List of Symbols

$\leq$ submodule

$\leq_e$ essential submodule

$\mathbb{M}_n(R)$ the ring of $n \times n$ matrices over the ring R

$\operatorname{ass} S$ associated subset $\{r \in R : sr = 0$ for some $s \in S\}$ of the m.c. subset S

$\zeta(R)$ the right singular ideal of R

$\mathrm{r.ann}(a)$ the right annihlator of a

$\mathrm{l.ann}(a)$ the left annihlator of a

a.c.c. ascending chain condition

$\mathcal{O}_r(I)$ the right order of fractional ideal I

$\mathcal{O}_l(I)$ the left order of fractional ideal I

R_S the right quotient ring with respect to S

$\mathrm{Hom}_R(M, N)$ the group of R–homomorphisms from M to N

$\mathscr{C}_R(0)$ the set of all regular elements of R

$Q(R)$ the right quotient ring of R

$\operatorname{Spec} R$ the prime spectrum of R

$\operatorname{ass}_M S$ the associated submodule of M w. r. t. S

$m^{-1}N$ the right ideal $\{r \in R : mr \in N\}$

$\mathrm{u.dim}\, M$ uniform dimension of the module M

1 Quotient Rings and Goldie's Theorem

1.1 Quotient Rings

It is well–known in Commutative Ring Theory that, for a given nontrivial commutative ring R and a multiplicatively closed subset S of R (i.e. a subset S of R which contains 1 and is closed under products), we can form a ring, say Q, with a ring homomorphism $\varphi : R \to Q$, which satisfies the following properties:

QR1 $\varphi(s)$ is unit in Q for every $s \in S$;

QR2 if $a \in \ker \varphi$, then there exists $s \in S$ such that $sa = 0$;

QR3 for every element $x \in Q$, there exist $r, s \in R$ with $s \in S$ such that $x = \varphi(r)\varphi(s)^{-1}$.

Q is called a **ring of fractions** of R. The properties **QR1**-**QR3** determine Q uniquely up to isomorphism. One concrete way of constructing the ring of fractions of a commutative ring R (with respect to a multiplicatively closed subset S) is to consider an equivalence relation on the Cartesian product set $R \times S$ defined by

$$(a, s) \sim (b, t) \text{ if and only if there exists } s' \in S \text{ such that } s'(at - bs) = 0.$$

Equivalence class of an element $(a, s) \in R \times S$ is usually denoted by a/s, a formal fraction with numerator a and denominator s. The set of all equivalence classes a/s, denoted by $S^{-1}R$, can be made into a commutative ring by putting addition and multiplication in a similar way as we do for constructing the field of rational numbers. Now the ring $S^{-1}R$ satisfies the conditions **QR1**-**QR3** and any ring satisfiying these conditions is isomorphic to $S^{-1}R$. The idea of ring of fractions has been proved crucial in the study of commutative rings since it was first thought by H. Grell [Gr] in 1926 . As a special case, one can consider the set complement of a prime ideal, which is clearly a multiplicatively closed subset, and form a in of fractions with respect to it, which is called the **localization** of the ring at that prime ideal. This idea, mainly based on inverting the elements that lie outside a prime ideal, gives rise to a local ring (a commutative ring with unique maximal ideal). There are some properties of rings and modules which can be detected from local rings. These properties are called **local properties**. For example flatness is a local property but being a free module is not. After Krull's introduction of fundamental notions, localization has become one of the most significant tools in commutative algebra and has been used effectively in this area. Although prime ideals are still important for non-commutative rings, the idea of localization is not always available, contrary to the case of commutative rings. A part of this note is devoted to give an account of localizations in the case of non-commutative rings. But before, we need to develop a theory for rings of fractions of non-commutative rings (which are called quotient rings in this particular setting).

Quotient rings of non-commutative rings were first thought after van der Waerden's question in his famous book [V]. He asked if every non-commutative domain is a subring of a division ring. The answer is no as the counter-example given by A. I. Malcev shows. Then O. Ore gave in 1931 a necessary and sufficient condition under which a non-commutative domain R could be embedded in a division ring Q (see [O]). Ore's approach gives a division ring which is, under an additional hypothesis, a quotient ring (indeed, the total quotient ring) of R. Now it is time to give a formal definition of a quotient ring of any non-commutative ring.

Let R be an associative ring with unity (as always will be). Let S be a multiplicatively closed (abbreviated as m.c.) subset of R. A **right quotient ring** of R with respect to S is a ring Q together with a homomorphism $\varphi : R \to Q$ such that the conditions **QR1**-**QR3** are satisfied. **Left quotient rings** are defined analogously. Notice that if $rs = 0$ for some $s \in S$, then $\varphi(r) = \varphi(rs)\varphi(s)^{-1} = 0$, and so

$$\ker \varphi = \{r \in R : rs = 0 \text{ for some } s \in S\}.$$

We denote this set by $\operatorname{ass} S$. Note that if $\operatorname{ass} S = 0$, then R can be embedded into Q (by means of φ), in which case we identify R as its image under φ, and write elements of Q in the form rs^{-1} where $r \in R$ and $s \in S$. Although the definition of quotient rings of non-commutative rings is given in exactly the same way as the one given for commutative rings, unlike the commutative case, the existence of Q is not guaranteed.

Proposition 1.1. *Suppose that there exists a right quotient ring Q of R with respect to a m.c. subset S together with the defining homomorphism φ. Then given a ring homomorphism $\mu : R \to R'$ with the property that $g(S)$ consists of units of R', there exists a unique homomorphism $\nu : Q \to R'$ such that $\nu\varphi = \mu$, i.e., the following diagram is commutative*

$$\begin{tikzcd} R \arrow[rr, "\mu"] \arrow[dr, "\varphi"'] & & R' \\ & Q \arrow[ur, dashed, "\nu"'] & \end{tikzcd}$$

This proposition immediately yields the following

Corollary 1.2. *If there exists a right quotient ring Q of R with respect to a m.c. subset S, then it is unique up to isomorphism. Additionally, if R has also a left quotient ring Q' with respect to S, then $Q \cong Q'$. In this case, we have*

$$\operatorname{ass} S = \{r \in R : sr = 0 \text{ for some } s \in S\}.$$

Following the uniqueness statement in Corolary 1.2, it is now reasonable to use the notation R_S for the right quotient ring Q of R with respect to S. This ring is also called the *right localization* of R at S.

Example 1.3. There are some circumstances in which the existence of a right quotient ring is immediate. For example if R is the first Weyl algebra $A_1(k)$ over a field k, which can be viewed as a skew polynomial ring $k[X][Y;\partial/\partial X]$, where X and Y are indeterminates (regarding the coefficients on the right), and S is taken to be the set minus $k[X]\setminus\{0\}$, then R_S exists and is equal to the ring $k(X)[Y;\partial/\partial Y]$, denoted by $B_1(k)$, with φ the natural embedding $A_1(k)\hookrightarrow B_1(k)$.

Similarly, if $R = A[X;\sigma]$ for some ring A and automorphism σ on A, and $S = \{x^n : n \geq 0\}$, then R_S exists and is equal to the ring $A[X, X^{-1};\sigma]$ of skew Laurent polynomials over k, and φ is the natural embeding.

Theorem 1.4 ((Ore's Theorem))**.** *Let S be a m.c. subset of the ring R. Then R_S exists if and only if the following conditions holds:*

(i) *If $r \in R$ and $s \in S$, then there exist $r' \in R$ and $s' \in S$ such that $rs' = sr'$.*

(ii) *If $r \in R$ and $s \in S$ with $sr = 0$, then there exists $t \in S$ such that $rt = 0$.*

Proof. Assume that R_S exists. Then there exists a homomorphism $\varphi : R \to R_S$ satisfying the consitions **QR1**-**QR3**. By **QR3**, there exist $r_1 \in R$ and $s_1 \in S$ such that

$$\varphi(s)^{-1}\varphi(r) = \varphi(r_1)\varphi(s_1)^{-1}.$$

Thus $\varphi(r)\varphi(s_1) = \varphi(s)\varphi(r_1)$, and so $rs_1 - sr_1 \in \ker\varphi = \operatorname{ass} S$. This gives that $(rs_1 - sr_1)s_2 = 0$ for some $s_2 \in S$. If we write $s' = s_1s_2$ and $r' = r_1s_2$, (i) follows. If $sr = 0$ for some $r \in R$ and $s \in S$, then since $\varphi(s)$ is unit in R_S by **QR2**, we have $r \in \ker\varphi$, which establishes part (ii).

Suppose now that the conditions (i) and (ii) hold. To construct the quotient ring R_S, consider the set

$$\mathscr{F} = \{I : I \text{ is a right ideal of } R \text{ with } I \cap S \neq \emptyset\}.$$

It is readily checked that (i) implies that if $I_1, I_2 \in \mathscr{F}$ and $\alpha : I_1 \to R$ is a homomorhism of R–modules, then

(a) $I_1 \cap I_2 \in \mathscr{F}$,and

(b) $\alpha^{-1}(I_2) \in \mathscr{F}$.

Now let $\overline{R} = R/\operatorname{ass} S$ and consider the set

$$\bigcup\{\operatorname{Hom}_R(\overline{I}, \overline{R}) : I \in \mathscr{F}\}$$

together with the equivalence relation given by $\alpha_1 \sim \alpha_2$ if and only if α_1 and α_2 coincide on some $\overline{J}$, where $J \in \mathscr{F}$, $J \subseteq I_1 \cap I_2$, for $\alpha_i \in \operatorname{Hom}_R(\overline{I_i}, \overline{R})$ $(i = 1, 2)$. We define addition and multiplication on the equivalence classes $[\alpha_i]$, for $\alpha_i \in \operatorname{Hom}_R(\overline{I_i}, \overline{R})$, as follows:

$[\alpha_1]+[\alpha_2]=[\beta]$, where β is the sum of the restrictions of α_1 and α_2 to $\overline{I_1\cap I_2}$.
$[\alpha_1][\alpha_2]=[\gamma]$, where γ is the composition

$$\alpha_2^{-1}(\overline{I_1}) \xrightarrow{\alpha_2} \overline{I_1} \xrightarrow{\alpha_1} \overline{R}\,.$$

It is easy to check that these operations are well-defined and make the set of equivalence classes into a ring. We denote this ring by R_S. Let $\varphi : R \to R_S$ be a function defined by $\varphi(r)=[\lambda_r]$, where $\lambda_r : \overline{R} \to \overline{R}$ is the homomorphism of R–modules defined by $x \mapsto \overline{rx}$ for all $x \in R$ (here we put bars over elements to denote the cosets in $\overline{R}$). It is clear that φ is a ring homomorphism. Let $s \in S$ and define $\alpha : \overline{sR} \to \overline{R}$ such that $\alpha(\overline{sx})=\overline{x}$. It is immediate that α is an R–homomorphism with $\varphi(s)[\alpha]=[\alpha]\varphi(s)=[1]$. This gives that $\varphi(s)$ is unit in R_S with inverse $[\alpha]$. On the other hand, if $\alpha \in \mathrm{Hom}_R(\overline{I},\overline{R})$ for some $I \in \mathscr{F}$, then $[\alpha]=\varphi(r)\varphi(s)^{-1}$ where $s \in I\cap S$ and $r=\alpha(s)$. Moreover, if $\varphi(a)=[0]$ for some $a \in R$, then $\lambda_a(\overline{I})=\{\overline{0}\}$. Since $I\cap S \neq \emptyset$, there exists $s \in S$ such that $as \in \operatorname{ass} S$. But in this case $a \in \operatorname{ass} S$. This establishes the theorem. $\square$

The condition *(i)* in Theorem 1.4 is called the **right Ore condition**. One can easily define the ***l*eft Ore condition** by symmetry. A m.c. subset S of R is called a **right denominator set** if it satisfies both conditions *(i)* and *(ii)* of Theorem 1.4.

Lemma 1.5. *Let S be a m.c. subset of the ring R which satisfy the right Ore condition. Then given $r_1,\dots,r_n \in R$ and $s_1,\dots,s_n \in S$, there exist $r'_1,\dots,r'_n \in R$ and $s' \in S$ such that $r_is'=s_ir'_i$ for each $i=1,\dots,n$.*

Proof. We use induction on n. Let $n=2$. Since S satisfies the right Ore condition there exist $r_1^*,r_2^* \in R$, $s_1^*,s_2^* \in S$ such that $r_1s_1^*=s_1r_1^*$ and $r_2s_2^*=s_2r_2^*$. Again applying the right Ore condition to s_1^* and s_2^*, we get $s_1^*t_1=s_2^*t_2$ for some $t_1 \in S$, $t_2 \in R$. Since S is multiplicatively closed, we have $s=s_1^*t_1=s_2^*t_2 \in S$. Also we have

$$r_is=r_is_i^*t_i=s_ir_i^*t_i$$

for $i=1,2$. Setting $r'_i=r_i^*t_i$ completes the proof when $n=2$. Now assume that $n>2$ and the assertion is true for positive integers less than n. By induction hypothesis, there exist $r''_1,\dots,r''_n \in R$ and $s'' \in S$ such that $r_is''=s_ir''_i$ for each $i=1,\dots,n-1$. Applying the right Ore condition to r_ns'' and s_n, we obtain $r_ns''t=s_nr'_n$ for some $t \in S$ and $r'_n \in R$. Setting $r'_i=r''_it$ for $i=1,\dots,n-1$ completes the inductive step. $\square$

An element $a \in R$ is called **right regular** if there are no nonzero elements $b \in R$ such that $ab=0$. In other words, if a is a right regular element of R and $ab=0$ for some $b \in R$, then $b=0$. **Left regular elements** are defined analogously. A **regular element** of R is an element which is both right and left regular. The set of regular elements of R is denoted by $\boldsymbol{\mathscr{C}_R(0)}$ and is a multiplicatively closed set of particular importance.

We simply call the right quotient ring of the ring R with respect to $\mathscr{C}_R(0)$, if it exists, the **right quotient ring** of R, and denote it by $\boldsymbol{Q(R)}$. An integral domain R is called a **right Ore domain** if $\mathscr{C}_R(0)$ is a right Ore set. Notice that if R is an Ore domain, then since $\operatorname{ass}\mathscr{C}_R(0)=0$, R can be embedded into the division ring $Q(R)$ by Theorem 1.4, which establishes one part of the following

Corollary 1.6. *An integral domain has a right quotient ring if and only if it is a right Ore domain.*

The following theorem says that there are plenty of Ore domains.

Theorem 1.7. *If R is a right Noetherian domain, then it is an Ore domain.*

Proof. We need to check if $\mathscr{C}_R(0)$ satisifies the Ore condition. Let $a, b \in R \setminus \{0\}$. We aim to show that there exists nonzero elements a' and b' such that $ab' = ba' \neq 0$. Assume the contrary. Suppose that there exist a positive integer n and elements $a_0, a_1, \dots, a_k$ of R such that

$$b^n ar_0 + b^{n+1}ar_1 + \cdots b^{n+k}ar_k = 0 \text{ and } b^n ar_0 \neq 0.$$

Since R is an integral domain, we have

$$0 \neq ar_0 = b(-ar_1 - \cdots - b^{k-1}ar_k),$$

a contradiction. It follows that the sum $\sum_{n\geq 1} b^n aR$ is direct. But this contradicts the fact that R is right Noetherian, completing the proof. □

It is now appropriate to make some convensions on rings of quotients. Let R be a ring, S a right denominator set, and $Q = R_S$. First, although the canonical homomophism $\varphi : R \to R_S$ need not be an embedding, we write the elements of Q in the form rs^{-1} (or r/s in some circumstances) where $r \in R$ and $s \in S$ instead of $\varphi(r)\varphi(s)^{-1}$ as given in the definition. We do this just for the sake of simplicity and does not mean that R is considered as a subring of Q.

Proposition 1.8. *Let S be a right denominator set in the ring R, and let $Q = R_S$. Then the folloiwng statements hold:*

(i) If $q_1, \dots, q_n \in Q$, then there exist $r_1, \dots, r_n \in R$ and $s \in S$ such that $q_i = r_i s^{-1}$ for each $i = 1, \dots, n$. In other words, any finite subset of Q has a common denominator.

(ii) Q is a flat left R–module.

Proof. **(i)** Let $q_i = a_i s_i^{-1}$ for each $i = 1, \dots, n$. Applying Lemma 1.5 by taking $r_1 = \ldots = r_n = 1$, we get $s_i b_i = s$ for some $b_i \in R$ $(1 \leq i \leq n)$ and $s \in S$. Set $r_i = a_i b_i$ for each $i = 1, \dots, n$. In Q, we can wirite $s_i^{-1} = b_i s^{-1}$, and so $q_i = a_i s_i^{-1} = a_i b_i s^{-1} = r_i s^{-1}$, as desired.

(ii) To see that ${}_RQ$ is a flat module, it is enough to check that if I is a right ideal of R, then the natural map $\kappa : I \otimes_R Q \to Q$ is injective. Suppose $\kappa(z) = 0$ where $z \in I \otimes_R Q$. Then z is a finite sum of simple tensors in $I \otimes_R Q$. Let $z = \sum a_i \otimes q_i$ for some $a_i \in I$ and $q_i \in Q$. By (i) above, there exist $r_i \in R$ $(1 \leq i \leq n)$ and $s \in S$ such that $q_i = r_i s^{-1}$. Then we have $z = a \otimes s^{-1}$, where $a = \sum a_i r_i \in I$. It follows that $0 = \kappa(z) = as^{-1}$, which means that $\varphi(a) = 0$, or in other words, $a \in \ker \varphi = \operatorname{ass} S$. Then $at = 0$ for some $t \in S$, and so $z = at \otimes t^{-1}s^{-1} = 0$. This completes the proof of part (ii). □

Remark 1.9. In view of the proof of Proposition (i), we have a useful fact to check if a given pair of elements of a quotient ring Q are equal. Namely, for elements $a_1s_1^{-1}$ and $a_2s_2^{-1}$ of Q, $a_1s_1^{-1} = a_2s_2^{-1}$ if and only if there exist $b_1, b_2 \in R$ such that $s_1b_1 = s_2b_2 \in S$ and $a_1b_1 = a_2b_2$. This shows that we can describe Q as $R \times S$ modulo the equivalence relation given by the latter condition.

For any subset A of Q, we denote the set $\{r \in R : r/1 \in A\}$ by $A \cap R$, not meaning the set theoretical intersection. Of course if φ is an embedding and R is regarded as its image under φ, then $A \cap R$ coincides with the ordinary intersection. If $B \subseteq R$, then the right ideal of Q generated by $\varphi(B) = \{b/1 : b \in B\}$ is denoted by BQ.

Proposition 1.10. *Let S be a right denominor set in the ring R, and let $Q = R_S$. Then the following statements hold:*

- ***(i)*** *If B is a right ideal of Q, then $B \cap R$ is a right ideal of R and $B = (B \cap Q)Q$.*
- ***(ii)*** *If A is a right ideal of R, the AQ is a right ideal of Q, $AQ = \{as^{-1} : a \in A,\ s \in S\}$, and $AQ \cap R = \{r \in R : rs \in A$ for some $s \in S\}$.*
- ***(iii)*** *If A_1, A_2 are right ideals of R with $A_1 \cap A_2 = 0$, then $A_1Q \cap A_2Q = 0$.*
- ***(iv)*** *If I is an ideal of R and Q is right Noetherian, then IQ is an ideal of Q.*
- ***(v)*** *If R is Noetherian, then there is a one-to-one correspodence between $\{P \in \operatorname{Spec} R : P \cap S = \emptyset\}$ and $\operatorname{Spec} Q$ defined by*

$$\operatorname{Spec} R \ni P \longmapsto PQ, \quad \operatorname{Spec} Q \ni P' \longmapsto P' \cap R.$$

Proof. ***(i)***, ***(ii)***, ***(iii)*** *These are straightforward.*

(iv) It is enough to show that given $s \in S$, then $s^{-1}IQ \subseteq IQ$. Since $sI \subseteq I$, we have $I \subseteq s^{-1}I$ in Q. This yields an ascending chain

$$s^{-1}I \subseteq s^{-2}I \subseteq \ldots \subseteq s^{-n}I \subseteq \ldots$$

of right ideals of Q, which mus,t by our Noetherian hypothesis, must stop. Thus $s^{-(n+1)}I = s^{-n}I$ for some positive integer n. Hence $s^{-1}IQ = IQ$, as desired, since s is unit in Q.

(v) Let P be a prime ideal of R with $P \cap S = \emptyset$. Let $P' = PQ \cap R$. Since R is Noetherian, P' is finitely generated as a left ideal of R, and so by (ii), there exists $s \in S$ such that $P's \subseteq P$. By primeness of P together with the property thet $P \cap S = \emptyset$, we have $P = P' = PQ \cap R$. In particular, this implies that $PQ \neq Q$. Our next step is to show that PQ is a prime ideal of Q. To see this, let A and B be ideals of Q with $AB \subseteq PQ$. Then

$$(A \cap R)(B \cap R) \subseteq AB \cap R \subseteq PQ \cap R \subseteq P.$$

Again by primeness of P, we have either $A \cap R \subseteq P$ or $B \cap R \subseteq P$. Assume, without loss of generality, that $A \cap R \subseteq P$. Then by (i) above

$$A = (A \cap R)Q \subseteq PQ.$$

This gives that $PQ \in \operatorname{Spec} Q$. Finally, we need to see that $P' \cap R$ is a prime ideal of R which does not meet S, for each prime ideal P' of Q. It is easy to see, using (i), that Q is right Noetherian. By (i), $P' \cap R \neq R$. Let $IJ \subseteq P' \cap R$. Then by (i) and (iv), we hve

$$(IQ)(JQ) = I(QJQ) = IJQ \subseteq (P' \cap R)Q = P'.$$

This implies that one of the IQ or JQ, say for instance IQ, lies in P'. Then $I \subseteq IQ \cap R \subseteq P' \cap R$. It follows that $P' \cap R$ is a prime ideal of R. Moreover, if $P' \cap R$ meets S, then P' contains a unit element of Q, contradicting the fact that P' is a proper ideal of Q as being prime. This completes the proof. $\square$

Given a right denominator set and a right R–module M, it is also possible to construct a **modue of quotients** M_S in a similar way as we do for R_S. To construct M_S, first define

$$\operatorname{ass}_M S = \{m \in M : ms = 0 \text{ for some } s \in S\},$$

which is clearly a submodule of M (called the **torsion submodule with respect to** S). Note also that we call M a **torsion module with respect to** S if $M = \operatorname{ass}_M S$. Next, let $\overline{M} = M/\operatorname{ass}_M S$ and consider $\{\operatorname{Hom}_R(\overline{I}, \overline{M}) : I \in \mathscr{F}\}$, where we again put bars over the ideals or elements of R to denote their images under the natural map $R \to R/\operatorname{ass} S$. If we put the same equivalence relation and operations on this set as in the contruction of R_S, we obtain an R_S–module, that we denote by M_S. Observe that each element of $M_S$has the form $\overline{ms}^{-1}$ for some $\overline{m} \in \overline{M}$ and $s \in S$. Also, M is a torsion module with respect to S if and only if $M_S = 0$. One can easily check that the map $\varphi_M : M \to M_S$ defined by $\varphi(m) = \overline{m}$ is an R–homomorphism with kernel $\operatorname{ass}_M S$ and has the following **universal property**.

Proposition 1.11. *Let S be a right denominator set in the ring R and let M be an R–module. If L is a R_S–module and $f : M \to L$ is an R–homomorphism then there exists a unique R_S–homomorphism $g : M_S \to L$ such that the following diagram is commutative:*

$$\begin{tikzcd} M \arrow[r, "\varphi_M"] \arrow[d, "f"'] & M_S \arrow[dl, dashed, "g"] \\ L & \end{tikzcd}$$

One natural way to produce an R_S–module from an R–module is to consider tensor products. Given an R–module M, the tensor product $M \otimes_R R_S$ is also a right R_S–module. The following proposition says that this is nothing but the module of quotients M_S of M with respect to S.

Proposition 1.12. *Let S be a right denominator set and let M be an R–module. Then*

(i) *$M \otimes_R R_S \cong M_S$.*

(ii) *The torsion submodule of M with respect to S is the kernel of the natural homomorphism $M \to M \otimes_R R_S$.*

(iii) *M is a torsion module with respect to S if and only if $M \otimes_R R_S = 0$.*

Proof. ***(i)*** *The universal property of M_S gives a homorphism $M_S \to M \otimes_R R_S$ whose inverse can be given by te universal property of $\otimes$.*

(ii) (iii) These follow immediately from (i) and explanation given before Prosition 1.11 above. □

1.2 Uniform (Goldie) Dimension

Let M be a module over the ring R and let N be a submodule of M. If every nonzero submodule L of M meets N nontrivially (i.e., $L \cap N \neq 0$), then we say that N is an **essential submodule** of M. In this case, we use the notation $N \leq_e M$.

If a right ideal I of R is an essential submodule of R_R, then I is called an **essential right ideal** of R. Note that in a right Ore domain, every nonzero right ideal is essential. The following lemma is key to some result in the sequel and will be used without giving a reference.

Lemma 1.13. *If R is a prime ring then every nonzero ideal of R is an essential right ideal of R.*

Proof. Let I be a nonzero ideal of R. If A is a nonzero right ideal of R, then by assumption, $0 \neq IA \subseteq I \cap A$. This completes the proof. □

The following lemma provides some basic properties about essential submodules.

Lemma 1.14. *Let $M, M_1, \ldots, M_k$ be modules of the ring R. Then the following hold.*

(i) *If $L \leq_e M$ and N is a submodule of M containing L, then $N \leq_e M$.*

(ii) *Being essential submodule is transitive, that is, whenever $L \leq_e N$ and $N \leq_e M$, then $L \leq_e M$.*

(iii) *If $N \leq_e M$ and $L \leq_e M$, then $N \cap L \leq_e M$.*

(iv) *If $N \leq_e M$ and $m \in M$, then $m^{-1}N \leq_e R$, where $m^{-1}N = \{r \in R : mr \in N\}$.*

(v) *If $N_i \leq_e M_i$ for each $i = 1 \ldots, k$, then $N_1 \oplus \cdots \oplus N_k \leq_e M_1 \oplus \cdots \oplus M_k$.*

(vi) *If $N \leq M$, then there exists $C \leq M$ such that $N \cap C = 0$ and $N \oplus C \leq_e M$.*

(vii) *M is a semisimple module if and only if the there are no essential submodules of M other than itself.*

Proof. The proofs are routine and can be found in many graduate texts involving Module Theory. □

A module U is called **uniform** if $U \neq 0$ and every nonzero submodule of U is an essential submodule. Clearly, U is a uniform module if and only if for any pair m_1, m_2 of nonzero elements of M, there exist $r_1, r_2 \in R$ such that $m_1 r_1 = m_2 r_2 \neq 0$. It is now immediate that if R is an integral domain, then R_R is a uniform module if and only if R is a right Ore domain.

A nonzero module M is said to have **finite uniform dimension** if it contains no infinite direct sum of nonzero submodules. Uniform modules and Noetherian modules are clearly of this type. Note also that nonzero submodules of a module of finite uniform dimension have finite uniform dimesion too.

Let M be a module of finite uniform dimension. Now, either M is uniform or M contains a direct sum of two nonzero submodules, say $M \supseteq M_1 \oplus N_1$. If we apply the same process, we get that either M_1 is uniform of M_1 contains a direct sum of nonzero submodules, say $M_1 \supseteq M_2 \oplus N_2$. Continuing in this fashion, we obtain a direct sum $N_1 \oplus N_2 \oplus \cdots$, which cannot have infinitely many summands. Thus the process must stop and so we must end up with a uniform submodule of M. Consequently, any module of finite uniform dimension contains a uniform submodule. An easy application of Zorn's Lemma shows that there exists an independent family $\{U_\lambda\}_{\lambda \in \Lambda}$ of uniform submodules of M, i.e., the sum $\sum_{\lambda \in \Lambda} U_\lambda$ is direct and there is no family of uniform submodules of M strictly containing $\{U_\lambda\}_{\lambda \in \Lambda}$ whose sum is direct. We claim that $\bigoplus_{\lambda \in \Lambda} U_\lambda \leq_e M$. Assume the contrary and let N be a nonzero submodule of M such that $\left(\bigoplus_{\lambda \in \Lambda} U_\lambda\right) \cap N = 0$. Since N has finite uniform dimension, N contains a uniform submodule, U say. This gives that $\{U_\lambda\}_{\lambda \in \Lambda} \cup \{U\}$ is an independent family of uniform submodules of M strictly containing $\{U_\lambda\}$, a contradiction. Therefore, $\bigoplus_{\lambda \in \Lambda} U_\lambda$ is an essential submodule of M. Since M has finite uniform dimension Λ must be finite. Let $|\Lambda| = n$ and write $\Lambda = \{1, \ldots, n\}$.Then we have

$$U_1 \oplus \cdots \oplus U_n \leq_e M,$$

where $U_1, \ldots, U_n$ are uniform submodules of M. The following theorem says that the number of U_i's is independent of the choice of the direct sum decomposition.

Theorem 1.15. *Let M be a module of finite uniform dimension and let $V = U_1 \oplus \cdots \oplus U_n$ be a finite direct sum of uniform submodules of M which is essential in M. Then*

(i) *any direct sum of nonzero submodules of M has at most n summands, and*

(ii) *a direct sum of uniform submodules of M is essential in M if and only if it has precisely n summands.*

Proof. ***(i)*** *Let $M_1 \oplus \cdots \oplus M_k$ be a direct sum of nonzero submodules of M. Let $N = M_2 \oplus \cdots \oplus M_k$. If $N \cap U_i \neq 0$for every $i = 1, \ldots, k$, then, Lemma 1.14 (v),*

$$(N \cap U_1) \oplus \cdots \oplus (N \cap U_k) \leq_e U_1 \oplus \cdots \oplus U_k \leq_e M,$$

and so, by (i) of the same lemma, $N \leq_e M$ since $(N \cap U_1) \oplus \cdots \oplus (N \cap U_k)$ is contained in N, a contradiction. It follows that $N \cap U_i = 0$ for some $i = 1, \ldots, n$, say for $i = 1$. Then *we have a direct sum $U_1 \oplus M_2 \oplus \cdots \oplus M_k$. If we repeat this process k times, we can replace all the M_i's by some of the U_i's, which implies that $k \leq n$.*

(ii) Let W be a direct sum of uniform submodules of M. If W is not essential, then there is a nonzero submodule of M which meet W trivially, and which contains a uniform submodule, say U. Then $W \oplus U$ is again a direct sum of uniform submodules. This process continues until we end up with an essential direct sum of uniform submodules. Thus if W has n summands, then W must be essential in M, by (i) above. Conversely, if $W \leq_e M$, then applying (i) for both V and W gives that W must have exactly n summands. This completes the proof of part (ii). □

Theorem 1.15 shows that if M is a module of finite uniform dimension, then there is a maximal nonnegative integer n such that there exists a direct sum of n submodules in M and there is no direct sum of submodules in M having more than n summands. The number n is called the **uniform dimension** (or **Goldie dimension**) of M, denoted by $\mathrm{u.dim}\, M$. If there is direct sums of infinitely many submodules in M, then we write $\mathrm{u.dim}\, M = \infty$. In case $M = R_R$, we write $\mathrm{u.dim}\, R_R$.

Corollary 1.16. *Let M be a module. Then*

(i) $\mathrm{u.dim}\, M = 0$ *if and only if* $M = 0$.

(ii) $\mathrm{u.dim}\, M = 1$ *if and only if* M *is uniform.*

(iii) *If* $N \leq M$ *and* $\mathrm{u.dim}\, M = n$, *then* $\mathrm{u.dim}\, N \leq n$ *with equality precisely when* $N \leq_e M$.

(iv) $\mathrm{u.dim}(M_1 \oplus M_2) = \mathrm{u.dim}\, M_1 + \mathrm{u.dim}\, M_2$.

Note that if R is an integral domain, then $\mathrm{u.dim}\, R = 1$ if and only if R is a right Ore domain, by the remarks after Lemma 1.14. In particular, this applies to any right Noetherian integral domain.

Example. Let k be a field and let $R = k[X,Y]/(X,Y)^n$. It is not difficult to check that the sum $\sum_{i=0}^{n-1} \overline{X}^i \overline{Y}^{n-1-i} k$ is a direct sum of uniform submodules of R and is essential in R_R (where $\overline{}$ is used to denote the homomorphic images in R). Thus $\mathrm{u.dim}\, R_R = n$.

Example 1.17. Another example comes form matrix rings. Let $\mathrm{u.dim}\, R = n$ and let $S = \mathbb{M}_t(R)$, the ring of $t \times t$ matrices over R. Let $\{e_{ij} : i, j = 1, \ldots, t\}$ be the standard set of matrix units of S. One can easily check that there is a one-to-one correspondence between the S–submodules of $e_{ij}S$ and right ideals of R which corresponds zero submodule and zero ideal and which preserves direct sums and intersections. Thus $\mathrm{u.dim}(e_{ij}S)_S = \mathrm{u.dim}\, R_R$. Since $S = \bigoplus_{i=1}^{t} e_{ii}S$, we have $\mathrm{u.dim}\, S_S = t(\mathrm{u.dim}\, R_R)$.

Lemma 1.18. *Let S be a right denominator set in the ring R consisting of regular elements of R and let $Q = R_S$. Let A and B be a right ideals of R and Q, respectively. Then*

(i) $A \leq_e R$ *if and only if* $AQ \leq_e Q$;

(ii) *$B \leq_e Q$ if and only if $B \cap R \leq_e R$;*

(iii) $\mathrm{u.dim}\, A_R = \mathrm{u.dim}\, AQ_Q = \mathrm{u.dim}\, AQ_R$*;*

(iv) $\mathrm{u.dim}\, Q_Q = \mathrm{u.dim}\, R_R$*.*

This lemma is not true for a general right denominator set S. For example if $R = k[X,Y]/(X^2, XY)$, $P = (\overline{X})$, and $S = R \setminus P$ where k is a field and bars over the elements of $k[X,Y]$ are used to indicate homomorphic images in R, then $Q = R_S = R_P$ is a commutative local ring with unique nonzero proper ideal PR_P (notice that the elements of P are in the form $c\overline{X}$ where $c \in k$, which shows that P is a minimal nonzero ideal of R). Hence $\mathrm{u.dim}\, Q_Q = 1$. However, $\mathrm{u.dim}\, R_R \geq 2$ since $\overline{X}R \cap \overline{Y}R = 0$.

1.3 Goldie's Theorem

The main objective of this section is to characterize those rings whose right quotient rings exist and are semisimple Artinian.

We first introduce the notion of right singular ideal, originated in [J].

Let $\mathscr{F}(R)$ denote the set of all essential right ideals of R and define

$$\begin{aligned}\zeta(R) &= \{a \in R : aE = 0 \text{ for some } E \in \mathscr{F}(R)\} \\ &= \{a \in R : \mathrm{r.ann}(a) \in \mathscr{F}(R)\}.\end{aligned}$$

Using Lemma 1.14, one can easily deduce that $\zeta(R)$ is an ideal of R, known as the **right singular ideal** of R. Here we have used the notation $\mathrm{r.ann}(a)$ to denote the right ideal $\{b \in R : ab = 0\}$, called the **right annihilator** of a. Similary, one can define the **left annihilator** of a, denoted by $\mathrm{l.ann}(a)$. In general given a subset X of R, we define the right annihilator of X, denoted by $\mathrm{r.ann}(X)$, to be the subset $\{b \in R : xb = 0 \text{ for all } x \in X\}$ of R, which is indeed a right ideal of R. We call a right ideal I of R a **right annihilator ideal** if there exists a subset X of R such that $I = \mathrm{r.ann}(X)$. The left annihilator of a subset X is defined analogously and is denoted by $\mathrm{l.ann}(X)$.

A ring R is called **right non-singular** when $\zeta(R) = 0$. Our next step is to show that if R is a **semiprime ring** (i.e. R contains no nonzero nilpotent ideals) with ascending chain condition (**a.c.c.** for short) on right annihilator ideals, then R is right non-singular. But before we need to give two lemmas.

Lemma 1.19. *Let R be a ring with a.c.c. on right annihlator ideals.*

(i) *For an element $b \in R$, there is an integer m such that, for all $n \geq m$, $\mathrm{r.ann}(b^m) = \mathrm{r.ann}(b^n)$; and then $\mathrm{r.ann}(b^n) \cap b^nR = 0$.*

(ii) *If ,aditionally, R has finite right uniform dimension, then $\mathrm{r.ann}(b^n) \oplus b^nR$ is an essential right ideal for all sufficiently large integers n.*

Proof. **(i)** Notice that given an element $b \in R$, we have an ascending chain

$$\mathrm{r.ann}(b) \subseteq \mathrm{r.ann}(b^2) \subseteq \ldots \subseteq \mathrm{r.ann}(b^i) \subseteq \mathrm{r.ann}(b^{i+1}) \ldots$$

of right annihilator ideals of R, which must stabilize by the assumption on R. It follow that there exists a positive integer m such that $\mathrm{r.ann}(b^m) = \mathrm{r.ann}(b^n)$ for each $n \geq m$. The rest of the part (i) follows easily from this fact.

(ii) By (i) above, for all sufficiently large integers n, $\mathrm{r.ann}(b^n) \cap b^n R = 0$, and so the sum is direct. Let I be a right ideal of R such that

$$I \cap (\mathrm{r.ann}(b^n) \oplus b^n R) = 0.$$

Consider the sum $\sum_{k=1}^{t} b^{kn} I$. We want to show that this sum is direct, which clearly completes the proof. Suppose by induction that this holds for $t-1$. Since $\mathrm{r.ann}(b^n) \cap b^n R = 0$, using the induction hypothesis gives that the sum $\sum_{k=2}^{t} b^{kn} I$ is direct. Let $x \in b^n I \cap \sum_{k=2}^{t} b^{kn} I$. Then $x = b^n i = b^{2n} j$ with $i \in I$, $j \in R$. Therefore $i - b^n j \in \mathrm{r.ann}(b^n)$, and so

$$i \in I \cap (\mathrm{r.ann}(b^n) \oplus b^n R) = 0.$$

Thus $x = 0$, as desired; hence the sum $\sum_{k=1}^{t} b^{kn} I$ is direct. □

Lemma 1.20. *Each nonzero nil left (or right) ideal of R contans a nonzero nilpotent left (resp. right) ideal.*

Proof. It can be readily checked that for $a \in R$, aR is nil (or nilpotent) if and only if Ra is nil (resp. nilpotent). Thus it is enough to consider only a nil left ideal L, say. Choose $0 \neq a \in L$ such that $\mathrm{r.ann}(a)$ is maximal among right annihilators of nonzero elements of L. If $x \in R$, then there exists a positive integer t such that $(xa)^t = 0$ and $(xa)^{t-1} \neq 0$. The maximality of $\mathrm{r.ann}(a)$ shows that $\mathrm{r.ann}(xa)^{t-1} = \mathrm{r.ann}(a)$, hence $axa = 0$. Thus $aRa = 0$, and so Ra is nilpotent. □

Now we are ready to prove the aformentioned result on semiprime rings with a.c.c. on right annihilators.

Theorem 1.21. *Let R be a semiprime ring with a.c.c. on anihilator ideals. Then R is a right non-singular ring.*

Proof. We complete the proof by showing that $A = \zeta(R)$ is a nilpotent ideal of R. By assumption, $\mathrm{r.ann}(A^n) = \mathrm{r.ann}(A^{n+1})$ for all sufficiently large integers n. Suppose $A^{n+1} \neq 0$. By assumption, we may choose a maximal element of the set

$$\{\mathrm{r.ann}(x) : x \in A \text{ and } A^n x \neq 0\},$$

say $\mathrm{r.ann}(y)$. If $b \in A$, the $\mathrm{r.ann}(b) \leq_e R_R$, so $yR \cap \mathrm{r.ann}(b) \neq 0$. Hence $byr = 0$ for some $r \in R$ with $yr \neq 0$. It follows that $\mathrm{r.ann}(by) \supsetneq \mathrm{r.ann}(y) = \mathrm{r.ann}(Y)$, which contradicts the choice of $\mathrm{r.ann}(Y)$ unless $A^n by = 0$. This shows that $A^{n+1}y = 0$, and so, by the the choice of n, $A^n y = 0$. Hence $A^{n+1} = 0$. □

Proposition 1.22. *If R is a semiprime right non-singular ring with finite right uniform dimension, and if $c \in R$ is right regular, then c is regular and $cR \leq_e R_R$.*

Proof. Since $cR \cong R$, we have $\mathrm{u.dim}\, cR = \mathrm{u.dim}\, R_R$. Therefore $cR \leq_e R_R$, by Corollary 1.16. Since

$$cR \subseteq \mathrm{r.ann}(\mathrm{l.ann}(cR))$$

we have $\mathrm{r.ann}(\mathrm{l.ann}(cR)) \leq_e R_R$. $\mathrm{l.ann}(cR) \subseteq \zeta(R) = 0$. Hence $\mathrm{l.ann}(c) = 0$. □

Proposition 1.23. *Let R be semiprime right non-singular ring of finite right uniform dimension, and let I be a right ideal of R.*

(i) *I contains an element c such that $\mathrm{r.ann}(c) \cap I = 0$.*

(ii) *I is essential in R_R if and only if I contains a regular element of R.*

Proof. ***(i)*** *First consider the case when I is uniform. Since $I^2 \neq 0$, $cd \neq 0$ for some $c, d \in I$. Let $V = \mathrm{r.ann}(c) \cap I$ and suppose $V \neq 0$. Since I is uniform, V is an essential submodule of I. Thus by Lemma 1.14 (iv), we have $d^{-1}V$ is essential in R_R. Since $cd(d^{-1}V) = 0$, we obtain $cd \in \zeta(R) = 0$, a contradiction. Therefore $\mathrm{r.ann}(c) \cap I = 0$.*

Now consider the general case. Choose a uniform right ideal $U_1 \subseteq I$, and an element $a_1 \in U_1$ such that $\mathrm{r.ann}(a_1) \cap U_1 = 0$. If $\mathrm{r.ann}(a_1) \cap I \neq 0$, then choose a uniform right ideal U_2 in $\mathrm{r.ann}(a_1) \cap I$, and choose $a_2 \in U_2$, with $\mathrm{r.ann}(a_2) \cap U_2 = 0$. So far, we have got

$$a_1R \oplus a_2R \oplus (\mathrm{r.ann}(a_1) \cap \mathrm{r.ann}(a_2) \cap I) \subseteq I.$$

If we continue in this fashion, after a finite step we obtain elements $a_1, \ldots, a_n$ of R such that

$$a_1R \oplus a_2R \oplus \cdots \oplus a_nR \oplus (\mathrm{r.ann}(a_1) \cap \mathrm{r.ann}(a_2) \cap \ldots \cap \mathrm{r.ann}(a_n) \cap I) \subseteq I.$$

Since $\mathrm{u.dim}\, R_R < \infty$, this process must terminate– say at this stage. This means that

$$\mathrm{r.ann}(a_1) \cap \mathrm{r.ann}(a_2) \cap \ldots \cap \mathrm{r.ann}(a_n) \cap I = 0.$$

Let $c = a_1 + a_2 + \cdots + a_n \in I$. Since the sum $\sum_{i=1}^n a_iR$ is direct, it follows that $\mathrm{r.ann}(c) = \bigcap_{i=1}^n \mathrm{r.ann}(a_i)$. Therefore $\mathrm{r.ann}(c) \cap I = 0$.

(ii) If I is essential, then $\mathrm{r.ann}(c) = 0$, and so by Proposition 1.22, c is regular. Conversely, if $c \in I$ is regular, then $cR \leq_e R_R$, by Poroposition 1.22. Hence $I \leq_e R_R$. □

Now we are ready to prove Goldie's Theorem. As seen below, Goldie's Theorem describes those rings whose ring of quotients exist and are semisimple Artinian.

Theorem 1.24 (Goldie's Theorem)**.** *The following statements on a ring R are equivalent:*

(i) *R is a semiprime ring with finite right uniform dimension which satisfies the a.c.c. on right annihilator ideals.*

(ii) *R is a semiprime, right non-singular ring with finite right unifom dimension.*

(iii) *R has a right quotient ring Q which is semisimple Artinian.*

Moreover; R is prime if and only if Q is simple.

Proof. (i)⇒(ii): By Theorem 1.21.

(ii)⇒(iii): Let $r, s \in R$ with s regular. By Proposition 1.22, $sR \leq_e R_R$. Then by Lemma 1.14 (iv), $r^{-1}(sR) \leq_e R_R$. It follows, from Proposition 1.23 (ii), that $r^{-1}(sR)$ contains a regular element s', say. Then $rs' = sr'$ for some $r' \in R$. Sice $\mathscr{C}_R(0)$ consists of regular elements of R, by Ore's Theorem (Theorem 1.4), R has a right quotient ring Q, say.

To see that Q_Q is a semisimple module, it suffices to show that Q has no essential right ideal other than itself. Now let J be an essential right ideal of Q. By Lemma 1.18 (ii), $J \cap R \leq_e R_R$, and so contains a regular element of R, by Proposition 1.23. But then J contains a unit element of Q since regular elements of R are unit in Q. This gives that $J = Q$, as desired.

(iii)⇒(i): Since Q_Q is Artinian, we have $\mathrm{u.dim}\, Q_Q < \infty$. Thus by Lemma 1.18 (iv), R has finite right uniform dimension. On the other hand it is not difficult to see that the property of a.c.c. on right annihilator ideals in Q is inherited by R. Therefore, it remains to show that R is semiprime. Suppose that N is a nilpotent ideal of R. Let X be a nonzero right ideal of R such that $X \cap \mathrm{l.ann}(N) = 0$. Since N is nilpotent, there exists a non-negative integer l such that $XN^l \neq 0$ and $XN^{l+1} = 0$ (where we assume $N^0 = R$). Thus XN^l lies in $X \cap \mathrm{l.ann}(N)$, a contradicion. It follows that $\mathrm{l.ann}(N)$ is an essential right submodule of R_R. Therefore by Lemma 1.18 (i), $\mathrm{l.ann}(N)Q$ is an essential right ideal of Q. However, since Q is semisimple, every right ideal of Q is a direct summand of Q. It follows that $\mathrm{l.ann}(N)Q = Q$. Hence, by Proposition 1.10 (ii), $1 = ac^{-1}$ for some $a \in \mathrm{l.ann}(N)$ and $c \in R$ with c regular. But then $a = c \in \mathrm{l.ann}(N)$, so $N = 0$.

For the last equivalence of the theorem, first assume that Q is simple Aritinian. Let A be a nonzero ideal of R. Then, by assumption on Q, $QAQ = Q$, and so, using Lemma 1.8, we get $1 = \sum_i r_i c_i^{-1} a_i d^{-1}$with $r_i \in R$, $a_i \in I$ and $c_i, d \in \mathscr{C}_R(0)$. Therefore, $d \in QA$, and hence $QA = Q$. Now, if J is another nonzero ideal of R, since $QIJ = QJ = Q$, we have $IJ \neq 0$.

Now let R be a prime ring with finite right uniform dimension which satisfies the a.c.c. on right annihilator ideals. We have already proved that Q is semisimple Artinian. Let X be a nonzero ideal of Q. Then $X \cap R$ is a nonzero ideal of R. By Lemma 1.13 (i), $X \cap R$ is essential in R_R. Hence, $X = Q$ as before, and so Q is simple. □

Any ring R satisfying one of the equivalent conditions of Theorem 1.24 is called a **semiprime right Goldie ring**.

Corollary 1.25. *Any semiprime right Noetherian is a right Goldie ring, and hence has a semisimple Artinian right quotient ring.*

Corollary 1.26. *Let S be a right denominator set in the ring R and assume that R_S is right Noetherian (for example, R could be right Noetherian). If Q is a semiprime ideal of R_S, then the factor ring $R/(Q \cap R)$ is a semiprime right Goldie ring, and the right quotient rings of $R/(Q \cap R)$ and R_S/Q are isomorphic.*

Proposition 1.27. *Let R be a semiprime right Goldie ring with the quotient ring Q.*

(i) *An annihilator right ideal of R is of the form $J \cap R$ for $J \leq Q_Q$.*

(ii) *A right ideal U of R is uniform if and only if UQ is a minimal right ideal of Q.*

(iii) *If U is a uniform right ideal of R and $0 \neq u \in U$, then*

a) $\operatorname{u.dim}(\operatorname{r.ann}(u)) = \operatorname{u.dim} R_R - 1$ *and*

b) if I is a right ideal of R with $\operatorname{r.ann}(u) \subsetneq I$, then $I \leq_e R_R$.

Proof. ***(i)*** *Let $I \neq R$ be an annihilator right ideal of R with $I = \operatorname{r.ann}(X)$ for some $X \subseteq R$. Let $a \in IQ \cap R$. Then by Proposition 1.10, $as \in I$ for some regular element $s \in R$. Hence $Xas = 0$, and so $Xa = 0$ by regularity of s. It follows that $I = IQ \cap R$, as desired.*

(ii) Let U be a uniform right ideal of R. By Lemma 1.18 (iii), UQ is a uniform right ideal of Q. But Q is semisimple Artinian, which gives that UQ is minimal.

(iii) By (ii), UQ is a minimal right ideal of Q; hence $UQ = uQ$. It follows that $\operatorname{r.ann}_Q(u)$ is a maximal right ideal of Q. This gives that $\operatorname{u.dim}(\operatorname{r.ann}_Q(u)) = \operatorname{u.dim} Q_Q - 1$. But since $\operatorname{r.ann}_R(u) = \operatorname{r.ann}_Q(u) \cap R$, by Lemma 1.18 (iii) and (iv), we obtain

$$\operatorname{u.dim}(\operatorname{r.ann}_R(u)) = \operatorname{u.dim}(\operatorname{r.ann}_Q(u)) = \operatorname{u.dim} Q_Q - 1 = \operatorname{u.dim} R_R - 1.$$

Now let I be a right ideal of R with $\operatorname{r.ann}(u) \subsetneq I$. Since

$$\operatorname{u.dim} R_R - 1 = \operatorname{u.dim}(\operatorname{r.ann}(u)) \leq \operatorname{u.dim} I_R \leq \operatorname{u.dim} R_R,$$

we have either $\operatorname{u.dim} I_R = \operatorname{u.dim}(\operatorname{r.ann}(u))$ or $\operatorname{u.dim} I_R = \operatorname{u.dim} R_R$. In the former case, we have $\operatorname{r.ann}(u) \leq_e I$. This gives a contradiction since

$$\operatorname{r.ann}_Q(u) = \operatorname{r.ann}_R(u)Q \leq IQ \lneq Q$$

together with the maximality of $\operatorname{r.ann}_Q(u)$ implies that $\operatorname{r.ann}_R(u) = IQ \cap R \supseteq I$. Therefore $\operatorname{u.dim} R_R = \operatorname{u.dim} I_R$, and so $I \leq_e R_R$. □

Proposition 1.28. *Let R be a semiprime right Goldie ring with the quotient ring Q, I a right ideal of R and $b \in R$. Then there exists $d \in I$ such that $\operatorname{u.dim}(b+d)R = \operatorname{u.dim}(bR + I)$.*

Proof. If $\operatorname{u.dim}(bR + I) = \operatorname{u.dim} bR$, then one can take $d = 0$. Otherwise there exits $U \subseteq I$, a uniform right ideal such that $bR \cap U = 0$. By induction, it is enough to find $u \in U$ such that $\operatorname{u.dim}(b+u)R = \operatorname{u.dim}(bR \oplus U) = \operatorname{u.dim}(bR) + 1$.

We first show that $\operatorname{r.ann}(b) \nsubseteq \operatorname{r.ann}(U)$. Suppose contrarily that $\operatorname{r.ann}(b) \subseteq \operatorname{r.ann}(U) = A$, and let $A' = \operatorname{l.ann}(A)$. Now A is an ideal of R and $A' \cap A = 0$ (since $(A' \cap A)^2 = 0$ and R is semiprime). Therefore $A' \cong bA' \subseteq A'$, and so $\operatorname{u.dim} A' = \operatorname{u.dim} bA'$, which yields $bA' \leq_e A'$. However, $U \subseteq A'$, and so $0 \neq bA' \cap U \subseteq bR \cap U$, a contradiction.

Therefore there is some $u \in U$ such that $\operatorname{r.ann}(b) \nsubseteq \operatorname{r.ann}(u)$. Then $\operatorname{r.ann}(b) + \operatorname{r.ann}(u)$ is an essential right ideal of R, by 1.27 (iii-b), and so contains a regular element c of R. Write $c = x + y$, where $bx = uy = 0$. Note that $bc = (b+u)y$ and $uc = (b+u)x$. Choose any element $br + us \in bR \oplus uR$. The right Ore condition gives elements $r', s' \in R$, $c' \in \mathscr{C}_R(0)$ such that $rc' = cr'$ and $sc' = cs'$ (see Lemma 1.5). But then

$$(br + us)c' = bcr' + ucs' = (b+u)(yr' + xs') \in (b+u)R.$$

Hence

$$\mathrm{u.dim}(b+u)R = \mathrm{u.dim}(bR \oplus uR) = \mathrm{u.dim}\, bR + 1.$$

□

Proposition 1.29. *If Q is a right Artinian ring, then every right regular element of Q is a unit.*

Proof. Let s be a right regular element in Q and consider the descending chain $\{s^n Q\}$ of right ideals. This stabilizes with, say, $s^n Q = s^{n+1}Q$. Thus $s^n = s^{n+1}q$ for some $q \in Q$. Since s is right regular, so too is s^n, and hence $sq = 1$. This shows, in particular, that s is also left regular. Finally,

$$s(qs-1) = (sq-1)s = 0.$$

Thus $sq = qs = 1$, as desired. □

Lemma 1.30. *Let R be a semiprime right Goldie ring with the quotient ring Q.*

(i) A right ideal E of R is essential if and only if $EQ = Q$.

(ii) A principal right ideal cR is essential if and only if $c \in \mathscr{C}_R(0)$.

(iii) If R is also left Goldie, then a right ideal E of R is essential if and only if $\mathrm{l.ann}(E) = 0$.

Proof. ***(i)*** *This follows from 1.18 since Q is the only essential right ideal of Q.*

(ii) If $cR \leq_e R_R$, then $cQ = Q$. Thus using the left analogue of Proposition 1.29, c is a unit in Q, and hence regular in R. The converse is clear by 1.23 (ii).

(iii) It is clear from (i) that if E is essential, then $\mathrm{l.ann}(E) = 0$. Conversely, □

Corollary 1.31. *Let R be a semiprime right Goldie ring with the quotient ring Q. Then each essential right ideal of R is generated by regular elements.*

Proof. Let E be an essential right ideal of R. By Proposition 1.23 (ii), there is a regular element $c \in E$. By Proposition 1.28, given any $b \in E$, there exists $d \in cR$ such that

$$\mathrm{u.dim}(b+d)R = \mathrm{u.dim}(bR + cR) = \mathrm{u.dim}\, R_R.$$

Therefore $(b+d)R \leq_e R_R$ and by Lemma 1.30 $b + d \in \mathscr{C}_R(0)$. Since $b \in (b+d)R + cR$, the result is proved. □

2 Multiplicative Theory of Ideals

2.1 Orders in Quotient Rings

Corollary 1.26 shows that there are some rings having the same right quotient rings. For another example, one can consider the pair of rings $k[X;\sigma]$ and $k[X,X^{-1};\sigma]$ where k is a field and σ is an automorphism on k. In this section we explore this phenomenon and its relations with earlier results.

A ring Q is called a **quotient ring** if every regular element of Q is a unit. The following proposition shows that there is a good supply of quotient rings.

Proposition 2.1. *Any right Artinian ring is a quotient ring.*

Proof. This is just a restatement of $\square$

Given a quotient ring Q, a subring R, not necessarily containing 1, is called a **right order** in Q if each $q \in Q$ has the form rs^{-1} for some $r, s \in R$. A **left order** is defined analously; and a left and right order is called an **order**. Since Q has 1, it does not matter to adjoin 1 to R. However, asssuming right orders without 1 is much more useful on occasion.

Proposition 2.2. *Let R be a subring (with 1) of a ring Q and let S be the set of all units of Q that lie in R.*

- ***(i)*** *If Q is the right quotient ring of R, then Q is a quotient ring, R is a right order in Q, and $S = \mathscr{C}_R(0)$.*
- ***(ii)*** *If Q is a quotient ring and R is a right order in Q, then $Q = R_S$. If, further, either R is also a left order in Q or Q is right Artinian, then $S = \mathscr{C}_R(0)$ and Q is the right quotient ring of R.*

Proof. ***(i)*** *If $q \in Q$ is regular, with $q = rs^{-1}$, where $r, s \in R$, $s \in \mathscr{C}_R(0)$, then $r = qs \in \mathscr{C}_R(0)$, and so it is a unit in Q. Hence q is a unit in Q and Q is a quotient ring. The fact that R is a right order in Q follows from the assumption.*

(ii) The first claim is an immediate consequence of the definitions. Notice that by our asumption, we already have $S \subseteq \mathscr{C}_R(0)$. Since R is a right order in Q, regular elements of R are right regular in Q. If R is also a left order in Q (or if Q is right Artinian, by Proposition 2.1), then regular elements of R are also left regular in Q. In this case, we have $\mathscr{C}_R(0) \subseteq \mathscr{C}_Q(0) \cap R = S$, and so $\mathscr{C}_R(0) = S$. Therefore Q is the right quotient ring of R. $\square$

Corollary 2.3. *Let Q be a semisimple Artinian ring and let R be a subring (with 1) of Q. If R is a right order in Q, then R is a semiprime right Goldie ring and Q is the quotient ring of R.*

Proof. Immediate from Proposition 2.2. □

Corollary 2.4. *R is a semiprime right Goldie ring with quotient ring Q if and only if $\mathbb{M}_n(R)$ is a semiprime right Goldie ring with right quotient ring $\mathbb{M}_n(Q)$.*

Proof. Let R be a semiprime right Goldie ring with right quotient ring Q. Since Q is semisimple Artinian, so too is $\mathbb{M}_n(Q)$. We will show that $\mathbb{M}_n(R)$ is a right order in $\mathbb{M}_n(Q)$. Let $x \in \mathbb{M}_n(Q)$. Taking common denominator, by Proposition 1.8, we can write x in the form $(a_{ij}c^{-1})$, where $a_{ij}, c \in R$. Using the natural embedding of Q into $\mathbb{M}_n(Q)$, this means that $x = ac^{-1}$ with $a, c \in \mathbb{M}_n(R)$. Thus $\mathbb{M}_n(R)$ is a right order in $\mathbb{M}_n(Q)$. By Colollary 2.3, $\mathbb{M}_n(R)$ is a semiprime right Goldie ring.

The converse is easy to prove. □

Proposition 2.5. *Let R be a right order in a quotient ring Q and let S be a subring of Q (not necessarily with 1).*

- ***(i)*** *If there are units a, b of Q such that $aRb \subseteq S$, then S is also a right order in Q. In particular, if $R \subseteq S \subseteq Q$, then S is also a right order in Q.*
- ***(ii)*** *If R is a prime right Goldie ring, A is a nonzero ideal of R such that $A \subseteq S \subseteq R$, then S is a prime right Goldie ring, and has the same right quotient ring as R.*

Proof. ***(i)*** *Given $q \in Q$, consider the element $a^{-1}qa$. Since R is a right order in Q, by definition, we have $a^{-1}qa = rt^{-1}$ for some $r, t \in R$. But then*

$$q = art^{-1}a^{-1} = arb(atb)^{-1},$$

so S is a right order in Q. The remainder follows by taking $a = b = 1$.

(ii) Since R is a prime ring, we have $A \leq_e R_R$, and therefore, by Proposition 1.23 (ii), A contains a regular element, c say. Then c is a unit in Q, and $cR \subseteq S$. Now the result follows from (i) above. □

Proposition 2.5 leads us to the ideal of equivalent orders as follows. Let R_1 and R_2 be right orders in a quotient ring Q. Define a relation

$$R_1 \sim R_2 \Leftrightarrow \text{there exist units } a_1, a_2, b_1, b_2 \in Q \text{ such that } a_1R_1b_1 \subseteq R_2 \text{ and } a_2R_2b_2 \subseteq R_1.$$

It is routine to check that this is an equivalence relation. The right orders R_1 and R_2 which lie in the same equivalence class is termed **equivalent** right orders.

For examples, we can give the matrix ring $\mathbb{M}_2(\mathbb{Z})$ and its subring

$$\begin{bmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix};$$

and the first Weyl algebra $A_1(k)$ and its subring $k + XA_1(k)$.

The following lemma is very useful when dealing with extensions of rings which are equivalent orders in a quotient ring.

Lemma 2.6. *Suppose R, S are equivalent right orders in Q with $R \subseteq S$. Then there are equivalent right orders T, T' in Q with*

$$R \subseteq T \subseteq S, \quad R \subseteq T' \subseteq S,$$

and units r_1, r_2 of Q contained in R such that

$$r_1 S \subseteq T, \ Tr_2 \subseteq R, \ \text{and} \ Sr_2 \subseteq T', \ r_1 T' \subseteq R.$$

In particular, $r_1 S r_2 \subseteq R$.

Proof. By definition, $aSb \subseteq R$ for some units a, b of Q. Say $a = r_1 s_1^{-1}$, $b = r_2 s_2^{-1}$, with $r_i, s_i \in R$. Then

$$r_1 S r_2 \subseteq r_1 s_1^{-1} S r_2 \subseteq R s_2 \subseteq R.$$

Now if we set

$$T = R + r_1 S + R r_1 S \quad \text{and} \quad T' = R + S r_2 + S r_2 R,$$

then we are done. □

2.2 Fractional Ideals

It is now appropriate to introduce the notion of fractional R-ideals by which we can produce right and left orders equivalent to R.

Let R be a right (or left) order in a quotient ring Q. A non-zero additive subgroup I of Q is a **fractional right R–ideal** provided (i) $IR \subseteq I$, (ii) I contains a unit of Q, and (iii) there exists a unit $b \in Q$ such that $bI \subseteq R$. If, further, $I \subseteq R$, then I is an **integral** right R–ideal. In the same way, one defines **fractional left R–ideals** and **fractional (two-sided) R–ideals**. In what follows we usually drop the term "fractional" and call these ideals simply (right, left, or two-sided) R–ideals. Let S be another right (or left) order in Q. If I is a right R–ideal and left S–ideal, then we often indicate this situation by saying that I is an (S, R)–ideal.

Remark 2.7. Let R be a semiprime right Goldie ring with quotient ring Q. Since any essential right ideal of R contains a regular element by 1.23, essential right ideals of R become integral R–ideals. Moreover, if I is any right R–ideal, then I is isomorphic to an integral right R–ideal. To see this let a and b be units of Q such that $a \in I$ and $bI \subseteq R$. Then bI is an integral right R–ideal since b is unit and $ba \in \mathscr{C}_R(0) \cap bI$.

For a concrete example of fractional ideals, one can consider the ring

$$R = \begin{bmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

Then $\mathbb{M}_2(\mathbb{Z})$ is an R–ideal.

Given a right R–ideal I, the **right order** and the **left order** of I are defined respectively to be the subrings of Q

$$\mathcal{O}_r(I) = \{q \in Q : Iq \subseteq I\},$$

and

$$\mathcal{O}_l(I) = \{q \in Q : qI \subseteq I\}.$$

For example if

$$I = \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix} \subset Q = \mathbb{M}_2(\mathbb{Q}),$$

then

$$\mathcal{O}_r(I) = \mathbb{M}_2(\mathbb{Z}) \quad \text{and} \quad \mathcal{O}_l(I) = \begin{bmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \frac{1}{2}\mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

Now we give the basic facts about right and left orders of I in the following

Lemma 2.8. *Let R be a right order in a quotient ring Q and let I be a right or left R–ideal. Then:*

- ***(i)*** *$\mathcal{O}_r(I)$ and $\mathcal{O}_l(I)$ are right orders in Q and are equivalent to R, and*
- ***(ii)*** *I is a $(\mathcal{O}_l(I), \mathcal{O}_r(I))$–ideal.*

Proof. We may assume that I is a right R–ideal since the alterative can be dealt with symmetrically.

(i) Let a and b be units of Q such that $a \in I$ and $bI \subseteq R$. Then

$$ba\mathcal{O}_r(I) \subseteq R \subseteq \mathcal{O}_r(I)$$

and

$$b\mathcal{O}_l(I)a \subseteq b\mathcal{O}_l(I)I \subseteq bI \subseteq R.$$

Also, since $aRbI \subseteq aR \subseteq I$, we have $aRb \subseteq \mathcal{O}_l(I)$. This establishes (i).

(ii) Obvious. □

We define the inverse of a right (or left) R–ideal I to be the set

$$I^{-1} = \{q \in Q : IqI \subseteq I\}.$$

Let $I \leq Q_R$. One can define the mapping $I \times Q \to IQ$ by $(a, rs^{-1}) \mapsto ars^{-1}$, which is clearly bilinear. This gives a homomorphism $I \otimes_R Q \to IQ$, which we shall denote by η_I. It is easy to see that any element of the right ideal IQ of Q has form as^{-1} for $a \in I$ and $s \in \mathcal{S}$. Now define a map $IQ \to I \otimes_R Q$ by $as^{-1} \mapsto a \otimes s^{-1}$ for all $a \in I$, $s \in \mathcal{S}$. This is a well-defined mapping. To see that choose $a, b \in R$ and $s, t \in \mathcal{S}$ such that $as^{-1} = bt^{-1}$. Since $Q = R_\mathcal{S}$, $\mathcal{S}$ is an Ore set. It follows that there exist $s_1 \in \mathcal{S}$ and $c \in R$ such that $ss_1 = tc$. Then $as_1 = as^{-1}tc = bt^{-1}tc = bc$, and so $a \otimes s^{-1} = a \otimes s_1 s_1^{-1} s^{-1} = as_1 \otimes s_1^{-1}s^{-1} = bc \otimes s_1^{-1}s^{-1} = b \otimes cs_1^{-1}s^{-1} = b \otimes t^{-1}$. Therefore we have two unambigious maps one from $I \otimes_R Q$ into IQ and one from IQ into $I \otimes_R Q$, whose composition is the identity map in either order. In summary, we have obtained that $\eta_I : I \otimes_R Q \to IQ$ is an isomorphism of right Q–modules. Let J be another right R–submodule of Q. Then we have an isomorphism

$$\mathrm{Hom}_Q(I \otimes_R Q, J \otimes_R Q) \longrightarrow \mathrm{Hom}_Q(IQ, JQ)$$

given by $\gamma \mapsto \eta_J \gamma \eta_I^{-1}$. Thus, we can define a homomorphism

$$\begin{aligned} \operatorname{Hom}_R(I, J) &\longrightarrow \operatorname{Hom}_Q(IQ, JQ) \\ \alpha &\longmapsto \tilde{\alpha} \end{aligned}$$

of abelian groups, where $\tilde{\alpha} : IQ \to JQ$ is defined by $\tilde{\alpha}(as^{-1}) = \alpha(a)s^{-1}$. This is clearly an embedding. Under this embedding $\operatorname{Hom}_R(I, J) \cong \{\beta \in \operatorname{Hom}_Q(IQ, JQ) : \beta I \subseteq J\}$. Assume, further, that I is a right R–ideal. Then $IQ = Q$, and so $\operatorname{Hom}_R(I, J) \cong \{\beta \in \operatorname{Hom}_Q(Q, JQ) : \beta I \subseteq J\}$. If $\beta : Q \to JQ$ is an R–homomorphism with $\beta(1) = q$, then for any $q' = rs^{-1} \in Q$, we have $\beta(rs^{-1})s = qr$, which implies that $\beta(q') = qq'$. It follows that any R–homomorphism from Q into JQ is a left multiplication by an element of Q. This shows that $\operatorname{Hom}_Q(Q, JQ) = \operatorname{Hom}_R(Q, JQ)$, and also that

$$\operatorname{Hom}_R(I, J) \cong \{q \in Q : qI \subseteq J\}$$

where the abelian group on the right is denoted by $(J : I)_l$. Moreover, if $\alpha \in \operatorname{Hom}_R(I, J)$ and q_α is the corresponding elements of Q in $(J : I)_l$, then α and q_α act on I in exactly the same way, i.e., for any $a \in I$, $\alpha(a) = q_\alpha(a)$.

One can also prove that if I is a right R–ideal of, then there is an isomorphism $\operatorname{End}_R(I) \cong \mathcal{O}_l(I)$ of rings. Note that if J is a (two-sided) R–ideal and I is a right R–ideal, then $(J : I)_l$ is a left R–ideal. Alternatively, one can consider the subgroup $(J : I)_r = \{q \in Q : Iq \subseteq J\}$ for which an analogous isomorphism can be given just as above under the hypothesis that R is a left order in Q and I is a left R–ideal. In the particular case when $J = R$ above, we write $(R : I)_l = I^*$ and $(R : I)_r = I^+$. It follows, from what we have just observed, that if I is a fractional right R–ideal, then I^* is a left R–ideal and $I^* \cong \operatorname{Hom}_R(I, R)$, as R–modules. Moreover, we may write I^*I is an ideal of R equal to the ideal $\operatorname{Hom}(I_R, R)I$. Similarly, if R is a left order in Q and I is a left R–ideal, then $I^+ \cong \operatorname{Hom}({}_RI, R)$ and II^+ is an ideal of R equal to the ideal $I \operatorname{Hom}({}_RI, R)$. To sum up, we can conclude that the following statements hold for a right R–ideal I:

(1) I is an $(\mathcal{O}_l(I), \mathcal{O}_l(I))$–ideal while I^{-1} is an $(\mathcal{O}_r(I), \mathcal{O}_l(I))$–ideal;

(2) $(\mathcal{O}_r(I) : I)_l = I^{-1} = (\mathcal{O}_l(I) : I)_r$;

(3) $I^* \subseteq I^{-1}$ (for a left R–ideal J, $J^+ \subseteq J^{-1}$);

(4) I^* is an $(R, \mathcal{O}_l(I))$–ideal, so that I^*I is an ideal of R and II^*, II^{-1} are both ideals of $\mathcal{O}_l(I)$ such that $II^* \subseteq II^{-1} \subseteq \mathcal{O}_l(I)$;

(4) II^+ is a right ideal of R and I^+I is an ideal of R; if, further, I is a left R–ideal, then $I^+$is an $(\mathcal{O}_r(I), R)$–ideal, and $II+$ is an ideal of R.

Let I be an R–ideal. We say that I is *invertible* if there exists an R–ideal J with $IJ = JI = R$. If I is an invertible R–ideal, then it is easy to see that $\mathcal{O}_r(I) = R = \mathcal{O}_l(I)$ and $I^* = I^+ = I^{-1}$; in this case, we have $I^{-1}I = II^{-1} = R$.

Now let I be a right R–ideal. Then I is also a right $\mathcal{O}_r(I)$–ideal. Since $\mathcal{O}_r(I)$ is a right order in Q, every element of Q has the form ab^{-1} for elements $a, b \in \mathcal{O}_r(I)$. Also, $IQ = Q$, and so each element of Q has the form $ac^{-1}, a \in I$, $c \in \mathcal{O}_r(I)$. Let I be projective over $\mathcal{O}_r(I)$. Then there exist families $\{a_j\}$ and $\{\beta_j\}$, where $a_j \in I$, $\beta_j \in \operatorname{Hom}_R(I, \mathcal{O}_r(I))$, such that for every $a \in I$, $\beta_j(a) = 0$ for all but a finite number of j and $a = \sum_j a_j\big(\beta_j(a)\big)$. We know, by above remarks, that $\mathrm{Hom}_R(I, \mathcal{O}_r(I)) \cong (\mathcal{O}_r(I) : I)_l = I^{-1}$. Let $\beta_j \mapsto q_j$ under this isomorphism. Then for any $a \in I$, $q_j a = \beta_j a = 0$ for all but a finite number of j. Choosing a to be reqular gives that $q_j = 0$ for all but a finite number of j, and

$$a = \sum_j a_j\big(\beta_j(a)\big) = \sum_j a_j(q_j a) = \Big(\sum_j a_j q_j\Big)a,$$

which implies that $1 = \sum_j a_j q_j \in II^{-1}$. It follows that $II^{-1} = \mathcal{O}_l(I)$. It is also immediate that I_R is finitely generated (by the set $\{a_j : q_j \neq 0\}$).

Converely, if $II^{-1} = \mathcal{O}_l(I)$ for a right R–ideal I, then it is easy to see that I is a projective right $\mathcal{O}_r(I)$–ideal. Thus we have partialy proved the following

Proposition 2.9. $(\boldsymbol{i})$ *Let I be a right R–ideal. Then $\mathcal{O}_l(I) = II^{-1}$ if and only if I is a projective right $\mathcal{O}_r(I)$–ideal; and then I is a finitely generated right $\mathcal{O}_r(I)$–ideal.*

$(\boldsymbol{ii})$ *If J is a left R–ideal such that $J^{-1}J = \mathcal{O}_r(J)$, then J is a finitely generated projective left $\mathcal{O}_l(J)$–ideal.*

Proof. (i) It remains only to show that if $\mathcal{O}_l(I) = II^{-1}$, then I is a projective right $\mathcal{O}_r(I)$–ideal. By ssumption $1 = \sum_{i=1}^n a_i q_i$ for some $a_i \in I$ and $q_i \in I^{-1}$. Define a map $f_i : I \to \mathcal{O}_r(I)$ by $f_i(a) = q_i a$. Then, clearly, $f_i \in \mathrm{Hom}_R(I, \mathcal{O}_r(I))$, and for any $a \in I$,

$$a = \Big(\sum_{i=1}^n a_i q_i\Big)a = \sum_{i=1}^n a_i(q_i a) = \sum_{i=1}^n a_i\big(f_i(a)\big).$$

This gives that I is projective as a right $\mathcal{O}_r(I)$–module.

(ii) The proof of (i) above can be slightly modified to give (ii). □

Note that a similar proposition can be given for a left R–ideal if we take R to be a left order in Q. This observation will be used in the proof of Corollary 2.11.

Corollary 2.10. *Let I be a right R–ideal. Then I is a projective right R–ideal if and only if and $II^* = \mathcal{O}_l(I)$; and in this case I is a finitely generated right R–ideal.*

Proof. In view of the proof of above proposition, we obtain that I_R is projective if and only if $1 \in II^*$, and that I is finitely generated as a right R–module. □

Corollary 2.11. *Let I be an R–ideal. If I is an invertible R–ideal, then $\mathcal{O}_r(I) = R = \mathcal{O}_l(I)$ and ${}_RI$, I_R are (finitely generated) projective.*

Moreover, if R is also a left order in Q, then the converse is true.

Proof. Assume first that I is an invertible R–ideal. We already know that $\mathcal{O}_r(I) = \mathcal{O}_l(I) = R$ and $I^{-1}I = II^{-1} = R$. By Proposition 2.9 (i) and (ii) we obtain that I is a finitely generated projective right and left R–module.

Now assume that R is a two-sided order in Q. Let I be an R–ideal such that $\mathcal{O}_r(I) = R = \mathcal{O}_l(I)$, and that I is projective as a right and left R–module. Then by Proposition 2.9 (i), $II^{-1} = R$. On the other hand, if we apply the same proposition for I as a left R-ideal with R being a left order, we also conclude that $I^{-1}I = R$. □

Notice that the above corollary shows that, for a general ring R, projectivity may not always lead to invertibility, in contrast with the commutative case.

2.3 Maximal Orders

Let R be a right (or left) order in a quotient ring Q. If R is not contained in any equivalent right order, then R is called a **maximal right order**. **Maximal left orders** and **maximal** (**two-sided**) **orders** are defined analogously. Let R be a maximal right or left order. If I is a right (resp. left) R–ideal, then, clearly, $\mathcal{O}_r(I) = R$ (resp. $\mathcal{O}_l(I) = R$). Moreover, for a two-sided R–ideal, we have $I^* = I^+ = I^{-1}$.

Proposition 2.12. *The following conditions on R are equivalent:*
(1) *R is a maximal right order.*
(2) *$\mathcal{O}_r(I) = \mathcal{O}_l(I) = R$ for all R–ideals.*
(3) *$\mathcal{O}_r(I) = \mathcal{O}_l(I) = R$ for all integral R–ideals.*

Proof. $(1) \Rightarrow (2)$: This part has been given before the proposition.
$(2) \Rightarrow (3)$: Trivial.
$(3) \Rightarrow (1)$: Let S be a right order in Q equivalent to R such that $S \supseteq R$. By [**?**, Lemma 3.1.10], we may choose a right order T and unit elements r_1 and r_2 in Q such that $R \subseteq T \subseteq S$, $r_1 S \subseteq T$ and $Tr_2 \subseteq R$. Set $I = \{r \in R : Tr \subseteq R\}$. Then I is evidently an integral right R–ideal and $T \subseteq \mathcal{O}_l(I)$. Therefore $T = R$. By symmetry, we also have $S = R$. This completes the proof. □

It is reasonable to ask what kind of commutative rings can be maximal orders. The following theorem answers this question.

Theorem 2.13. *commutative rings which are maximal orders*

Theorem 2.14. *Let R be a maximal right order in a quotient ring Q. Then the following hold:*
(i) *Let I be a right R–ideal. Then I is a projective right R–ideal if and only if $II^{-1} = \mathcal{O}_l(I)$.*
(ii) *Let T be a (two-sided) R–ideal. Then T is a projective right R–ideal if and only if $TT^{-1} = R$; and then T is a finitely generated right R–ideal.*

Let R be any ring and M_R a right R–module. Let $M^* = \mathrm{Hom}_R(M, R)$, the dual of M. M_R is said to be *torsionless* if given any $0 \neq m \in M$, there exists $\alpha \in M^*$ such that $\alpha(m) \neq 0$. It is easy to see that this is equivalent to saying that M embeds in some direct product of copes of R. Note that for any module M_R, ${}_RM^*$ is torsionless since if $0 \neq \alpha \in M^*$ and $m \in M$ is such that $\alpha(m) \neq 0$, then the map $M^* \to R$ by $\beta \mapsto \beta(m)$ is a required.

We now define $M^{**} = \mathrm{Hom}_R(M^*, R)$. There is an obvious homomorrphism $M \to M^{**}$ this being an embedding if and only if M is torsionless. When this is the case, it is convenient to indentify M with its image in M^{**}. The module M_R is called *reflexive* if M is torsionless and $M = M^{**}$. It is routine to check that a finite direct sum of reflexive modules is again reflexive, and that any direct summand of a reflexive module is again reflexive. This shows, in particular, that every finitely generated projective module is reflexive.

We continue to assume R is a right order in a quotient ring Q. Let I be a right R–ideal. Then I is a torsionless right R–module since if b is a unit of Q such that $bI \subseteq R$, then the mapping $I \to R$ defined by $x \mapsto bx$ is an injective R–homomorphism. (Similarly every left R–ideal is a torsionless left R–module.) We say that I is a *reflexive right R–ideal* if it is reflexive as a right R–module. Similarly reflexive left R–ideals are defined. A (two-sided) R–ideal is called a *reflexive R–ideal* if it is reflexive as both left and right R–module.

Proposition 2.15. *Let R be a right order in a quotient ring Q, and let I be a right R–ideal. If I is a reflexive right R–ideal, then $I = I^{*+}$.*

Moreover, if R is a two-sided order in Q, then the converse is also true.

Proof. We already know that $I \subseteq I^{*+}$. For the reverse inclusion, let $q \in I^{*+}$. Recall that $I^* \cong \mathrm{Hom}_R(I, R)$. Let q_f denote the element of I^* corresponding to $f \in \mathrm{Hom}_R(I, R)$ under this isomorphism. Define

$$\begin{aligned} \hat{q} : \mathrm{Hom}_R(I, R) &\longrightarrow R \\ f &\longmapsto q_f \cdot q. \end{aligned}$$

Clearly, $\hat{q}$ is an R–homomorphism. Since I_R is reflexive, there exists $a \in I$ such that $\hat{q}(f) = f(a)$ for all $f \in \mathrm{Hom}_R(I, R)$. But we know that $f(a) = q_f a$; so that $q_f a = q_f q$ for every $f \in \mathrm{Hom}_R(I, R)$. Since I^* is a left R–ideal, we may choose q_f to be regular; so that $q = a \in I$. This establishes the first statement.

Now let R be a two-sided order in Q, and let I be a right R–ideal such that $I = I^{*+}$. Let $\varphi : \mathrm{Hom}_R(I, R) \to R$ be a homomorphism of left R–modules. There is an R–isomorphism $\mathrm{Hom}_R(I, R) \to I^*$ which maps every $f \in \mathrm{Hom}_R(I, R)$ to an element in I^*, denoted by q_f, and whose inverse is given by $y \mapsto \hat{y}$, where $\hat{y} : I \to R$ is defined by $a \mapsto ya$. Define $\varphi' : I^* \to R$ by $\varphi'(y) = \varphi(\hat{y})$ for all $y \in I^*$. Then φ' is an R–homomorphism of left R–modules. Since R is also a left order in Q and I^* is a left R–ideal, we have $\mathrm{Hom}_R(I^*, R) \cong I^{*+} = I$. Let a be the element of I corresponding to φ' under this isomorphism. It follows that for every $f \in \mathrm{Hom}_R(I, R)$

$$\varphi(f) = \varphi(\hat{q_f}) = \varphi'(q_f) = q_f a = f(a),$$

which completes the proof. □

For a right R–ideal I, we always have $I^* = I^{*+*}$; so I^* is always a reflexive left R–ideal when R is a two-sided order in Q. Similarly, if J is a left R–ideal, then J^+ is a reflexive right R–ideal under the same assumption.

A right order R in Q whose R–ideals form a group under multiplication is called an *Asano right order*. Thus an Asano right order is precisely a right order whose R–ideals are all invertible.

Theorem 2.16. *Let R be a right order in a quotient ring Q. Then the following conditions on R are equivalent:*

(1) *R is an Asano right order.*

(2) *R is a maximal right order and every integral R–ideal is a projective right R–ideal.*

(3) *The R–ideals form an abelian group under multiplication.*

Proof. $(1) \Rightarrow (2)$: Let I be an R–ideal. Since, by assumption, I is invertible, we must have $\mathcal{O}_r(I) = \mathcal{O}_l(I) = R$. Since I is arbitrary, Proposition 2.12 shows that R is a maximal right order. Also by Corollary 2.11, I is a projective right R–ideal.

$(2) \Rightarrow (3)$: Since the integral R–ideals are all finitely generated (by Theorem 2.14) R satisfies the ascending chain condition for integral R–ideals. Let M be a maximal proper integral R–ideal. Notice that M is a maximal ideal of R. Then by Theorem 2.14, $MM^{-1} = R$. Let $M^{-1}M = M$. Multiplying both sides from right by M^{-1} gives $M^{-1} = M^{-1}MM^{-1} = MM^{-1} = R$. But this implies that $R = MM^{-1} = MR = M$, a contradiction. It follows that $M^{-1}M \neq M$. But $R \supseteq M^{-1}M \supset M$, and M is maximal. Hence $R = M^{-1}M$.

Next, let I be any integral R–ideal. By the maximal condition on proper integral R–ideals, there exists a maximal proper integral R–ideal M_1 such that $I \subseteq M_1$. Then $I \subseteq M_1^{-1}I \subseteq R$. If $I = M_1^{-1}I$, then $R = II^{-1} = M_1^{-1}II^{-1} = M_1^{-1}R = M_1^{-1}$, a contradiction. Thus $I \subset M_1^{-1}I$. If $M_1^{-1}I \neq R$, then $M_1^{-1}I \subseteq M_2$ for some maximal integral R–ideal M_2, and $M_1^{-1}I \subset M_2^{-1}M_1^{-1}I$. Using the ascending chain condition, it follows that, for some integer n, $M_n^{-1} \dots M_2^{-1}M_1^{-1}I = R$, where the M_i are maximal. Hence $I = M_1M_2 \dots M_n$.

We have showed that each proper integral R–ideal is a product of maximal proper integral R–ideals. Let M_1, M_2 be two maximal proper integral R–ideals. If we prove that $M_1M_2 = M_2M_1$, then it follows that multiplication of integral R–ideals is commutative. It $M_1 = M_2$, then there is nothing to prove. Thus assume that $M_1 \neq M_2$. Since $M_1 \cap M_2 \subseteq M_1$ and $M_1^{-1}M_1 = R$, we have $M_1^{-1}(M_1 \cap M_2) \subseteq R$. Since M_2 is a prime ideal of R such that $M_1 \nsubseteq M_2$, and $M_1\left[M_1^{-1}(M_1 \cap M_2)\right] = M_1 \cap M_2 \subseteq M_2$, we must have $M_1^{-1}(M_1 \cap M_2) \subseteq M_2$. It follows that $M_1 \cap M_2 = M_1M_2$. By symmetry, we have $M_1M_2 = M_2M_1$.

Now let T be any R–ideal, and let $J = \{r \in R : rT \subseteq R\}$. There exists a unit element q of Q such that $qT \subseteq R$. Let $q = ab^{-1}$ for $a, b \in R$. As q is unit in Q, a is a regular element of R. Also $bT \subseteq T$ implies that $T \subseteq b^{-1}T$, which leads to $aT \subseteq ab^{-1}T \subseteq R$, proving that J is an integral R–ideal. It follows that JT is also an integral R–ideal. Say $J = Q_1 \dots Q_m$ and $ST = P_1 \dots P_n$,where the P_i, Q_i are maximal integral R–ideals. Then $T = Q_m^{-1} \dots Q_1^{-1}P_1 \dots P_n$. Thus multiplication of R–ideals is commutative.

The proof is now complete since the part $(3) \Rightarrow (1)$ is trivial. □

Remark. If R is a two-sided order in Q, then R is an Asano right order if and only if it is Asano left order, if and only if R is a maximal order whose R–ideals are projective as either right or left R–module.

Corollary 2.17. *Let R be a prime Goldie ring with quotient ring Q. Then the following conditions are equivalent:*

(1) *R is an Asano order in Q.*

(2) *R is a maximal order in Q such that each non-zero two-sided ideal I of R is a projective left and projective right R–module.*

(3) *R is a maximal right order and every two-sided ideal of R is reflexive.*

(3)′ *R is a maximal right order and every R–ideal is reflexive.*

(4) *Each non-zero two-sided ideal of R is invertible.*

Proof. $(1) \Rightarrow (2)$ follows directly from Theorem 2.16 $(1) \Rightarrow (2)$ together with the above remark.

$(2) \Rightarrow (3)$: This is clear since every projective right R–module is reflexive. Since R is a maximal right order, the reflexivity of an R–ideal as a right R–module leads to its reflexivity as a left R–module as well.

$(3) \Rightarrow (4)$: Let I be an ideal of R, and let $J = I^*I$. By assumption, $J^{**} = J$. Let $q \in Q$ be such that $qJ \subseteq R$. Then $qI^* \subseteq I^*$, and so $q \in \mathcal{O}_l(I^*) = R$. It follows that $J^* = R$. This gives that $J = J^{**} = R^* = R$. It can also be poved that $II^* = R$. Therefore I is an invertible R–ideal.

$(4) \Rightarrow (1)$: Let T be any R–ideal and set $J = \{r \in R : rT \subseteq R\}$. As in the proof of $(2) \Rightarrow (3)$ in Theorem 2.16, both J and JT are integral R–ideals. Then by assumption, J and JT are invertible. It follows that

$$[(JT)^{-1}J]T = R.$$

On the other hand since $JT(JT)^{-1}J = J$, we have

$$T[(JT)^{-1}J] = R.$$

This gives that T is invertible. □

Theorem 2.18. *Let R be an Asano right order. Then the following statements hold:*
(i) *R satisfies the accending chain condition on integral R–ideals.*
(ii) *Prime integral R–ideals are maximal.*
(iii) *Every integral R–ideal is a unique product of primes.*
(iv) *There are only a finite number of integral R-ideals containing a fixed integral R–ideal.*
(v) *For each R–ideal T, $TT^{-1} = T^{-1}T = R$.*
(vi) *Every R–ideal is finitely generated and projective both as a right ideal and as a left ideal.*

Proof. Only part (iv) needs to be proved since the other statements are easily obtained from the proof of Theorem 2.16. Let I be a fixed proper integral R–ideal. By (ii) and (iii), there exist distict maximal ideals $\mathfrak{M}_1, \dots, \mathfrak{M}_n$ of R and positive integers $t_1, \dots, t_n$ such that $I = \mathfrak{M}_1^{t_1} \dots \mathfrak{M}_n^{t_n}$. Let J be a proper integral R–ideal containing I. Let $J = \mathfrak{Q}_1^{s_1} \dots \mathfrak{Q}_m^{s_m}$ for some distinct maximal ideals $\mathfrak{Q}_1, \dots, \mathfrak{Q}_m$ of R and positive integers $s_1, \dots, s_m$. Since for every $i = 1, \dots, m$, $\mathfrak{Q}_i \supseteq J \supseteq I = \mathfrak{M}_1^{t_1} \dots \mathfrak{M}_n^{t_n}$ and $\mathfrak{Q}_i$ is a prime ideal of R, we have $\mathfrak{Q}_i = \mathfrak{M}_j$ for some $j = 1, \dots, n$. It follows that $\{\mathfrak{Q}_1, \dots, \mathfrak{Q}_m\} \subseteq \{\mathfrak{M}_1, \dots, \mathfrak{M}_n\}$. Without loss of generality, assume that $\mathfrak{Q}_1 = \mathfrak{M}_1$. If $s_1 > t_1$, then $\mathfrak{Q}_1^{s_1 - t_1}\mathfrak{Q}_2^{s_2} \dots \mathfrak{Q}_m^{s_m} \supseteq \mathfrak{M}_2^{t_2} \dots \mathfrak{M}_n^{t_n}$, a contradiction since $s_1 - t_1 \neq 0$ and $\mathfrak{Q}_1 \notin \{\mathfrak{M}_2, \dots, \mathfrak{M}_n\}$. Therefore $s_1 \leq t_1$. Similarly, if $\mathfrak{Q}_i = \mathfrak{M}_j$, we must have $s_i \leq t_j$. This means that there is only a finite number of choices to make up an integral R–ideal containing I. □

Let R be an Asano right order, and let I be a projective right R–ideal with $\mathcal{O}_l(I) = S$. Then by Theorem 2.14, $II^{-1} = S$. Now let $T = I^{-1}I$. Then T is an integral R–ideal, and so $T^{-1}T = R$. Thus $T^{-1}I^{-1}I = R$. This implies that $T^{-1}I^{-1} \subseteq I^{-1}$, or in other words, $T^{-1} \subseteq \mathcal{O}_l(I^{-1}) = R$. But we also have $R \subseteq T^{-1}$, which gives that $T^{-1} = R$; so that $I^{-1}I = R$. Now define maps

$$T_1 \mapsto IT_1I^{-1}; \qquad T_2 \mapsto I^{-1}T_2I$$

from R–ideals into S–ideals and from S–ideals into R–ideals, respectively. It is easy to check that these maps are inverses of each other, giving a one-to-one correspondence between R–ideals and S–ideals.

Proposition 2.19. *Let R be an Asano right order, and let I be a projective right R–ideal. Then $\mathcal{O}_l(I)$ is also an Asano right order.*

Proof. Since correspondence given before the proposition clearly preserves products and inverses, the S–ideals form a group isomorphic to the group of R–ideals. Therefore S is an Asano right order. □

A ring R is called *right (resp. left)* $hereditary$ if each right (resp. left) ideal of R is a projective right (resp. left) R–module. We say that R is $hereditary$ if it is both right and left hereditary. If R is a hereditary Noetherian prime ring with the right (and hence left) quotient ring Q which is a maximal order in Q, then R is called a *Dedekind prime ring*. Thus, it follows, from Corollary 2.17, that a ring R is a Dedekind prime ring if and only if R is a hereditary Noetherian prime ring which is an Asano order in its quotient ring.

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