2. Completion and Artin-Rees Lemma

Reminder:

Let X be any nonempty set. Suppose that for every $x \in X$, we have a system $\mathcal{B}_x$ of subsets of X such that

(i) for every $B \in \mathcal{B}_x$, $x \in B$,

(ii) for every pair B_1 and B_2 of elements of $\mathcal{B}_x$, there exists $B_3 \in \mathcal{B}_x$ such that $B_3 \subseteq B_1 \cap B_2$.

If τ is the collection of subsets U of X with the property that for each $u \in U$, there exists $B \in \mathcal{B}_u$ such that $B \subseteq U$, then τ is a topology on X. With respect to this topology the elements of $\mathcal{B}_x$ for each $x \in X$ are all open subsets of X. The systems $\mathcal{B}_x$ are called systems of neighborhoods of $x \in X$.

Let R be a ring, and M an R-module. Suppose that $\mathcal{F} = \{M_\lambda\}_{\lambda \in \Lambda}$ is a family of submodules of M, where Λ is a directed set such that $\lambda < \mu$ implies $M_\mu \subset M_\lambda$. Then taking $\mathcal{F}$ as a system of neighborhoods of 0 we form a topology on M for which the open subsets of M are those subsets A of M satisfying the property that for any $x \in A$, there exists $\lambda \in \Lambda$ such that $x + M_\lambda \subseteq A$. With respect to this topology, addition and subtraction in M are continuous. When $M = R$, each M_λ is an ideal, so that multiplication is also continuous since

$$(a + M_\lambda)(b + M_\lambda) = ab + M_\lambda$$

for any $a, b \in M$ and $\lambda \in \Lambda$. To see this let U be an open subset of M containing ab. Then there exist $\lambda \in \Lambda$ such that $ab + M_\lambda$. Now set $V = (a + M_\lambda) \times (b + M_\lambda)$, an open subset of $M \times M$. Then V contains $(a, b) \in M \times M$ and is mapped into U by multiplication of M. This type of topology is called a <u>linear</u> <u>topology</u> on M. The linear topology, defined by $\mathcal{F}$, is Hausdorff if and only if $\bigcap_{\lambda \in \Lambda} M_\lambda = 0$. Since each coset $x + M_\lambda \subset M$ is an open set and $M \setminus M_\lambda$ is a union of cosets, the subset $M \setminus M_\lambda$ (for any $\lambda \in \Lambda$) is both an open and closed subset of M. Also, the quotient module M/M_λ is discrete in the quotient topology.

$M/\bigcap_\lambda M_\lambda$ is called the <u>separated</u> <u>module</u> <u>associated</u> with M. Moreover, since for $\lambda < \mu$ there is a natural linear map $\varphi_{\lambda\mu} : M/M_\mu \to M/M_\lambda$, we can construct the inverse system $\{M/M_\lambda ; \varphi_{\lambda\mu}\}$ of R-modules; its inverse limit $\varprojlim M/M_\lambda$ is called the <u>completion</u> of M, and is written $\hat{M}$. We give each M/M_λ the discrete topology and $\prod_\lambda M/M_\lambda$ the product topology, and $\hat{M}$ the subspace topology in $\prod_\lambda M/M_\lambda$.

Reminder

Definition Product Topology (Source: Wikipedia)

Given X such that

$$X := \prod_{i\in I} X_i,$$

is the Cartesian product of the topological spaces X_i, indexed by $i \in I$, and the **canonical projections** $p_i : X \to X_i$, the **product topology** on X is defined to be the coarsest topology (i.e. the topology with the fewest open sets) for which all the projections p_i are continuous. The product topology is sometimes called the **Tychonoff topology**.

The open sets in the product topology are unions (finite or infinite) of sets of the form $\prod_{i\in I} U_i$, where each U_i is open in X_i and $U_i \neq X_i$ for only finitely many i. In particular, for a finite product (in particular, for the product of two topological spaces), the products of base elements of the X_i gives a basis for the product $\prod_{i\in I} X_i$.

The product topology on X is the topology generated by sets of the form $p_i^{-1}(U)$, where i is in I and U is an open subset of X_i. In other words, the sets $\{p_i^{-1}(U)\}$ form a subbase for the topology on X. A subset of X is open if and only if it is a (possibly infinite) union of intersections of finitely many sets of the form $p_i^{-1}(U)$. The $p_i^{-1}(U)$ are sometimes called open cylinders, and their intersections are cylinder sets.

In general, the product of the topologies of each X_i forms a basis for what is called the box topology on X. In general, the box topology is finer than the product topology, but for finite products they coincide.

Let $\psi : M \to \hat{M}$ be the natural R-linear map; i.e. $\psi(m) = (m + M_\lambda)_{\lambda\in\Lambda}$ for every $m \in M$.

<u>2. Exercise</u> Show that the natural R-linear map $\psi : M \to \hat{M}$ is continuous. Also show that $\psi(M)$ is dense in $\hat{M}$.

Write $p_\lambda : \hat{M} \to M/M_\lambda$ for the projection, and set $\operatorname{Ker} p_\lambda = M_\lambda^*$.

<u>3. Exercise.</u> Prove that the topology on $\hat{M}$ coincides with the linear topology defined by $\{M_\lambda^*\}_{\lambda\in\Lambda}$.

Since $p_\lambda(\psi(M)) = M/M_\lambda$, p_λ is surjective; so that

$$\hat{M}/M_\lambda^* \cong M/M_\lambda,$$

and the completion of $\hat{M}$ coincides with $\hat{M}$ itself. If $\psi: M \to \hat{M}$ is an isomorphism, we say that M is <u>complete</u>.

If $\mathcal{F}' = \{M'_\gamma\}_{\gamma\in\Gamma}$ is another family of submodules of M indexed by a directed set Γ, then $\mathcal{F}$ and $\mathcal{F}'$ give the same topology on M if and only if for each M_λ, there exists a $\gamma\in\Gamma$ such that $M'_\gamma \subseteq M_\lambda$, and for every M'_γ there exists a $\mu\in\Lambda$ such that $M_\mu \subseteq M'_\gamma$. It is then easy to see that there is an isomorphism of topological modules $\varprojlim M/M_\lambda \cong \varprojlim M/M'_\gamma$. Thus $\hat{M}$ depends only on the topology of M, as does the question of whether M is complete.

When $M = R$, $\{M/M_\lambda \,;\, \varphi_{\lambda\mu}\}$ becomes an inverse system of rings, $\hat{M} = \hat{R}$ is a ring, and $\psi: R \to \hat{R}$ a ring homomorphism. $M_\lambda^* \subseteq \hat{R}$ is not just an R-submodule, but an ideal of $\hat{R}$; this is clear from the fact that $p_\lambda : \hat{R} \to R/M_\lambda$ is a ring homomorphism.

If $N \subseteq M$ is a submodule, then the closure $\bar{N}$ of N in M is given by the following formula:

$$\bar{N} = \bigcap_{\lambda\in\Lambda} (N + M_\lambda).$$

Indeed,

$$x \in \bar{N} \Leftrightarrow (x+M_\lambda) \cap N \neq \phi \text{ for all } \lambda$$
$$\Leftrightarrow x \in N+M_\lambda \text{ for all } \lambda.$$

If we write M'_λ for the image of M_λ in the quotient module M/N, the quotient topology of M/N is just the linear topology defined by $\{M'_\lambda\}_{\lambda\in\Lambda}$. In fact, let $G \subseteq M$ be the inverse image of $G' \subseteq M/N$. Then

G' is open in the quotient topology of M/N

$\Leftrightarrow$ G is open in M

$\Leftrightarrow$ for every $x \in G$, there is an M_λ such that $x+M_\lambda \subseteq G$

$\Leftrightarrow$ for every $x' \in G'$, there is an M'_λ such that $x'+M'_\lambda \subseteq G'$.

Hence the condition for M/N to be separated is that $\bigcap_{\lambda\in\Lambda} M'_\lambda = 0$, that is $\bigcap_{\lambda\in\Lambda} N+M_\lambda = N$, or in other words, that N is closed in M. Moreover, the subspace topology of N is clearly the same thing as the linear topology defined by $\{N \cap M_\lambda\}_{\lambda\in\Lambda}$. Set $M/N = M'$. Then

$$0 \to N/(N\cap M_\lambda) \to M/M_\lambda \to M/(N+M_\lambda) \cong M'/M'_\lambda \to 0$$

is an exact sequence, so that taking the inverse limit, we see that

$$0 \to \hat{N} \to \hat{M} \to (M/N)^\wedge$$

is exact. If we view $\hat{N}$ as a submodule of $\hat{M}$, the condition that $x = (x_\lambda + M_\lambda)_{\lambda\in\Lambda} \in \hat{M}$ belongs to $\hat{N}$ is that each $x_\lambda + M_\lambda$ can be represented by an element of N, or in other words, that $x \in \psi(N) + M_\lambda^*$ for each λ. Thus $\hat{N}$ is the same thing as the closure of $\psi(N)$ in $\hat{M}$. In general it is not clear whether $\hat{M} \to (M/N)^\wedge$ is surjective but this holds when $\Lambda = \{1, 2, \dots\}$. In fact, then

$$(M/N)^\wedge = \varprojlim (M/N+M_n) \, ;$$

given an element $x' = (x'_n + (N+M_n))_{n=1}^{\infty} \in (M/N)^\wedge$, we have $x'_2 - x'_1 \in N + M_1$; so that we can write

$$x'_2 - x'_1 = t + m_1$$

for some $t \in N$ and $m_1 \in M_1$. Set $x_2 = x'_2 - t$ and $x_1 = x'_1$. Then $x_2 - x_1 \in M_1$ and $x_2 + (N+M_2) = x'_2 + (N+M_2)$. By induction we can find $x_1, x_2, x_3, \dots \in M$ such that $x_{n+1} - x_n \in M_n$ and $x_n + (N+M_n) = x'_n + (N+M_n)$ for every $n = 1, 2, \dots$. This gives us an element $(x_n + M_n)_{n=1}^{\infty} \in \hat{M}$ which maps to the element x' in $(M/N)^\wedge$. This proves the following

34. Theorem. Let R be a ring, M an R-module with a linear topology, and N a submodule of M. We give N the subspace topology and M/N the quotient topology. Then these are both linear topologies, and we have:

(i) $0 \to \hat{N} \to \hat{M} \to (M/N)^\wedge$ is an exact sequence, and $\hat{N}$ is the closure of $\psi(N)$ in $\hat{M}$, where $\psi: M \to \hat{M}$ is the natural map.

(ii) If, moreover, the topology of M is defined by a decreasing chain of submodules $M_1 \supset M_2 \supset \dots$, then

$$0 \longrightarrow \hat{N} \longrightarrow \hat{M} \longrightarrow (M/N)^\wedge \longrightarrow 0$$

is exact. In other words, $(M/N)^\wedge \cong \hat{M}/\hat{N}$.

Suppose now that M and N are two R-modules with linear topologies, and let $f: M \longrightarrow N$ be a continuous linear map. If the topologies of M and N are given by $\{M_\lambda\}_{\lambda\in\Lambda}$ and $\{N_\gamma\}_{\gamma\in\Gamma}$, then for any $\gamma\in\Gamma$ there exists $\lambda\in\Lambda$ such that $M_\lambda \subseteq f^{-1}(N_\gamma)$. Define φ_λ to be the composite

$$\varphi_\gamma : \hat{M} \longrightarrow \hat{M}/M_\lambda^* \cong M/M_\lambda \longrightarrow N/N_\gamma,$$

where the first arrow is the natural map, and the second is induced by f. One sees at once that φ_γ does not depend on the choice of λ for which $M_\lambda \subseteq f^{-1}(N_\gamma)$. Also, for $\gamma<\gamma'$ if we let $\psi_{\gamma\gamma'}$ denote the natural map $N/N_{\gamma'} \longrightarrow N/N_\gamma$, it is easy to see that $\varphi_\gamma = \psi_{\gamma\gamma'} \circ \varphi_{\gamma'}$, i.e. the diagram

$$\begin{tikzcd}
\hat{M} \arrow[r, "\varphi_\gamma"] \arrow[d, "\varphi_{\gamma'}"'] & N/N_\gamma \\
N/N_{\gamma'} \arrow[ur, "\psi_{\gamma\gamma'}"'] &
\end{tikzcd}$$

is commutative. Hence there is a continuous linear map $\hat{f}: \hat{M} \to \hat{N}$ defined by the $(\varphi_\gamma)_{\gamma\in\Gamma}$, and the following diagram is commutative (the vertical arrows are the natural maps):

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ \hat{M} & \xrightarrow[\hat{f}]{} & \hat{N} \end{array}$$

Similarly, if R and R' are rings with linear topologies, and $f: R \to R'$ is a continuous ring homomorphism, then f induces a continuous ring homomorphism $\hat{f}: \hat{R} \to \hat{R}'$.

Let I be an ideal of R and M an R-module. The topology on M defined by $\{I^n M\}_{n=1}^{\infty}$ is called the <u>I-adic topology</u>. If we also give R the I-adic topology, the completions $\hat{R}$ and $\hat{M}$ of R and M are called <u>I-adic completions</u>. It is easy to see that $\hat{M}$ is an $\hat{R}$-module: for $\alpha = (r_1, r_2, \ldots) \in \hat{R}$ with $r_n \in R/I^n$ and $x = (x_1, x_2, \ldots) \in \hat{M}$ with $x_n \in M/I^n M$ (for all n), we can just set

$$ax = (r_1 x_1, r_2 x_2, \ldots) \in \hat{M}.$$

One can easily show that M is complete for the I-adic topology if and only if for every sequence $x_1, x_2, \ldots$ of elements of M satisfying $x_i - x_{i+1} \in I^i M$ for all i, there exists a unique $x \in M$ such that $x - x_i \in I^i M$ for all i. We shall say that a sequence $\{x_n\}$ in M "converges to x" or "has x as a limit" (with respect to the I-adic topology) if the following condition is satisfied: given any integer $k \geq 0$, there always exists a positive integer N such that $x - x_n \in I^k M$ whenever $n > N$. In these cases we often write $x_n \to x$.

4. Exercise. Let I be an ideal of R and M an R-module. Suppose that $x_n \to x$ and $y_n \to y$ in M with respect to the I-adic topology.

(1) Show that when the I-adic topology on M is Hausdorff, limits are unique.

(2) Prove that

$$(x_n + y_n) \to x + y$$
$$(x_n - y_n) \to x - y$$
$$r x_n \to r x$$

for every $r \in R$.

(3) Suppose that $f : M \to M'$ is a continuous mapping

into another R-module M' (not necessarily a homomorphism). Show that if a sequence $\{x_n\}$ in M converges to x, then the sequence $\{f(x_n)\}$ in M' converges to $f(x)$, i.e.,

$$x_n \longrightarrow x \quad \text{implies} \quad f(x_n) \longrightarrow f(x).$$

We say that a sequence $\{x_n\}$ in M is a <u>Cauchy sequence</u> provided that for every positive integer k, there exist N such that $x_{n+1} - x_n \in I^k M$ whenever $n > N$. Now completeness can be expressed as saying that every Cauchy sequence has a unique limit.

Let R be a ring and I an ideal of R. Let $a \in I$. Set $a_n = 1 - a + a^2 - a^3 + \cdots + (-1)^{n-1} a^{n-1} = \sum_{i=0}^{n-1} (-1)^i a^i$. Then $a_{n+1} - a_n = (-1)^n a^n \in I^n$; so that $(a_n + I^n)_{n=1}^{\infty} \in \hat{R}$. If, further, R is I-adically complete there exists a <u>unique</u> $b \in R$ such that $b - a_n \in I^n$ for every $n > 0$. On the other hand, since $a_{n+1} = 1 - a a_n$ for each $n > 0$, we have

$$\begin{aligned} (1-ab) - a_n &= 1 - a a_n + a a_n - ab - a_n \\ &= (a_{n+1} - a_n) + a(a_n - b) \in I^n \end{aligned}$$

for each $n > 0$. Since $b \in R$ is unique with this property, we must have $1 - ab = b$; so that $(1+a)b = 1$. In such a case, we say that the series

$$1 - a + a^2 - a^3 + \cdots = \sum_{i=0}^{\infty} (-1)^n a^i$$

converges in R to b and b is an inverse of $1+a$. Hence $1+a$ is a unit of R. This means that $I \subseteq J(R)$, where $J(R)$ is the Jacobson radical of R. This establishes the first part of the following

<u>35. Theorem.</u> Let R be a ring, I an ideal and M an R-module.

(i) If R is I-adically complete, then $I \subseteq J(R)$;

(ii) If M is I-adically complete and $a \in I$, then multiplication by $1+a$ is an automorphism of M.

<u>proof</u>. We need to prove only part (ii). Since $M \cong \hat{M}$, M is also an $\hat{R}$-module. Note that the image of $1+a$ in $\hat{R}$ is a unit in $\hat{R}$ (why?). Since the multiplication on M by $1+a$ is the same thing as the multiplication on M by the image of $1+a$ in $\hat{R}$, we are done. $\square$

<u>36. Theorem.</u> (Hensel's lemma) Let $(R, \mathfrak{m})$ be a local ring with the residue field k, and suppose that R is $\mathfrak{m}$-adically complete. Let $F(x) \in R[X]$ be a monic polynomial, and let $\bar{F} \in k[X]$ be the polynomial obtained by reducing the coefficients of F modulo $\mathfrak{m}$. If there are monic polynomials $g, h \in k[X]$ with $(g,h)=1$ and such that $\bar{F} = gh$, then there exist monic polynomials $G, H \in R[X]$

such that $F = GH$, $\bar{G} = g$ and $\bar{H} = h$.

<u>proof</u>. We can take polynomials $G_1, H_1 \in R[X]$ such that $g = \bar{G}_1$, $h = \bar{H}_1$. Then $F \equiv G_1 H_1 \mod \mathfrak{m}[X]$. Suppose by induction that monic polynomials G_n, H_n have been constructed such that $F \equiv G_n H_n \mod \mathfrak{m}^n[X]$, and $\bar{G}_n = g$, $\bar{H}_n = h$. Then we can write

$$F - G_n H_n = \sum a_i P_i(X), \text{ with } a_i \in \mathfrak{m}^n \text{ and } \deg P_i < \deg F.$$

Since $(g,h) = 1$ we can find $v_i, w_i \in k[X]$ such that $\bar{P}_i = g v_i + h w_i$. Replacing v_i by its remainder modulo h, and making the corresponding correction to w_i we can assume $\deg v_i < \deg h$. Then

$$\deg h w_i = \deg(\bar{P}_i - g v_i) < \deg F.$$

It follows that $\deg w_i < \deg g$. Choose $V_i, W_i \in R[X]$ such that $\bar{V}_i = v_i$, $\deg V_i = \deg v_i$, $\bar{W}_i = w_i$, $\deg W_i = \deg w_i$, and set $G_{n+1} = G_n + \sum a_i W_i$, $H_{n+1} = H_n + \sum a_i V_i$. Then

$$F - G_{n+1} H_{n+1} = F - G_n H_n - G_n \sum a_i V_i - H_n \sum a_i W_i - (\sum a_i W_i)(\sum a_i V_i)$$
$$= \sum a_i P_i - \sum a_i (G_n V_i + H_n W_i) - (\sum a_i W_i)(\sum a_i V_i)$$

Since

$$(G_n V_i + H_n W_i) \equiv (g v_i + h w_i) \mod \mathfrak{m}[X]$$
$$\equiv P_i \mod \mathfrak{m}[X]$$

we have

$$a_i(G_nV_i + H_nW_i) \equiv a_iP_i \mod \mathcal{M}^{n+1}[X]$$

for every i; so that

$$F \equiv G_{n+1}H_{n+1} \mod \mathcal{M}^{n+1}[X].$$

We construct in this way sequences of polynomials G_n, H_n for $n = 1, 2, \ldots$. Since all the G_n's (resp. H_n's) have the same degree and R is $\mathcal{M}$-adically complete, $\lim G_n = G$ (resp. $\lim H_n = H$) exists. Since R is $\mathcal{M}$-adically complete

$$\bigcap_{n\geq 1} \mathcal{M}^n = 0;$$

so that

$$\bigcap_{n\geq 1} \mathcal{M}^n[X] = 0$$

in $R[X]$. On the other hand given any $n > 0$,

$$\begin{aligned} F - GH &= (F - G_kH_k) + (G_kH_k - GH) \\ &= (F - G_kH_k) + (G_kH_k - GH_k + GH_k - GH) \\ &= (F - G_kH_k) + [(G_k - G)H_k + G(H_k - H)] \in \mathcal{M}^n[X] \end{aligned}$$

for all sufficiently large k.

It follows that $F = GH$. Obviousy,

$$\overline{G} = \overline{G}_1 = g, \quad \overline{H} = \overline{H}_1 = h.$$

This completes the proof. $\square$

37. Theorem. Let R be a ring, I an ideal, and M an R-module. Suppose that R is I-adically complete, and M is separated for the I-adic topology. If M/IM is generated over R/I by $m_1+IM, \ldots, m_n+IM$ for some $m_1, \ldots, m_n \in M$, then M is generated over R by $m_1, \ldots, m_n$.

proof. By assumption $M = \sum_{i=1}^{n} Rm_i + IM$, so that $IM = I(\sum Rm_i + IM) = \sum Im_i + I^2M$, and similarly $I^kM = \sum I^k m_i + I^{k+1}M$ for all $k>0$. For any $u \in M$, write $u = \sum_{i=1}^{n} a_i m_i + u_1$ for some $a_i \in R$ and $u_1 \in IM$. Then $u_1 = \sum a_{i1} m_i + u_2$ for some $a_{i1} \in I$ and $u_2 \in I^2M$. Now choose successively $a_{ik} \in I^k$ and $u_k \in I^kM$ such that

$$u_k = \sum a_{ik} m_i + u_{k+1} \text{ for } k=1,2,\ldots.$$

Then $a_i + a_{i1} + a_{i2} + \cdots$ converges in R. If we set b_i for this sum then

$$u - \sum_{i=1}^{n} b_i m_i \in \bigcap_{k>0} I^kM = 0.$$

$\square$

38. Theorem. (the Artin-Rees lemma) Let R be a Noetherian ring, M a finitely generated R-module, $N \subseteq M$ a submodule, and I an ideal of R. Then there exists a positive integer k such that for every $n>k$, we have $I^nM \cap N = I^{n-k}(I^kM \cap N)$.

proof. It is clear that

$$I^nM \cap N \supseteq I^{n-k}(I^kM \cap N).$$

Suppose that I is generated by $a_1, \ldots, a_r \in R$, and M by $m_1, \ldots, m_s$. An element of I^nM can be written as $\sum_{i=1}^{s} f_i(a)m_i$, where $f_i(x) = f_i(X_1, \ldots, X_r)$ is a homogeneous polynomial of degree n with coefficients in R and $a = (a_1, \ldots, a_r)$. Now set $R_1 = R[X_1, \ldots, X_r]$, and for each $n > 0$ set

$$J_n = \left\{ (f_1, \ldots, f_s) \in R_1^s : \begin{array}{l} f_i \text{ are homogeneous of degree } n \\ \text{and } \sum_{i=1}^{s} f_i(a)m_i \in N \end{array} \right\}.$$

Let C be the R_1-submodule of R_1^s generated by $\bigcup_{n>0} J_n$. Since R_1 is Noetherian, C is a finitely generated R_1-module, so that $C = \sum_{j=1}^{t} R_1 u_j$, where each u_j is a linear combination of elements of $\bigcup J_n$. Therefore C is generated by finitely many elements of $\bigcup J_n$. Suppose

$$C = R_1 u_1 + \cdots + R_1 u_t, \text{ where } u_j = (u_{j1}, \ldots, u_{js}) \in J_{d_j}$$

for $1 \le j \le t$. Set $k = \max\{d_1, \ldots, d_t\}$. Now if $\alpha \in I^nM \cap N$, then we can write $\alpha = \sum f_i(a)m_i$ with $(f_1, \ldots, f_s) \in J_n$, and hence

$$(f_1, \ldots, f_s) = \sum p_j(x) u_j, \text{ with } p_j \in R_1.$$

The left hand side is a vector made up of homogeneous polynomials of degree n only, so that the terms of degree other than n on the right hand side must cancel out to give 0. Hence we can suppose that the $p_j(X)$ are homogeneous of degree $n-d_j$. Then

$$\alpha = \sum f_i(a)m_i = \sum_j p_j(a) \sum_i u_{ji}(a) m_i,$$

and

$$\sum_i u_{ji}(a)m_i \in I^{d_j}M \cap N$$

so that if $n>k$, then

$$p_j(a) \in I^{n-d_j} = \underset{a_j}{I^{n-k}} \cdot \underset{b_j}{I^{k-d_j}}$$

which gives that

$$\alpha \in I^{n-k}(I^k M \cap N)$$

for any $n>k$. □

<u>39. Theorem.</u> In the notation of above theorem, the I-adic topology of N coincides with the topology induced by the I-adic topology of M on the subspace $N \subseteq M$.

<u>proof.</u> By the preceding theorem, for $n>k$, we have $I^nN \subseteq I^nM \cap N \subseteq I^{n-k}N$. The topology of N as a subspace of M is the linear topology defined by $\{I^nM \cap N\}_{n=1}^{\infty}$, and the above formula says that this

defines the same topology as $\{I^n N\}_{n=1}^{\infty}$.

$\square$

<u>40. Theorem.</u> Let R be a Noetherian ring, I an ideal, and M a finite R-module. Writing $\hat{M}$, $\hat{R}$ for the I-adic completions of M and R we have

$$M \otimes_R \hat{R} \cong \hat{M}.$$

<u>proof</u>. By Theorems 34 and 39, the I-adic completion of an exact sequence of finitely generated R-modules is again exact. Now given M, let $R^p \to R^q \to M \to 0$ be an exact sequence. The commutative diagram

$$\begin{tikzcd}
\widehat{R^p} \arrow[r] & \widehat{R^q} \arrow[r] & \hat{M} \arrow[r] & 0 \\
R^p \otimes \hat{R} \arrow[r] \arrow[u] & R^q \otimes \hat{R} \arrow[r] \arrow[u] & M \otimes \hat{R} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

has exact rows. Here the vertical arrows are natural maps. Since completion commutes with direct sums, the two left-hand arrows are obviously isomorphisms.

$$\begin{tikzcd}
= R^p \otimes \hat{R} \arrow[r] \arrow[rr, bend left, "\cong"] & \widehat{R^p} \arrow[r, "\cong"] & (\hat{R})^p
\end{tikzcd}$$

$$y \otimes (r_i) \mapsto (r_i y) \mapsto \Big(\underbrace{(r_i y_j)_i}_{\substack{\downarrow \\ (r_i)_i y_j}}\Big)_{j=1}^{p}$$

Hence the right hand arrow is an isomorphism, as required.

$\square$

41. Theorem. Let R be a Noetherian ring, I an ideal, and $\hat{R}$ the I-adic completion of R. Then $\hat{R}$ is flat over R.

proof. By Theorem 26, it is enough to show that $J \otimes \hat{R} \to \hat{R}$ is injective for every ideal J of R. But $J \otimes \hat{R} \cong \hat{J}$, and by Theorems 34 and 39, $\hat{J} \to \hat{R}$ is injective. $\square$

42. Theorem. (Krull) Let R be a Noetherian ring, I an ideal, and M a finitely generated R-module. Set $\bigcap_{n>0} I^n M = N$. Then there exists $a \in R$ such that $1-a \in I$ and $aN = 0$.

proof. By the Artin-Rees Lemma, $I^n M \cap N \subseteq IN$ for sufficiently large n. By definition of N we have $N = IN$. Since M is Noetherian as R-module, N is finitely generated, say by $n_1, \dots, n_t$. Then we can write

$$n_j = a_{1j} n_1 + \dots + a_{tj} n_t \quad (j = 1, \dots, t)$$

for some $a_{ij} \in R$. This gives that

$$A . [n_1 \dots n_t]^T = [n_1 \dots n_t]^T,$$

where $A = (a_{ij})$ the $t \times t$ matrix with the (i,j)-entry a_{ij}. Now, $(I - A)[n_1 \dots n_t]^T = 0$. Multiplying both sides by the adjoint matrix of $I - A$, we get

$$\det(I-A) n_j = 0$$

for every $j=1,\dots,t$. Observe that $\det(I-A)=1-b$ for some $b\in I$. Set $a=1-b$. Then a is as required.

$\square$

<u>43. Theorem.</u> (the Krull intersection theorem)

(i) Let R be a Noetherian ring and I an ideal of R with $I\subseteq J(R)$. Then for any finitely generated R-module the I-adic topology is separated, and any submodule is a closed set.

(ii) If R is a Noetherian integral domain and I a proper ideal of R, then

$$\bigcap_{n>0} I^n = 0.$$

<u>proof.</u> (i) In this case the a of the previous theorem is a unit of R, so that $N=0$, and M is separed. If $M'\subseteq M$ is a submodule, then M/M' is also I-adically separated, which is the same as saying that M' is closed in M.

(ii) Setting $M=R$ in the previous theorem, from $1\notin I$ we get that $a\neq 0$, and since a is not a zero-divisor, $N=0$.

$\square$

<u>44. Theorem.</u> Let R be a Noetherian ring, I and J ideals of R, and M a finitely generated R-module.

Write $\hat{}$ for the completion of an R-module in the I-adic topology, and $\psi : M \to \hat{M}$ for the natural map. Then

$$(JM)^\wedge = J\hat{M} = \text{the closure of } \psi(JM) \text{ in } \hat{M},$$

and

$$(M/JM)^\wedge \cong \hat{M}/J\hat{M}.$$

proof. By Theorems 34 and 39, $(JM)^\wedge$ is the kernel of $\hat{M} \to (M/JM)^\wedge$, and this is equal to the closure of $\psi(JM)$ in $\hat{M}$ by Theorem 34. Now suppose that $J = \sum_{i=1}^{r} a_i R$ and define $\varphi : M^r \to M$ by $(m_1, \ldots, m_r) \mapsto \sum a_i m_i$. Then the sequence

$$M^r \xrightarrow{\varphi} M \xrightarrow{\mu} M/JM \to 0,$$

where μ is the natural map, is exact. The I-adic completion,

$$(\hat{M})^r \xrightarrow{\hat{\varphi}} \hat{M} \xrightarrow{\hat{\mu}} (M/JM)^\wedge \to 0,$$

is again exact. On the other hand, $\hat{\varphi}$ is given by the formula $(y_1, \ldots, y_r) \mapsto \sum a_i y_i$. Hence

$$(JM)^\wedge = \ker \hat{\mu} = \mathrm{Im}\, \hat{\varphi} = \sum a_i \hat{M} = J\hat{M}. \qquad \square$$

As easily seen, the $(X_1, \ldots, X_n)$-adic completion of the polynomial ring $R[X_1, \ldots, X_n]$ over R can be identified with the formal power series ring $R[[X_1, \ldots, X_n]]$.

45. Theorem. Let R be a Noetherian ring, and $I=(a_1,\dots,a_n)$ an ideal of R. Then the I-adic completion $\hat{R}$ of R is isomorphic to $R[[X_1,\dots,X_n]]/(X_1-a_1,\dots,X_n-a_n)$. Hence $\hat{R}$ is a Noetherian ring.

proof. Let $R_1=R[X_1,\dots,X_n]$, and set $I'=\sum X_iR_1$, $J=\sum(X_i-a_i)R_1$. Then $R_1/J\cong R$, and the I'-adic topology on R considered as the R_1-module R_1/J coincides with the I-adic topology of R. Now writing $\wedge$ for the I'-adic completion of R_1-modules, we have

$$\hat{R}\cong \hat{R}_1/\hat{J}=\hat{R}_1/J\hat{R}_1=R[[X_1,\dots,X_n]]/(X_1-a_1,\dots,X_n-a_n).$$

$\square$

46. Theorem. Let R be a Noetherian ring, I an ideal, M a finitely generated R-module, and $\hat{M}$ the I-adic completion of M. Then the topology of $\hat{M}$ is the I-adic topology of $\hat{M}$ as an R-module, and is the $I\hat{R}$-adic topology of $\hat{M}$ as an $\hat{R}$-module.

proof. If we let M_n^* be the kernel of the projection map $\varprojlim(M/I^nM)\longrightarrow M/I^nM$, the topology of $\hat{M}$ is that defined by $\{M_n^*\}$. Thus it is enough to prove that $M_n^*=I^n\hat{M}$. It is easy to see that $(M/I^nM)^\wedge=M/I^nM$ and the kernel of $\hat{M}\to(M/I^nM)^\wedge$

is $I^n\hat{M}$ by Theorem 44. Therefore $M_n^* = I^n\hat{M}$. Moreover, $I^n\hat{M}$ can also be written as $(I^n\hat{R})\hat{M}$, and $I^n\hat{R} = (I\hat{R})^n$, so that the topology of $\hat{M}$ is also the $I\hat{R}$-adic topology.

$\square$

<u>47. Theorem</u> Let R be a Noetherian ring and I an ideal. If we consider R with the I-adic topology, the following conditions are equivalent:

(1) $I \subseteq J(R)$;

(2) every ideal of R is a closed set;

(3) the I-adic completion $\hat{R}$ of R is faithfully flat over R.

proof. We have already seen (1) $\Rightarrow$ (2). (see Theorem 43.)
(2) $\Rightarrow$ (3): Since $\hat{R}$ is flat over R, we need only prove that $m\hat{R} \neq \hat{R}$ for every maximal ideal m of R. By assumption, $\{0\}$ is closed in R, so that we can assume that $R \subseteq \hat{R}$, by Theorem 44, $m\hat{R}$ is the closure of m in $\hat{R}$. However, m is closed in R, so that $m\hat{R} \cap R = m$, and hence $m\hat{R} \neq \hat{R}$.
(3) $\Rightarrow$ (1): By Theorem 24, $m\hat{R} \cap R = m$ for every maximal ideal m of R. Now $m\hat{R} \subseteq \hat{R}$ is a closed set by

Theorems 35 (i) and 43 (i), and since the natural map $R \longrightarrow \hat{R}$ is continuous, $\mathfrak{M} = \mathfrak{M}\hat{R} \cap R$ is closed in R. If $I \not\subseteq \mathfrak{M}$, then $I^n + \mathfrak{M} = R$ for every $n > 0$, so that $\mathfrak{M}$ is not closed. Thus $I \subseteq \mathfrak{M}$. □

If the conditions of the above theorem are satisfied, the topological ring R is said to be <u>Zariski ring</u>, and I an <u>ideal of definition</u> of R. An ideal of definition is not uniquely determined; any ideal defining the same topology will do. The most important example of a Zariski ring is a (Noetherian) local ring $(R, \mathfrak{M})$ with the $\mathfrak{M}$-adic topology. When discussing the completion of a local ring, we will mean the $\mathfrak{M}$-adic completion unless otherwise specified.

<u>48. Theorem.</u> Let R be a quasi-semilocal ring with maximal ideals $\mathfrak{M}_1, \ldots, \mathfrak{M}_s$, and set $I = J(R) = \mathfrak{M}_1 \ldots \mathfrak{M}_s$. Then the I-adic completion $\hat{R}$ of R decomposes as a direct product

$$\hat{R} = \hat{R}_1 \times \cdots \times \hat{R}_s,$$

where $R_i = R_{\mathfrak{M}_i}$, and $\hat{R}_i$ is the completion of the quasi-local ring R_i.

<u>proof</u>. Since for $i \neq j$ and any $n > 0$ we have $\mathfrak{M}_i^n + \mathfrak{M}_j^n = R$,

we have

$$R/I^n \cong R/\mathfrak{m}_1^n \times \dots \times R/\mathfrak{m}_s^n \quad \text{for } n>0.$$

Hence taking the limit we get

$$\hat{R} = \varprojlim R/I_n = (\varprojlim R/\mathfrak{m}_1^n) \times \dots \times (\varprojlim R/\mathfrak{m}_s^n).$$

If we set R_i for the localization of R at $\mathfrak{m}_i$, then, since $R/\mathfrak{m}_i^n$ is already quasi-local

$$R/\mathfrak{m}_i^n \cong (R/\mathfrak{m}_i^n)_{\mathfrak{m}_i} \cong R_i/(\mathfrak{m}_i R_i)^n,$$

and so $\varprojlim R/\mathfrak{m}_i^n$ can be identified with $\hat{R}_i$.

$\square$

It follows from the results of this section that the following statements hold for a local ring $(R, \mathfrak{m})$:

(1) $\bigcap_{n\geq 1} \mathfrak{m}^n = (0)$;

(2) If M is a finitely generated R-module, then

$$\bigcap_{n\geq 1} (N + \mathfrak{m}^n M) = N$$

for every submodule N of M;

(3) The completion $\hat{R}$ of R is faithfully flat over R; hence $R \subseteq \hat{R}$, and $I\hat{R} \cap R = I$ for any ideal I of R;

(4) $\hat{R}$ is also a local ring with maximal ideal $\mathfrak{m}\hat{R}$, and it has the same residue class field as R; moreover, $\hat{R}/\mathfrak{m}^n\hat{R} \cong R/\mathfrak{m}^n$ for all $n>0$.

(5) If R is a complete local ring, then for any ideal $I \neq R$, R/I is again a complete local ring.

Remarks

(i) Even if R is complete, the localization R_P of R at a prime P may not be.

(ii) An Artinian local ring $(R, \mathfrak{m})$ is complete; in fact, there exists $\nu > 0$ such that $\mathfrak{m}^\nu = 0$, so that

$$\hat{R} = \varprojlim R/\mathfrak{m}^n R = R.$$

5. Exercise If R is a Noetherian ring, I and J are ideals of R, and R is complete both for the I-adic and J-adic topologies, then R is also complete for the $(I+J)$-adic topology.

6. Exercise Let R be a Noetherian ring, and $I \supseteq J$ ideals of R. If R is I-adically complete, prove that it is also J-adically complete.

7. Exercise Let R be a Noetherian ring and I a proper ideal of R. Consider the multiplicatively closed subset

$$S = 1 + I = \{1 + a : a \in I\}$$

of R. Then show that $S^{-1}R$ is a Zariski ring with ideal of definition $IS^{-1}R$, and its completion coincides with the I-adic completion of R.

8. Exercise Let $(R, \mathfrak{m})$ be a complete local ring, and $I_1 \supset I_2 \supset \cdots$ a chain of ideals of R for which

$$\bigcap_{n \geq 1} I_n = 0.$$

Then the $\mathfrak{m}$-adic topology on R is weaker than the linear topology defined by $\{I_n\}_{n\geq 1}$. (See the 1943-paper by C. Chevalley "On the theory of local rings", or try by yourself.)

9. Exercise Let R be a Noetherian ring and $P \in \mathrm{Ass}(R)$ (i.e., $P = \mathrm{ann}(a)$ for some non-zero $a \in R$). Then there is an integer $c > 0$ such that $P \in \mathrm{Ass}(R/I)$ for every ideal $I \subseteq P^c$. (Hint: Use the Artin-Rees Lemma.)