = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ of M lies in the span of I_1 (which we shall denote by F_1). Then $P \subseteq F_1$ but $M \not\subseteq \operatorname{span} F_1$ since M is not countable. It follows that there exists $e \in I_2$ such that for the natural projection $f : F \to e\mathbb{Z} \cong \mathbb{Z}$, we have $f(P) = 0$ but $f(M) \neq 0$. We shall take $\operatorname{im}(f) = \mathbb{Z}$ w.l.o.g.

Now consider the following two subgroups of M:

$$A = \{ (2a_1, 2^2 a_2, \ldots, 2^n a_n, \ldots) : a_i \in \mathbb{Z} \},$$

$$B = \{ (3b_1, 3^2 b_2, \ldots, 3^n b_n, \ldots) : b_i \in \mathbb{Z} \}.$$

Clearly, $A + B = M$. Since any element of A has the form

$$(2a_1, \ldots, 2^{n-1} a_{n-1}, 0, 0, \ldots) + 2^n x,$$

where $x \in M$, and $f(P) = 0$, we have $f(A) \subseteq 2^n \mathbb{Z}$, so that $f(A) = 0$. Similarly, we obtain $f(B) = 0$, a contradiction.

$\square$

Let R and S be two commutative rings with R being a subring of S. For two R-submodules A and B of S we define

$$AB = \left\{ \sum_{finite} a_i b_i \ : \ a_i \in A,\ b_i \in B \right\}$$

and

$$A^{-1} = \{ s \in S \ : \ sA \subseteq R \}.$$

Clearly, these are also R-submodules of S and we always have $A^{-1}A \subseteq R$ for any R-submodule A of S. We say that A is an **invertible** R-submodule if we have the equiality $A^{-1}A = R$. Observe that an R-submodule A of S is invertible if and only if there exists an R-submodule B of S such that $AB = R$.

Let A be an invertible R-submodule of S. Then

$$1 = y_1 x_1 + \cdots + y_k x_k$$

for some $y_1, \ldots, y_k \in A^{-1}$ and $x_1, \ldots, x_k \in A$. Let $a \in A$. Then $y_i a \in R$ for every $i = 1, \ldots, k$ and

$$a = (y_1 a) x_1 + \cdots + (y_k a) x_k.$$

Define $f_i : A \longrightarrow R$ by $f_i(x) = y_i x$ for all $x \in A$. Then f_i is an R-homomorphism and

$$a = f_1(a) x_1 + \cdots + f_k(a) x_k.$$

This gives, by the dual basis lemma, that A is a projective R-module.

1.2. Tensor Product of Modules

Let R be a ring, M a right R-module, and N a left R-module. Denote by $Z(M,N)$ the free $\mathbb{Z}$-module with basis the Cartesim product set $M\times N$. Thus the elements of $Z(M,N)$ are formal finite sums

$$n_1(x_1,y_1)+\cdots+n_k(x_k,y_k),$$

where $n_1,\dots,n_k\in\mathbb{Z}$, $(x_1,y_1),\dots,(x_k,y_k)\in M\times N$. Let $Y(M,N)$ be the subgroup of $Z(M,N)$ generated by the set of all elements of the form

$$(x_1+x_2,y)-(x_1,y)-(x_2,y),$$

$$(x,y_1+y_2)-(x,y_1)-(x,y_2),$$

$$(xa,y)-(x,ay),$$

where $(x,y),(x_1,y_1),(x_2,y_2)\in M\times N$ and $a\in R$. The factor group $Z(M,N)/Y(M,N)$ is called the <u>tensor product</u> of M and N, and is denoted by $M\otimes_R N$. Note that with the tensor product $M\otimes_R N$ we have produced an abelian group. If $x\in M$ and $y\in N$, we denote the element $(x,y)+Y(M,N)$ of the quotient $M\otimes_R N=Z(M,N)/_{Y(M,N)}$ by $x\otimes y$, and call such an element of $M\otimes_R N$ a simple tensor. The following properties hold for $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$, and $a \in R$:

(i) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$,

(ii) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$,

(iii) $xa \otimes y = x \otimes ay$.

With the help of these properties, we also get:

(iv) $n(x \otimes y) = nx \otimes y = x \otimes ny$ for any integer n,

(v) $0 \otimes y = x \otimes 0 = 0$,

and

(vi) $(-x) \otimes y = -(x \otimes y) = x \otimes (-y)$.

Notice that the set of simple tensors $\{x \otimes y : x \in M, y \in N\}$ generates $M \otimes_R N$ over $\mathbb{Z}$. This is equivalent to saying that every element of $M \otimes_R N$ can be written as a sum of simple tensors since

$$\sum_{i=1}^{k} n_i (x_i \otimes y_i) = \sum_{i=1}^{k} (n_i x_i) \otimes y_i .$$

<u>Definition</u>. Let M be a right R-module and N a left R-module. A function $f : M \times N \longrightarrow G$, where G is an abelian group, is called **bilinear** if

$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y),$$

$$f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2),$$

$$f(xa, y) = f(x, ay)$$
for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$, and $a \in R$.

<u>6. Proposition.</u> Let M be a right R-module and N a left R-module. If $f : M \times N \longrightarrow G$ is a bilinear map into an abelian group G, then there exists a unique homomorphism $g : M \otimes_R N \longrightarrow G$ such that $g(x \otimes y) = f(x, y)$ for all $x \in M$, $y \in N$. $\square$

<u>7. Proposition.</u> If N is a left R-module, then there is an isomorphism $g : R \otimes_R N \longrightarrow N$ such that $g(r \otimes n) = rn$ for all $r \in R$ and $n \in N$. $\square$

<u>8. Proposition.</u> Let I be a right ideal of the ring R and let M be a left R-module. Then there is an isomorphism of abelian groups

$$R/I \otimes_R M \cong M/IM.$$

If, further, I is a two-sided ideal, then above isomorphism is an R-isomorphism if we regard $R/I \otimes_R M$ as an R-module with the scalar multiplication defined by $a(r+I \otimes m) = ar + I \otimes m$ for all $a, r \in R$ and $m \in M$.

proof. Define a mapping $f: R/I \times M \longrightarrow M/IM$ by $f(r+I, m) = rm + IM$ for all $r \in R$ and $m \in M$. This is a well-defined mapping for if $r_1 + I = r_2 + I$, then $r_1 - r_2 \in I$, and so $r_1 m + IM = r_2 m + IM$ for any $m \in M$. By Proposition 6, there is a group homomorphism

$$\tilde{f} : R/I \otimes_R M \longrightarrow M/IM$$

such that $\tilde{f}(r+I \otimes m) = rm + IM$ for all $r \in R$ and $m \in M$. It is clear that $\tilde{f}$ is onto. Let $\sum (r_i + I) \otimes m_i \in \operatorname{Ker} \tilde{f}$. Then $\sum r_i m_i = \sum a_j n_j$ for some $a_j \in I$ and $n_j \in M$. Then $\sum (r_i + I) \otimes m_i = \sum (1+I) \otimes r_i m_i = (1+I) \otimes \sum r_i m_i = (1+I) \otimes \sum a_j n_j = \sum (1+I) \otimes a_j n_j = \sum 0 \otimes n_j = 0$. The proof of last statement is left to the student.

Let M, M' be right R-modules, N, N' be left R-modules, and $f: M \longrightarrow M'$ and $g: N \longrightarrow N'$ be homomorphisms. It is routine to check that the mapping $\varphi : M \times N \longrightarrow M' \otimes_R N'$ defined by

$$\varphi(m, n) = f(m) \otimes g(n)$$

for all $m \in M$ and $n \in N$ is bilinear. It follows, by Proposition 6, that we can define a group homomorphism from $M \otimes_R N$ into $M' \otimes_R N'$ which maps $m \otimes n$ onto $f(m) \otimes g(n)$ for all $m \in M$, $n \in N$. We denote this homomorphism by $f \otimes g$.

<u>9. Proposition.</u> Let M, M' be right R-modules, and N, N' be left R-modules. If $1_M : M \longrightarrow M$ and $1_N : N \longrightarrow N$ are the identity maps on M and N, then $1_M \otimes 1_N$ is the identity map on $M \otimes_R N$. If $f : M \longrightarrow M'$, $f' : M' \longrightarrow M''$, $g : N \longrightarrow N'$, and $g' : N' \longrightarrow N''$ are homomorphisms, then

$$(f' \otimes g')(f \otimes g) = f'f \otimes g'g.$$

$\square$

<u>10. Proposition.</u> If $f : M \to M'$ and $g : N \to N'$ are surjective homomorphisms of right and left R-modules, respectively, then $f \otimes g$ is surjective. $\square$

<u>11. Proposition.</u> If $f : M \to M'$ and $g : N \to N'$ are surjective homomorphisms of right and left R-modules, respectively, then $\mathrm{Ker}(f \otimes g)$ consists of all finite sums of the form $\sum x_i \otimes y_i$, where $x_i \in \mathrm{Ker} f$ or $y_i \in \mathrm{Ker}\, g$.

<u>proof.</u> (sketch)

- Let K denote the set of all sums $\sum x_i \otimes y_i$ in $M \otimes_R N$, where $x_i \in \mathrm{Ker} f$ or $y_i \in \mathrm{Ker}\, g$. Clearly,

$K \subseteq \mathrm{Ker}(f \otimes g)$. It follows that there is homomorphism $h : (M \otimes_R N)/K \longrightarrow M' \otimes_R N'$ such that

$$h(x \otimes y + K) = f(x) \otimes g(y)$$

for all simple tensors $x \otimes y \in M \otimes_R N$. Furthermore,

$$\mathrm{Ker}\, h = \mathrm{Ker}(f \otimes g)/K.$$

- Define a mapping $j : M' \times N' \longrightarrow (M \otimes_R N)/K$ by $j(x', y') = x \otimes y + K$, where $f(x) = x'$ and $g(y) = y'$, and show that j is well-defined.

- Prove that j is bilinear, which in turn yields a (unique) homomorphism $k : M' \otimes N' \longrightarrow (M \otimes_R N)/K$ such that $k(x' \otimes y') = x \otimes y + K$, where $f(x) = x'$ and $g(y) = y'$.

- Showing that kh and hk are identity maps, conclude that $\mathrm{Ker}\, h = 0$, i.e., $K = \mathrm{Ker}(f \otimes g)$.

<u>12. Theorem.</u> (i) Let

$$0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$$

be an exact sequence of left R-modules and let M be a right R-module. Then the sequence

$$M \otimes_R N' \xrightarrow{1_M \otimes f} M \otimes_R N \xrightarrow{1_M \otimes g} M \otimes_R N'' \longrightarrow 0$$

is exact.

(ii) Let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be an exact sequence of right R-modules and let N be a left R-module. Then the sequence

$$M' \otimes_R N \xrightarrow{f \otimes 1_N} M \otimes_R N \xrightarrow{g \otimes 1_N} M'' \otimes_R N \longrightarrow 0$$

is exact. $\square$

<u>13. Theorem</u>. Let $\{M_\alpha\}$ be a family of right R-modules, and let N be a left R-module. Then we have

$$(\oplus M_\alpha) \otimes_R N \cong \oplus (M_\alpha \otimes_R N).$$

$\square$

Let R and S be rings. Let M be a right R-module and a left S-module. Then we say that M is an $(S\text{-}R)$-bimodule if $s(mr) = (sm)r$ for all $m \in M$, $r \in R$, and $s \in S$. If M is an (S,R)-bimodule and N is a left R-module then the abelian group $M \otimes_R N$ can be made into a left S-module with the scalar multiplication defined by

$$s(m \otimes n) = sm \otimes n$$

for all $m \in M$, $n \in N$, and $s \in S$. Similarly if N is an (R,S)-bimodule, one can turn $M \otimes_R N$ into a right S-module via a similar action as above.

<u>14. Theorem.</u> Let R and S be rings, M an (S,R)-bimodule, and N a left R-module. If L is a right S-module, then

$$L \otimes_S (M \otimes_R N) \cong (L \otimes_S M) \otimes_R N. \quad \square$$

If R is a commutative ring and M, N are R-modules, then one can turn the abelian group $M \otimes_R N$ into an R-module with the scalar product defined on simple tensors by

$$r(m \otimes n) = rm \otimes n = m \otimes rn$$

for all $r \in R$, $m \in M$ and $n \in N$. If $f : R \to R'$ is a homomorphism of commutative rings, then R' has an R-module structure. If M is an R-module, then the abelian group $M \otimes_R R'$ can be turned into an R'-module with the scalar multiplication $z(x \otimes y) = x \otimes zy$ for all $x \in M$, $y, z \in R'$. We can apply above discussion by taking $R' = S^{-1}R$, where S is any multiplicatively closed subset of R.

<u>15. Theorem.</u> Let R be commutative ring and let M be

an R-module. Then for any multiplicatively closed subset S of R, we have $S^{-1}M \cong M \otimes_R S^{-1}R$.

<u>proof</u>. Define a mapping $M \times S^{-1}R \longrightarrow S^{-1}M$ by $(m, r/s) \longmapsto rm/s$, which is clearly bilinear. So there exists an R-homomorphism $\alpha : M \otimes_R S^{-1}R \longrightarrow S^{-1}M$ such that $\alpha(m \otimes r/s) = rm/s$. Conversely, we can define $\beta : S^{-1}M \longrightarrow M \otimes S^{-1}R$ by $\beta(m/s) = m \otimes 1/s$. It is easy to show that β is well-defined and that α and β are inverses of each other; so that they are R-isomorphisms. However, one can easily see that α (and β) is $S^{-1}R$-linear, which completes the proof. $\square$

Let R, S be rings, and let ${}_RA_S$, ${}_RB$ be modules. Then the abelian groups

$$\mathrm{Hom}_R(A, B) \text{ and } \mathrm{Hom}_R(B, A)$$

can be made into left and right S-modules, respectively, with the following respective S-actions:

if $s \in S$ and $f \in \mathrm{Hom}_R(A, B)$, we define

$$sf : a \longmapsto f(as)$$

for all $a \in A$, and

if $s \in S$ and $g \in \mathrm{Hom}_R(B, A)$, we define

$$gs : b \longmapsto g(b)s$$

for all $b \in B$.

Similarly, one can define modules ${}_SA_R$ and B_R and make the abelian groups $Hom_R(A,B)$ and $Hom_R(B,A)$ into right and left S-modules, respectively.

<u>16. Theorem</u>. (Adjoint Isomorphism) Given modules A_R, ${}_RB_S$, and C_S, where R and S are rings, there is an isomorphism $\tau_{A,B,C} : Hom_S(A \otimes_R B, C) \longrightarrow Hom_R(A, Hom_S(B,C))$ such that $\tau_{A,B,C} : f \longmapsto \tau(f)$, where $\tau(f)(a) : b \longmapsto f(a \otimes b)$.

Moreover, fixing any two of A, B, C each $\tau_{A,B,C}$ yields a natural isomorphism:

$$Hom_S(- \otimes_R B, C) \approx Hom_R(-, Hom_S(B,C)),$$
$$Hom_S(A \otimes_R -, C) \approx Hom_R(A, Hom_S(-,C)),$$
$$Hom_S(A \otimes_R B, -) \approx Hom_R(A, Hom_S(B,-)).$$

1.3. Some Homological Algebra

For a ring R, a <u>complex</u> (or R-complex) (C,d) is a set $C = \{C_i\}$ idexed by $\mathbb{Z}$ together with an indexed set $d = \{d_i : i \in \mathbb{Z}\}$ of R-homomorphisms $d_i : C_i \to C_{i-1}$ such that $d_{i-1} d_i = 0$ for all i. If (C,d) and (C',d') are R-complexes, a (chain) <u>homomorphism</u> of C into C' is an indexed set $\alpha = \{\alpha_i : i \in \mathbb{Z}\}$ of homomorphisms

$\alpha_i : C_i \to C_i'$ such that we have the commutative diagram

$$\begin{array}{ccccccccc}
C: & \cdots \longrightarrow & C_{i+1} & \xrightarrow{d_{i+1}} & C_i & \xrightarrow{d_i} & C_{i-1} & \longrightarrow \cdots \\
\alpha \downarrow & & \downarrow \alpha_{i+1} & & \downarrow \alpha_i & & \downarrow \alpha_{i-1} & \\
C' & \cdots \longrightarrow & C'_{i+1} & \xrightarrow[d'_{i+1}]{} & C'_i & \xrightarrow[d'_i]{} & C'_{i-1} & \longrightarrow \cdots
\end{array}$$

More briefly we write $\alpha d = d'\alpha$. With these definitions, R-complexes together with chain homomorphisms form a category which we denote <u>R-comp.</u>

Let (C, d) be a complex and let $Z_i(C) = \ker d_i$ and $B_i(C) = \operatorname{Im} d_{i+1}$ for every $i \in \mathbb{Z}$. Since $d_i d_{i+1} = 0$, we have $B_i(C) \subseteq Z_i(C)$ for every $i \in \mathbb{Z}$. (Here the elements of $Z_i(C)$ are called i-<u>cycles</u> and the elements of $B_i(C)$ are called the i-<u>boundaries</u>.) The module

$$H_i = H_i(C) := Z_i(C)/B_i(C)$$

is called the ith <u>homology module</u> of the complex (C, d). Evidently, $C_{i+1} \to C_i \to C_{i-1}$ is exact if and only if $H_i(C) = 0$.

Let α be a chain homomorphism of (C, d) into the complex (C', d'). It is easy to see that

$$\alpha_i B_i \subseteq B_i' = B_i(C')$$

and

$$\alpha_i Z_i \subseteq Z_i' = Z_i(C').$$

Hence the map $Z_i \longrightarrow H_i' = H_i(C') = Z_i'/B_i'$ defined by $z_i \longmapsto \alpha_i z_i + B_i'$ is a homomorphism with kernel containing B_i. This gives a homomorphism $\tilde{\alpha}_i$ of $H_i(C)$ into $H_i(C')$ such that $z_i + B_i \longmapsto \alpha_i z_i + B_i'$. It is now trivial to check that the maps $(C,d) \mapsto H_i(C)$, $\hom(C,C') \longrightarrow \hom(H_i(C), H_i(C'))$, where the latter is $\alpha \mapsto \tilde{\alpha}_i$, define a fuctor from R-comp to R-mod. We call this functor the ith <u>homology functor</u> from R-comp to R-mod. (In fact $H_i(-)$ is an additive functor for each i.)

In the sequel, the complexes we shall deal with will have either $C_i = 0$ for $i < 0$ or $C_i = 0$ for $i > 0$. In the first case, the complexes are called <u>positive</u> or <u>chain complexes</u> and in the second, <u>negative</u> or <u>cochain complexes</u>. In the latter case it is usual to denote C_{-i} by C^i and d_{-i} by d^i. With this notation, a cochain complex has the appearance

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots$$

if we drop the C^{-i}, $i > 1$. It is usual in this case to denote $\ker d^i$ by Z^i and $\operatorname{Im} d^{i-1}$ by B^i. $H^i = Z^i/B^i$ is the ith <u>cohomology group</u>. In the case of H^0, we have

$H^0 = Z^0$. A chain complex has the form

$$\cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0 .$$

In this case $H^0 = C_0 / d_1 C_1 = \operatorname{coker} d_1$.

By a <u>short exact sequence of complexes</u> we mean a sequence of complexes and chain homomorphisms $C' \xrightarrow{\alpha} C \xrightarrow{\beta} C''$ such that $0 \to C_i' \xrightarrow{\alpha} C_i \to C_i'' \to 0$ is exact for every $i \in \mathbb{Z}$. We shall indicate this by saying that $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$ is exact. We have the commutative diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_{i+1}' \arrow[r, "\alpha_{i+1}"] \arrow[d, "d_{i+1}'"'] & C_{i+1} \arrow[r, "\beta_{i+1}"] \arrow[d, "d_{i+1}"'] & C_{i+1}'' \arrow[r] \arrow[d, "d_{i+1}''"'] & 0 \\
0 \arrow[r] & C_i' \arrow[r, "\alpha_i"] \arrow[d, "d_i'"'] & C_i \arrow[r, "\beta_i"] \arrow[d, "d_i"'] & C_i'' \arrow[r] \arrow[d, "d_i''"'] & 0 \\
0 \arrow[r] & C_{i-1}' \arrow[r, "\alpha_{i-1}"] \arrow[d] & C_{i-1} \arrow[r, "\beta_{i-1}"] \arrow[d] & C_{i-1}'' \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

in which the rows are exact.

<u>17. Theorem</u>. Let $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$ be an exact sequence of complexes. Then for each $i \in \mathbb{Z}$ we can define a module homomorphism $\delta_i : H_i(C'') \longrightarrow H_{i-1}(C')$ so that the infinite sequence of homology modules

$$\cdots \longrightarrow H_i(C') \xrightarrow{\tilde{\alpha}_i} H_i(C) \xrightarrow{\tilde{\beta}_i} H_i(C'') \xrightarrow{\delta_i} H_{i-1}(C') \xrightarrow{\alpha_{i-1}} H_{i-1}(C) \longrightarrow \cdots$$

is exact. The homomorphism δ_i is called the <u>connecting homomorphism</u> of $H_i(C'')$ into $H_{i-1}(C')$ and the above infinite sequence of homology modules the <u>long exact homology sequence</u> determined by the short exact sequence of complexes $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$. Note that the connecting homomorphism δ_i is natural in the sense that if we have a diagram of homomorphisms of complexes

$$\begin{tikzcd}
0 \arrow[r] & C' \arrow[r, "\alpha"] \arrow[d, "f'"'] & C \arrow[r, "\beta"] \arrow[d, "f"'] & C'' \arrow[r] \arrow[d, "f''"'] & 0 \\
0 \arrow[r] & D' \arrow[r, "\mu"'] & D \arrow[r, "\nu"'] & D'' \arrow[r] & 0
\end{tikzcd}$$

which is commutative and has exact rows. Then

$$\begin{tikzcd}
H_i(C'') \arrow[r, "\delta_i"] \arrow[d, "\tilde{f}''_i"'] & H_{i-1}(C') \arrow[d, "\tilde{f}'_{i-1}"] \\
H_i(D'') \arrow[r, "\delta_i"'] & H_{i-1}(D').
\end{tikzcd}$$

Let M be a module over the ring R. We define a <u>complex over M</u> as a positive complex $C = (C, d)$ together with a homomorphism $\varepsilon : C_0 \longrightarrow M$, called an augmentation, such that $\varepsilon d_1 = 0$. Thu we have the sequence of homomorphisms

$$(*)\quad \cdots \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

where the product of any two successive hom. is 0. The complex C, ε over M is called a <u>resolution</u> of M if $(*)$ is exact. A complex C, ε over M is called projective if every C_i is projective.

Projective resolution of a module M always exists: Let F_0 be a free R-module s.t. $F_0 \xrightarrow{\varepsilon} M \to 0$ is exact. Denote $\ker \varepsilon$ by K_0. Then there exists a free module F_1 s.t. $F_1 \xrightarrow{f_1} K_0 \to 0$ is exact. Let $K_1 = \ker f_1$. Continuing in this way we obtain

$$\begin{tikzcd}
 & 0 \arrow[dr] & & 0 & & & & \\
 & & K_1 \arrow[ur] \arrow[dr, "j"] & & & & & \\
\cdots \arrow[r] & F_2 \arrow[ur, "f_2"] \arrow[rr, "d_2"] & & F_1 \arrow[rr, "d_1"] \arrow[dr, "f_1"'] & & F_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\
\cdots \arrow[ur] & & & & K_0 \arrow[ur, "j"'] \arrow[dr] & & & \\
 & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where i denote the inclusion map and $d_i = j f_i$ for each $i \geq 1$. It is easy to see that the sequence

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is exact, and hence is a projective resolution of M.

Let F be an additive functor from a category of R-mod to the category of $\underline{Ab}$. Let M be an R-module and let

$$\cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

be a projective resolution of M. Applying the functor F we obtain a sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow FC_1 \xrightarrow{F(d_1)} FC_0 \xrightarrow{F(\varepsilon)} FM \longrightarrow 0.$$

Since F is additive $F(\varepsilon)F(d_1) = 0 = F(d_i)F(d_{i+1})$ for every i, and so $FC = \{FC_i\}$, $Fd = \{F(d_i)\}$ with the augmentation $F\varepsilon$ is a (positive) complex over FM. Now put

$$(L_nF)M = H_n(FC), \quad n \geqslant 0.$$

This definition gives

$$H_0(FC) = FC_0 / F(d_1)FC_1$$

since we are taking the terms $FC_i = 0$ if $i < 0$. It can be proved that the above definitions are essentially independent of the choice of the resolutions. It can also be proved that L_nF defines indeed a functor from R-mod to Ab. This functor is called the nth <u>left derived functor</u> of the given functor F.

<u>18. Theorem.</u> Let F be an additive functor from R-mod to Ab. Then for any short exact sequence of R-modules $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ and any $n = 1, 2, \ldots$ there exists a connecting homomorphism

$$\delta_n : L_n F M'' \longrightarrow L_{n-1} F M'$$

such that

$$\cdots \longrightarrow L_1 F M'' \xrightarrow{\delta_1} L_0 F M' \xrightarrow{L_0 F(\alpha)} L_0 F M \xrightarrow{L_0 F(\beta)} L_0 F M'' \longrightarrow 0$$

is exact.

Let M be a right module over the ring, then $M \otimes_R -$ is an additive right exact functor from R-mod to Ab. The nth derived functor of $M \otimes_R -$ is denoted as $\mathrm{Tor}_n(M, -)$ (or $\mathrm{Tor}_n^R(M, -)$). More precisely, given a left R-module N, we choose a projective resolution of N

$$\mathcal{P} : \quad \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} N \longrightarrow 0,$$

and form the chain complex

$$M \otimes_R \mathcal{P}_N : \quad \cdots \longrightarrow M \otimes_R P_1 \xrightarrow{1_M \otimes d_1} M \otimes_R P_0 \longrightarrow 0,$$

where $\mathcal{P}_N$ is the deleted complex $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$, and then define $\mathrm{Tor}_n(M, N)$ to be the nth homology group $H_n(M \otimes \mathcal{P}_N)$ of $M \otimes \mathcal{P}$. By definition $\mathrm{Tor}_0(M,N) = (M \otimes_R P_0)/\mathrm{im}(1_M \otimes d_1)$. Since $M \otimes_R -$ is right exact, $M \otimes_R N \cong (M \otimes_R P_0)/\mathrm{im}(1_M \otimes d_1) = \mathrm{Tor}_0(M, N)$.

The isomorphism $Tor_0(M,N) \cong M \otimes N$ and the long exact sequence of homology imply that if $0 \to N' \to N \to N'' \to 0$ is exact, then

$$\cdots \longrightarrow Tor_1(M,N') \longrightarrow Tor_1(M,N) \longrightarrow Tor_1(M,N'') \longrightarrow$$
$$M \otimes_R N' \longrightarrow M \otimes_R N \longrightarrow M \otimes_R N'' \longrightarrow 0$$

is exact. We note that we can define functors $\overline{Tor_n}(M,N)$ using a projective resolution of the first argument M. Moreover, we can prove that $Tor_n(M,N) \cong \overline{Tor_n}(M,N)$.

Finally we note that if A and P are right R-modules with P projective, and if B and Q are left R-modules with Q projective, then for all $n \geq 1$,

$$Tor_n^R(P,B) = 0 \text{ and } Tor_n^R(A,Q) = 0.$$

1.4. Flat Modules

Let R be a ring (not necessarily commutative) and N a left R-module. Let

$$\mathcal{S}: \quad \cdots \longrightarrow M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \longrightarrow \cdots$$

be a sequence of right R-modules (indexed by integers). We shall denote the sequence

$$\cdots \longrightarrow M_{i+1} \otimes_R N \xrightarrow{f_{i+1} \otimes 1_N} M_i \otimes_R N \xrightarrow{f_i \otimes 1_N} M_{i-1} \otimes_R N \longrightarrow \cdots$$

by $\mathcal{S} \otimes_R N$. We say that N is **flat** over R if for every exact sequence $\mathcal{S}$ of right R-modules

$\mathcal{S} \otimes_R N$ is an exact sequence of abelian groups.

N is **faithfully flat** over R if for every sequence of right R-modules

$$\mathcal{S} \text{ is exact} \iff \mathcal{S} \otimes_R N \text{ is exact}.$$

Given an exact sequence $\mathcal{S}$

$$\cdots \longrightarrow M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \longrightarrow \cdots$$

one can break up $\mathcal{S}$ into short exact sequences as follows:

$$\begin{tikzcd}
 & 0 \arrow[dr] & & 0 \arrow[dr] & & 0 & & \\
\cdot \arrow[dr, dotted] & & M_{i+1}/\mathrm{Ker}\, f_{i+1} \arrow[ur] \arrow[dr] & & M_i/\mathrm{Ker}\, f_i \arrow[ur] \arrow[dr] & & \cdot & \\
\cdots \arrow[r] & M_{i+1} \arrow[rr, "f_{i+1}"] \arrow[ur] \arrow[dr] & & M_i \arrow[rr, "f_i"] \arrow[ur] \arrow[dr] & & M_{i-1} \arrow[rr] \arrow[ur, dotted] \arrow[dr] & & \cdots \\
\cdot \arrow[ur, dotted] & & \mathrm{im}\, f_{i+1} \arrow[ur] \arrow[dr] & & \mathrm{im}\, f_i \arrow[ur] \arrow[dr] & & \mathrm{im}\, f_{i-1} \arrow[dr] & \\
 & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & & 0
\end{tikzcd}$$

Thus in the definition of flatness it is enough to consider only short exact sequences. Moreover, since tensor product is right exact by Theorem 11, we can restrict our attension to exact sequences of the form

$0 \longrightarrow M' \longrightarrow M$ and need only to check if the induced sequence $0 \longrightarrow M' \otimes_R N \longrightarrow M \otimes_R N$ is exact.

If a left R-module N is a direct sum of $\{N_i\}_{i \in I}$, then the functor $- \otimes_R N$ is the direct sum of the functors $- \otimes_R N_i$, and so $- \otimes_R N$ is exact if and only if each of the $- \otimes_R N_i$ is. Therefore N is flat if and only if each of the N_i is. This implies that every projective module is flat.

From now on all rings will be considered to be commutative unless stated otherwise.

Let $f: R \longrightarrow R'$ be a homomorphism of (commutative) rings and let R' be flat as an R-module (induced by f). Then we say that f is a flat homomorphism and R' is a flat R-algebra. For example, the localization $S^{-1}R$ of R with respect to any multiplicatively closed subset S of R is a flat R-algebra. (This can be seen by Theorem 12 and the fact that $S^{-1}\_$ is a covariant exact functor from the category of R-modules to the category of $S^{-1}R$-modules.)

Let R' be an R-algebra and M an R'-module.

Then the following hold:

(1) If R' is flat over R and M is flat over R', then M is flat over R;

(2) If R' is faithfully flat over R and M is faithfully flat over over R', then M is faithfully flat over R.

(3) If M is faithfully flat over R' and flat over R, then R' is flat over R.

(4) If M is faithfully flat over both R and R', then R' is faithfully flat over R.

Under the hypothesis that M is only an R-module, we also have

(5) If M is flat over R, then $M \otimes_R R'$ is flat over R'.

(6) If M is faithfully flat over R, then $M \otimes_R R'$ is faithfully flat over R'.

<u>19. Theorem</u> Let $f : R \to R'$ be a homomorphism of rings and M an R'-module. Then M is flat over R if and only if for every $P \in \operatorname{spec}(R')$, the localization M_P is flat over R_q where $q = P^c = f^{-1}(P) \in \operatorname{spec}(R)$ (or the same condition for every maximal ideal P of R').

proof. First, note that if $S \subseteq R$ is a multiplicatively closed subset, then for any $S^{-1}R$-modules M and N, we have $M \otimes_{S^{-1}R} N \cong M \otimes_R N$. This follows from the fact that in $M \otimes_R N$ we have

$$\frac{a}{s} x \otimes y = \frac{ax}{s} \otimes \frac{sy}{s} = \frac{sx}{s} \otimes \frac{ay}{s} = x \otimes \frac{a}{s} y,$$

for $x \in M$, $y \in N$, $a \in R$ and $s \in S$.

Assume now that M is R-flat. $f: R \to R'$ induces $\tilde{f}: R_q \to R'_P$ and M_P is an R'_P-module, and so an R_q-module. Let $\mathcal{S}$ be an exact sequence of R_q-modules. Then

$$\mathcal{S} \otimes_{R_q} M_P \cong \mathcal{S} \otimes_R M_P \cong (\mathcal{S} \otimes_R M) \otimes_{R'} R'_P,$$

and the right hand side is an exact sequence, so that M_P is R_q-flat.

Conversely, suppose that M_P is flat over R_q for every maximal ideal P of R'. Let $0 \to N' \to N$ be an exact sequence of R-modules, and write K for the kernel of the R'-homomorphism $N' \otimes_R M \to N \otimes_R M$. Then we have an exact sequence $0 \to K \to N' \otimes_R M \to N \otimes_R M$ of R'-modules. For any $P \in \mathrm{Max}(R')$, the localization gives an exact sequence

$$0 \to K_P \to N' \otimes_R M_P \to N \otimes_R M_P$$

of R_p-modules. Since $N' \otimes_R M_p \cong N' \otimes_R (R_q \otimes_{R_q} M_p) \cong N'_q \otimes_{R_q} M_p$, and similarly $N \otimes_R M_p \cong N_q \otimes_{R_q} M_p$, we have $K_p = 0$ by hypothesis. Therefore we have $K=0$. ∎

$$\begin{tikzcd}
0 \arrow[r] & K_p \arrow[r] \arrow[d, "\wr"] & N' \otimes_R M_p \arrow[r] \arrow[d, equal] & N \otimes_R M_p \arrow[d, equal] \\
& 0 \arrow[r] & N'_q \otimes_{R_q} M_p \arrow[r] & N_q \otimes_{R_q} M_p
\end{tikzcd}$$

20. Remark. Let $f: A \longrightarrow B$ be an R-homomorphism and let M be an R-module.

(1) We always have the isomorphism

$$\frac{B \otimes_R M}{\mathrm{Im}(f \otimes 1_M)} \cong B/_{\mathrm{Im} f} \otimes_R M.$$

To see this consider the commutative diagram

$$\begin{tikzcd}
A \otimes_R M \arrow[r] \arrow[d, equal] & B \otimes_R M \arrow[r] \arrow[d, equal] & B/_{\mathrm{Im} f} \otimes_R M \arrow[r] \arrow[d, dashed, "\cong"] & 0 \\
A \otimes_R M \arrow[r] & B \otimes_R M \arrow[r] & \dfrac{B \otimes_R M}{\mathrm{Im}(f \otimes 1_M)} \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact.

(2) Now let M be flat over R. We have an exact

sequence $0 \to \mathrm{Ker} f \xrightarrow{i} A \xrightarrow{f} B$, which gives rise to the exact sequence $0 \to \mathrm{Ker} f \otimes_R M \xrightarrow{i \otimes 1_M} A \otimes_R M \xrightarrow{f \otimes 1_M} B \otimes_R M$.
It follows that $\mathrm{Ker}(f \otimes 1_M) = \mathrm{Im}(i \otimes 1_M) \cong \mathrm{Ker} f \otimes_R M$.
To sum up, we conclude that for any R-homomorphism $f: A \to B$, $\mathrm{Ker}(f \otimes 1_M) \cong \mathrm{Ker} f \otimes_R M$. On the other hand, if we consider the exact sequence $A \xrightarrow{f} B \xrightarrow{p} B/\mathrm{Im} f \to 0$, we get the exact sequence

$$A \otimes_R M \xrightarrow{f \otimes 1_M} B \otimes_R M \longrightarrow B/\mathrm{Im} f \otimes_R M \longrightarrow 0,$$

which gives that

$$\mathrm{Im}(f \otimes 1_M) = \mathrm{Ker}(p \otimes 1_M) \cong \mathrm{Ker} p \otimes_R M = \mathrm{Im} f \otimes_R M.$$

(3) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of R-modules and R-homomorphisms such that $gf = 0$ (i.e. $\mathrm{Im} f \subseteq \mathrm{Ker} g$). Then $\mathrm{Im}(f \otimes 1_M) \subseteq \mathrm{Ker}(g \otimes 1_M)$ since $(g \otimes 1_M) \circ (f \otimes 1_M) = (g \circ f) \otimes 1_M = 0$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Im} f \otimes_R M & \longrightarrow & \mathrm{Ker} g \otimes_R M & \longrightarrow & \dfrac{\mathrm{Ker} g}{\mathrm{Im} f} \otimes_R M & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \vdots & & \\
0 & \longrightarrow & \mathrm{Im}(f \otimes 1_M) & \longrightarrow & \mathrm{Ker}(g \otimes 1_M) & \longrightarrow & \dfrac{\mathrm{Ker}(g \otimes 1_M)}{\mathrm{Im}(f \otimes 1_M)} & \longrightarrow & 0
\end{array}$$

whose rows are exact. It follows that

$$\frac{\mathrm{Ker} g}{\mathrm{Im} f} \otimes_R M \cong \frac{\mathrm{Ker}(g \otimes 1_M)}{\mathrm{Im}(f \otimes 1_M)}.$$

21. Theorem. Let R be a ring and M an R-module. Then the following conditions are equivalent:

(1) M is faithfully flat over R;

(2) M is flat over R and $M \otimes_R N \neq 0$ for any non-zero R-module N;

(3) M is flat over R and $\mathfrak{m}M \neq M$ for every maximal ideal $\mathfrak{m}$ of R.

proof.

(1) $\Rightarrow$ (2): Let $\mathcal{S}$ be the sequence $0 \to N \to 0$. If $N \otimes M = 0$, then $\mathcal{S} \otimes M$ is exact, which implies that $\mathcal{S}$ is exact, by assumption, and so $N = 0$.

(2) $\Rightarrow$ (3): This is clear from $M/\mathfrak{m}M \cong R/\mathfrak{m} \otimes_R M$.

(3) $\Rightarrow$ (2): Let N be a nonzero R-module. Choose $0 \neq x \in N$. We know that $Rx \cong R/\operatorname{ann}(x)$. Since $Rx \neq 0$, there exists a maximal ideal $\mathfrak{m}$ of R containing $\operatorname{ann}(x)$. It follows that $M \neq \mathfrak{m}M \supseteq \operatorname{ann}(x)M$, and hence $Rx \otimes_R M \cong M/\operatorname{ann}(x)M \neq 0$.

(2) $\Rightarrow$ (1): Consider a sequence of R-modules

$$\mathcal{S}: \quad N' \xrightarrow{f} N \xrightarrow{g} N''.$$

If

$$\mathcal{S} \otimes M: \quad N' \otimes_R M \xrightarrow{f_M} N \otimes_R M \xrightarrow{g_M} N'' \otimes_R M$$

is exact, then $g_M \circ f_M = (g \circ f)_M = 0$, so that by flatness,

$Im(g\circ f)\otimes M = Im(g_M\circ f_M) = 0$. By assumption, we have $Im(g\circ f)=0$, that is $g\circ f=0$; hence $Ker g \supseteq Im f$. If we set $H = Ker g / Im f$, then by flatness, $H\otimes M = Ker(g_M)/Im(f_M) = 0$, so that the assumption gives $H=0$. Therefore $\mathcal{S}$ is exact. $\square$

Let $f: R\to R'$ be a ring homomorphism. Then f induces a map ${}^a f: Spec(R')\longrightarrow Spec(R)$, under which a point $q\in Spec(R)$ has an inverse ${}^a f^{-1}(q) = \{P\in Spec(R') : f^{-1}(P)=q\}$. Set $T = R'\otimes_R \kappa(q)$, where $\kappa(q)$ is the residue field of q. Let $S = R\setminus q$, and define $g: R'\longrightarrow T$ by $g(x) = x\otimes 1$. Use extension and contraction notation with reference to the homomorphism f. Since $\kappa(q)\cong R/q\otimes_R R_S$ (note that we sometimes denote the ring $S^{-1}R$ by R_S especially when it is more convenient to do so), we have

$$T = R'\otimes_R \kappa(q) \cong R'\otimes_R R/q\otimes_R R_S \cong (R'/q^e)_{\bar f(S)},$$

where $\bar f: R\longrightarrow R'/q^e$ is the natural homomorphism induced by f. It follows that ${}^a g: Spec(T)\longrightarrow Spec(R')$ has the image $\{P\in Spec(R') : P\supseteq q^e \text{ and } P\cap f(S)=\phi\} = \{P\in Spec(R') : P^c = q\} = {}^a f^{-1}(q)$. This means that for any $P^*\in spec(T)$ with $P = g^{-1}(P^*)\in spec(R')$, then $P^c = q$.

22. Theorem. Let $f: R \to R'$ be a ring homomorphism and M an R'-module. Then

(i) If M is faithfully flat over R, then ${}^af(\mathrm{Supp}(M)) = \mathrm{Spec}(R)$.

(ii) If M is finitely generated over R', then M is R-flat and ${}^af(\mathrm{Supp}(M)) \supseteq \mathrm{Max}(R)$ if and only if M is faithfully flat over R.

proof. (i) Use contraction notation with reference to f.
Let $q \in \mathrm{Spec}(R)$. Since M is faithfully flat over R, we have $M \otimes_R \kappa(q) \neq 0$. Let T denote (as above) the ring $R' \otimes_R \kappa(q)$. Set $M' = M \otimes_R \kappa(q) \cong M \otimes_{R'} T$. Since the T-module $M' \neq 0$, there exists $P^* \in \mathrm{Spec}(T)$ such that $M'_{P^*} \neq 0$. Now set $P = g^{-1}(P^*)$, where $g: R' \to T$ is defined as above. Then

$$\begin{aligned} M'_{P^*} \cong M' \otimes_T T_{P^*} &\cong M \otimes_{R'} T \otimes_T T_{P^*} \\ &\cong M \otimes_{R'} T_{P^*} \\ &\cong M \otimes_{R'} (R'_P \otimes_{R'_P} T_{P^*}) \\ &\cong M_P \otimes_{R'_P} T_{P^*} \end{aligned}$$

so that $M_P \neq 0$, that $P \in \mathrm{Supp}(M)$. But $P^* \in \mathrm{Spec}(T)$, so that as we have seen before the theorem, $P^c = q$. Therefore $q \in {}^af(\mathrm{Supp}(M))$.

(ii) It is enough to show that $M/\mathfrak{m}M \neq 0$ for any maximal ideal $\mathfrak{M}$ of R. By assumption there is a prime ideal P of R' such that $M_P \neq 0$ and $P^c = f^{-1}(P) = \mathfrak{M}$. By Nakayama's Lemma, since M_P is finitely generated over R'_P, we have $M_P/PM_P \neq 0$. Then $M_P/\mathfrak{M}M_P \cong (M/\mathfrak{M}M)_P$ is non-zero, which implies that $M/\mathfrak{M}M \neq 0$.

$\square$

Let $(R, \mathfrak{M})$ and $(R', \mathfrak{M}')$ be local rings and let $f: R \longrightarrow R'$ be a ring homomorphism. We say that f is a local homomorphism if $f(\mathfrak{M}) \subseteq \mathfrak{M}'$. In this case, by Theorem 21, f is a flat homomorphism if and only if f is a faithfully flat homomorphism.

Let S be a multiplicatively closed subset of R. Then the map $\mathrm{Spec}(R_S) \longrightarrow \mathrm{Spec}(R)$ (defined in a natural way) is surjective only if S consists of units, that is $R = R_S$. Thus from the above theorem, if $R \neq R_S$, then R_S is flat but not faithfully flat over R.

<u>23. Theorem.</u> (i) Let R be a ring, M a flat R-module, and N_1, N_2 two submodules of an R-module N. Then as submodules of $N \otimes_R M$ we have

$$(N_1 \cap N_2) \otimes_R M = (N_1 \otimes_R M) \cap (N_2 \otimes_R M).$$

(ii) Let $R \to R'$ be a *flat* ring homomorphism, and let I_1 and I_2 be ideals of R. Then

$$(I_1 \cap I_2)R' = I_1R' \cap I_2R'.$$

(iii) If, in addition, I_2 is finitely generated, then

$$(I_1 : I_2)R' = (I_1R' : I_2R').$$

<u>proof</u> (i) Define $\varphi : N \to N/N_1 \oplus N/N_2$ by $\varphi(x) = (x+N_1, x+N_2)$. Then the sequence

$$0 \to N_1 \cap N_2 \to N \to N/N_1 \oplus N/N_2$$

is exact.

Now consider the inclusion maps $i_1 : N_1 \to N$ and $i_2 : N_2 \to N$. Since M is flat, the homomorphisms $i_1 \otimes 1_M$ and $i_2 \otimes 1_M$ are injections. We identity the images of these maps in $N \otimes_R M$ with $N_1 \otimes_R M$ and $N_2 \otimes_R M$, respectively. We also view $(N_1 \cap N_2) \otimes M$ in $N \otimes M$ in the same way. By Remark 20 (1), we have isomorphisms

$$\frac{N \otimes M}{N_1 \otimes M} \cong N/N_1 \otimes M \quad \text{and} \quad \frac{N \otimes M}{N_2 \otimes M} \cong N/N_2 \otimes M.$$

Now we have a commutative diagram

$$\begin{array}{ccccccc}
0 \to & (N_1 \cap N_2) \otimes M & \to & N \otimes M & \to & (N/N_1 \oplus N/N_2) \otimes M \\
& \| & & \| & & \downarrow \cong \\
0 \to & (N_1 \cap N_2) \otimes M & \to & N \otimes M & \to & \dfrac{N \otimes M}{N_1 \otimes M} \oplus \dfrac{N \otimes M}{N_2 \otimes M}
\end{array}$$

whose first row is exact, and then the second row of the diagram is also exact, which gives the assertion in (i).

(ii) This a particular case of (i) with $N=R$, $M=R'$, in view of the fact that for an ideal I of R the subset $I\otimes_R R'$ of $R\otimes_R R' = R'$ coincides with IR'.

(iii) If $I_2 = Ra_1 + \cdots + Ra_n$, then since $(I_1 : I_2) = \bigcap_i (I_1 : a_i)$, we can use (ii) to reduce to the case that I_2 is principal. For $a\in R$, we have the exact sequence

$$0 \longrightarrow (I_1 : Ra) \longrightarrow R \xrightarrow{a} R/I_1 ,$$

and tensoring this with R' gives the assertion.

$\square$

<u>Example.</u> Let k be a field, and consider the subring $R=k[X^2,X^3]$ of the polynomial ring $R'=k[X]$ in an indeterminate X. Then $X^2R\cap X^3R$ is the set of polynomials made up of terms of degree ≥ 5 in X, so that $(X^2R\cap X^3R)R' = X^5R'$. But on the other hand $X^2R'\cap X^3R' = X^3R'$. Therefore by the above theorem, R' is not flat over R.

<u>24. Theorem.</u> Let $f : R \longrightarrow R'$ be a faithfully flat ring hom.

(i) For any R-module M, the map $M \longrightarrow M\otimes_R R'$ defined by $m \longmapsto m\otimes 1$ is injective; in particular $f : R \longrightarrow R'$ is itself injective.

(ii) If I is an ideal of R, then $IR' \cap R = I$.
(Here we use $IR' \cap R$ to denote $f^{-1}(IR')$ considering that f is an embedding.)

proof. (i) Let $0 \neq m \in M$. Then $Rm \otimes_R R'$ is an R'-submodule of $M \otimes_R R'$ which can be identified with $(m \otimes 1)R'$. But by Theorem 21, $(Rm) \otimes_R R' \neq 0$, so that $m \otimes 1 \neq 0$.

(ii) follows by applying (i) to $M = R/I$, using $(R/I) \otimes R' \cong R'/IR'$.

$$\begin{tikzcd} 0 \arrow[r] & R/I \arrow[r, "g"] & R/I \otimes R' \cong R'/IR' \\ & R \arrow[u, "p"] \arrow[r, "f"] & R' \arrow[u, "p'"'] \\ 0 \arrow[r] & R \arrow[r, "f"] & R' \end{tikzcd}$$

$$\underbrace{\ker(g \circ p)}_{I} = \underbrace{\ker(p' \circ f)}_{f^{-1}(IR') = IR' \cap R}$$

25. Theorem. Let R be a ring and M a flat R-module. If $a_{ij} \in R$ and $x_j \in M$ (for $1 \leq i \leq r$ and $1 \leq j \leq n$) satisfying

$\sum_j a_{ij} x_j = 0$ for all i,

then there exist an integer s and $b_{jk} \in R$, $y_k \in M$ (for $1 \leq j \leq n$ and $1 \leq k \leq s$) such that

$\sum_j a_{ij} b_{jk} = 0$ for all i, k and $x_j = \sum_j b_{jk} y_k$ for all j.

Conversely, if the above conclusion holds for the case of a single equation (that is for $r = 1$), then M is flat.

proof. Set $\varphi : R^n \longrightarrow R^r$ for the linear map defined by the matrix (a_{ij}), and let $\varphi_M : M^n \longrightarrow M^r$ be the same thing for M. Then $\varphi_M = \varphi \otimes 1_M$.

$$\begin{tikzcd} R^n \otimes M \arrow[r, "\varphi\otimes 1_M"] \arrow[d, "\wr\|"] & R^r \otimes M \arrow[d, "\wr\|"] \\ M^n \arrow[r, "\varphi_M"'] & M^r \end{tikzcd}$$

Setting $K = \operatorname{Ker}\varphi$ and tensoring the exact sequence $K \xrightarrow{i} R^n \xrightarrow{\varphi} R^r$ with M, we get the exact sequence $K \otimes M \xrightarrow{i\otimes 1} M^n \xrightarrow{\varphi_M} M^r$. By assumption $\varphi_M(x_1, \dots, x_n) = 0$, so that we can write

$$(x_1, \dots, x_n) = (i \otimes 1)\left(\sum_{k=1}^{s} \beta_k \otimes y_k\right) \text{ with } \beta_k \in K \text{ and } y_k \in M.$$

If we write out β_k as an element of R^n in the form $\beta_k = (b_{1k}, \dots, b_{nk})$ with $b_{ik} \in R$, then the conclusion follows. The converse will be proved after the next theorem. □

<u>26. Theorem.</u> Let R be a ring and M an R-module. Then M is flat over R if and only if for every finitely generated ideal I of R, the canonical map $I \otimes_R M \longrightarrow R \otimes_R M$ is injective, and therefore $I \otimes M \cong IM$.

<u>proof</u>. The 'only if' part is obvious, and we prove the 'if' part. Firstly, every ideal of R is a direct limit of the finitely generated ideals contained in it. Since tensoring with a direct limit is a direct limit of tensors and an exact sequence of

direct systems remains exact after taking the direct limits, $I \otimes M \to M$ is injective for every ideal I. Moreover, if N is an R-module and $N' \subseteq N$ a submodule, then since N is the direct limit of modules of the form $N'+F$, with F finitely generated, to prove that $N' \otimes M \to N \otimes M$ is injective we can assume that $N = N' + Rn_1 + \cdots + Rn_t$. Then setting $N_i = N' + Rn_1 + \cdots + Rn_i$ (for $1 \le i \le t$), we need only show that each step in the chain

$$N' \otimes M \to N_1 \otimes M \to N_2 \otimes M \to \cdots \to N \otimes M$$

is injective, and finally that if $N = N' + Rn$, then $N' \otimes M \to N \otimes M$ is injective. Now set $I = (N' : n)$, and get the exact sequence

$$0 \to N' \to N \to R/I \to 0.$$

This induces a long exact sequence

$$\cdots \to \operatorname{Tor}_1^R(R/I, M) \to N' \otimes M \to N \otimes M \to R/I \otimes M \to 0;$$

hence it is enough to prove that $\operatorname{Tor}_1^R(R/I, M) = 0$. For this, consider the short exact sequence

$$0 \to I \to R \to R/I \to 0$$

and the induced long exact sequence

$$\operatorname{Tor}_1^R(R, M) \to \operatorname{Tor}_1^R(R/I, M) \to I \otimes M \to M \to \cdots.$$

Since $I \otimes M \to M$ is injective and $\operatorname{Tor}_1^R(R, M) = 0$, we must have $\operatorname{Tor}_1^R(R/I, M) = 0$. $\square$

By Theorem 26, we can prove the converse of Theorem 25. Indeed, if $I = Ra_1 + \cdots + Ra_n$ is a finitely generated ideal of R, then an element $\tau \in I \otimes M$ can be written as $\tau = \sum_{i=1}^{n} a_i \otimes m_i$ with $m_i \in M$. Suppose that $\sum_{i=1}^{n} a_i m_i = 0$. Now if the conclusion of Theorem 25 holds for M, there exist $b_{ij} \in R$ and $y_j \in M$ such that

$$\sum_i a_i b_{ij} = 0 \text{ for all } j, \text{ and } m_i = \sum b_{ij} y_j \text{ for all } i.$$

Then $\tau = \sum a_i \otimes m_i = \sum_i \sum_j a_i b_{ij} \otimes y_j = \sum_j \left(\sum_i a_i b_{ij} \otimes y_j \right) = 0$, so that $I \otimes M \to M$ is injective, and therefore M is flat.

<u>27. Theorem</u>. Let R be a ring and M an R-module. The following conditions are equivalent:

(1) M is flat;

(2) for every R-module N, we have $\mathrm{Tor}_1^R(M, N) = 0$;

(3) $\mathrm{Tor}_1^R(M, R/I) = 0$ for every finitely generated ideal I.

<u>proof</u>. (1) $\Rightarrow$ (2): If we let

$$\cdots \to L_i \to L_{i-1} \to \cdots \to L_0 \to N \to 0$$

be a projective resolution of N, then

$$\cdots \to L_i \otimes M \to L_{i-1} \otimes M \to \cdots \to L_0 \otimes M$$

is exact, so that $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$.

(2) $\Rightarrow$ (3): obvious.

(3) $\Rightarrow$ (1): The short exact sequence $0 \to I \to R \to R/I \to 0$

induces a long exact sequence

$$\underbrace{\mathrm{Tor}_1^R(M, R/I)}_{0} \longrightarrow M \otimes I \longrightarrow M \longrightarrow M \otimes R/I \longrightarrow 0$$

and hence $I \otimes M \longrightarrow M$ is injective. Therefore by the previous theorem, M is flat. $\square$

<u>28. Theorem.</u> Let $(R, \mathfrak{m})$ be a local ring and M a flat R-module. If $x_1, \dots, x_n \in M$ are such that their images $\bar{x}_1, \dots, \bar{x}_n \in \bar{M} = M/\mathfrak{m}M$ are linearly independent over the field $R/\mathfrak{m}$, then $x_1, \dots, x_n$ are linearly independent over R.

<u>proof.</u> We use induction on n. let $n=1$ and $a_1 x_1 = 0$. Since M is flat, by Theorem 25, there exist $b_i \in R$ and $y_i \in M$ $(1 \le i \le s)$ such that $a_1 b_1 = \dots = a_1 b_s = 0$ and $b_1 y_1 + \dots + b_s y_s = x_1$. Since, by assumption, $x_1 \notin \mathfrak{m}M$, $b_j \notin \mathfrak{m}$ for some $j = 1, \dots, s$. Now b_j is a unit of R and so $a_1 = 0$.

Now assume that $n > 1$ and that the assertion is true for all positive integers smaller than n. let

$$a_1 x_1 + \dots + a_n x_n = 0.$$

By Theorem 25, there exist $b_{ij} \in R$ and $y_j \in M$ $(i = 1, \dots, n$ and $j = 1, \dots, s)$ such that $a_1 b_{1j} + \dots + a_n b_{nj} = 0$ for all j and $b_{i1} y_1 + \dots + b_{is} y_s = x_i$ for all i. Since $x_n \notin \mathfrak{m}M$, there exists

$j=1,\dots,s$ such that $b_{nj} \notin \mathfrak{M}$ (i.e. b_{nj} is a unit of R). Thus a_n is a linear combination of $a_1,\dots,a_{n-1}$ over R. Let $a_n = \sum_{i=1}^{n-1} c_i a_i$ for some $c_1,\dots,c_{n-1} \in R$. This gives that

$$a_1(x_1+c_1x_n)+a_2(x_2+c_2x_n)+\cdots+a_{n-1}(x_{n-1}+c_{n-1}x_n)=0.$$

It is easy to see that $\overline{x_1+c_1x_n},\dots,\overline{x_{n-1}+c_{n-1}x_n}$ are linearly independent over $R/\mathfrak{M}$. By induction, we have $a_1=\cdots=a_{n-1}=0$ and so $a_n=0$. This completes the proof. $\square$

29. Corollary. Let $(R,\mathfrak{M})$ be a local ring and M a finitely generated R-module. Then the following are equivalent:

(1) M is a flat R-module;

(2) M is a projective R-module;

(3) M is a free R-module.

proof. Exercise!

1. Exercise. Prove Corollary 29.

We say that a module M over a ring R is finitely presented if there exists an exact sequence

$$R^k \to R^n \to M \to 0.$$

30. Theorem. Let R be a ring, M and N two R-modules and R' a flat R-algebra. If M is finitely presented, then we have

$$\mathrm{Hom}_R(M,N)\otimes_R R' \cong \mathrm{Hom}_{R'}(M\otimes_R R', N\otimes_R R').$$

proof. Fixing N and B, we define contravariant functors F and G of an R-module M by

$$F(M) = \mathrm{Hom}_R(M,N)\otimes_R R'$$

and

$$G(M) = \mathrm{Hom}_{R'}(M\otimes_R R', N\otimes_R R');$$

then we can define a morphism of functors $\lambda : F \to G$ by

$$\lambda(f\otimes b) = b\cdot(f\otimes 1_B) \text{ for } f\in \mathrm{Hom}_R(M,N) \text{ and } b\in B.$$

Both F and G are left-exact functors.

Now if M is finitely presented there is an exact sequence of the form $R^k \to R^n \to M \to 0$, and from this we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \to & F(M) & \longrightarrow & F(R^n) & \longrightarrow & F(R^k) \\ & & \lambda\downarrow & & \lambda\downarrow & & \lambda\downarrow \\ 0 & \to & G(M) & \longrightarrow & G(R^n) & \longrightarrow & G(R^k) \end{array}$$

having two exact rows. Now $F(R^k) = N^k\otimes R'$ and $G(R^k) = (N\otimes B)^k$, the right-hand λ is an isomorphism, and similarly the middle λ is an isomorphism. Thus as one sees easily, the left-hand λ is also an isomorphism.

31. Corollary. Let R, M and N be as in Theorem 30, and let P be a prime ideal of R. Then

$$\mathrm{Hom}_R(M,N)\otimes_R R_P \cong \mathrm{Hom}_{R_P}(M_P,N_P).$$

32. Theorem. Let R be a ring and M a finitely presented R-module. Then M is a projective R-module if and only if $M_{\mathfrak{M}}$ is a free $R_{\mathfrak{M}}$-module for every maximal ideal $\mathfrak{M}$ of R.

proof. Assume first that M is projective. Then it is direct summand of a free R-module and this property is preserved by localization. Now use Corollary 29.

Next, assume that $M_{\mathfrak{M}}$ is free over $R_{\mathfrak{M}}$ for every maximal ideal $\mathfrak{M}$ of R. Let $N_1 \to N_2 \to 0$ be an exact sequence of R-modules. Write C for the cokernel of

$$\mathrm{Hom}_R(M,N_1) \longrightarrow \mathrm{Hom}_R(M,N_2).$$

Then for any maximal ideal $\mathfrak{M}$ of R we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(M,N_1)\otimes R_{\mathfrak{M}} \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_R(M,N_2)\otimes R_{\mathfrak{M}} \arrow[r] \arrow[d, "\cong"'] & C\otimes R_{\mathfrak{M}} \arrow[r] \arrow[d, "\cong"'] & 0 \\
\mathrm{Hom}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}},(N_1)_{\mathfrak{M}}) \arrow[r] & \mathrm{Hom}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}},(N_2)_{\mathfrak{M}}) \arrow[r] & C_{\mathfrak{M}} \arrow[r] & 0
\end{tikzcd}$$

of $R_{\mathfrak{M}}$-modules and $R_{\mathfrak{M}}$-homomorphism, in which the exactness of the first row implies that of the second row. It follows that

$$C_{\mathfrak{M}} \cong \mathrm{Coker}\{\mathrm{Hom}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}}, (N_1)_{\mathfrak{M}}) \longrightarrow \mathrm{Hom}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}}, (N_2)_{\mathfrak{M}})\} = 0.$$

Hence $C = 0$.

33. Corollary. If R is a ring and M is a finitely presented R-module, then M is flat if and only if it is projective.

proof. We already know that every projective module is flat. The converse follows from Theorems 19, 32 and Corollary 29.