Computational Geometry

Lecture 5: Casting a polyhedron

CAD/CAM systems

CAD/CAM systems allow you to design objects and test how they can be constructed

Many objects are constructed used a mold

Casting

Casting

A general question: Given an object, can it be made with a particular design process?

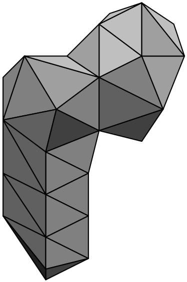
For casting, can the object be removed from its cast without breaking the cast?

Casting

Objects to be made are 3D polyhedra

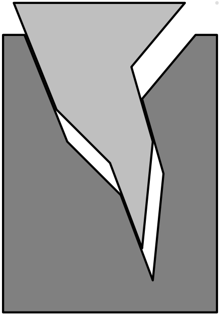
The boundary is like a planar graph, but the coordinates of vertices are 3D

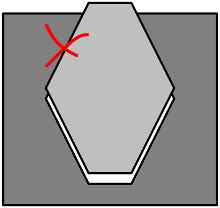
We can use a doubly-connected edge list with three coordinates in each vertex object



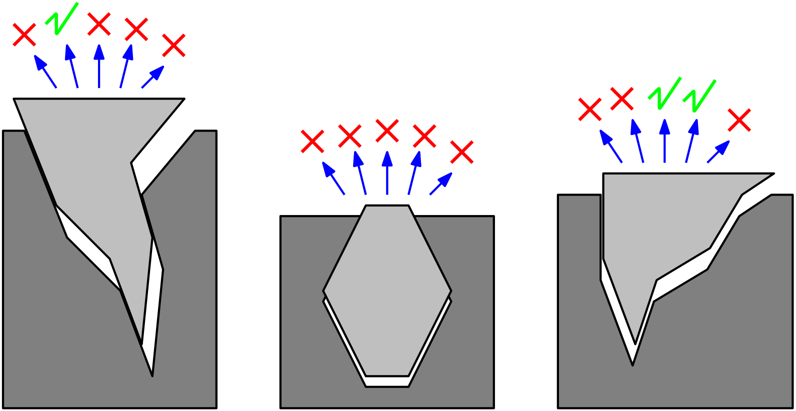
Casting in 2D

First the 2D version: can we remove a 2D polygon from a mold?





Casting in 2D



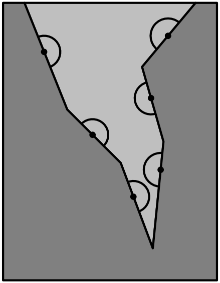
Certain removal directions may be good while others are not

Casting in 2D

What top facet should we use?

When can we even begin to move the object out?

What kind of movements do we allow?

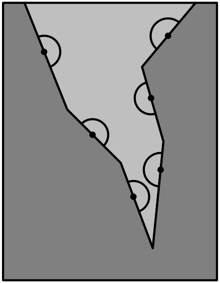


Casting in 2D

Assume the top facet is fixed; we can try all

Let us consider *translations* only

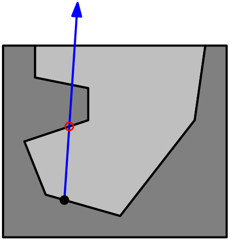
An edge of the polygon should not *directly* run into the coinciding mold edge



Casting in 2D

Observe: For a given top facet, if the object can be translated over some (small) distance, then it can be translated all the way out

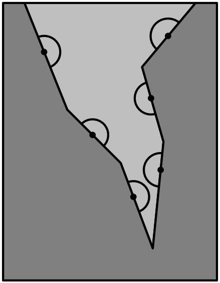
Consider a point p that at first translates away from its mold side, but later runs into the mold . . .



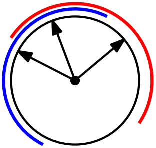
Casting in 2D

A polygon can be removed from its cast *by a single translation* if and only if there is a direction so that every polygon edge does not cross the adjacent mold edge

Sequences of translations do not help; we would not be able to construct more shapes than by a single translation

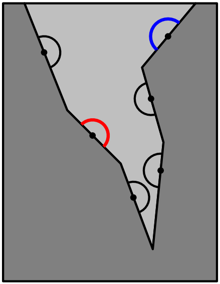


Circle of directions



We need a representation of directions in 2D

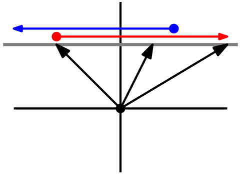
Every polygon edge requires the removal direction to be in a semi-circle



⇒ compute the common intersection of a set of circular intervals (semi-circles)

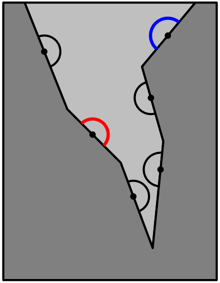
Line of directions

We only need to represent upward directions: we can use points on the line $y = 1$



Every polygon edge requires the removal direction to be in a half-line

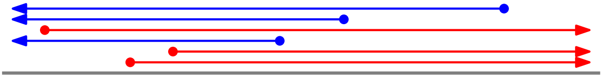
$\Rightarrow$ compute the common intersection of a set of half-lines in 1D



Common intersection of half-lines

The common intersection of a set of half-lines in 1D:

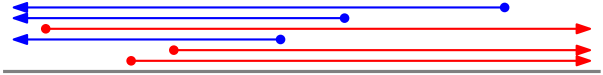
- Determine the endpoint p_l of the rightmost left-bounded half-line
- Determine the endpoint p_r of the leftmost right-bounded half-line
- The common intersection is $[p_l, p_r]$ (can be empty)



Common intersection of half-lines

The algorithm takes only $O(n)$ time for n half-lines

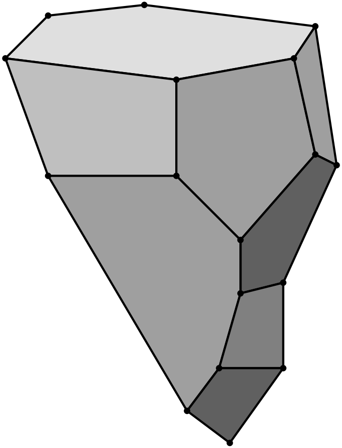
Note: we need not sort the endpoints



Casting in 3D

Can we do something similar in 3D?

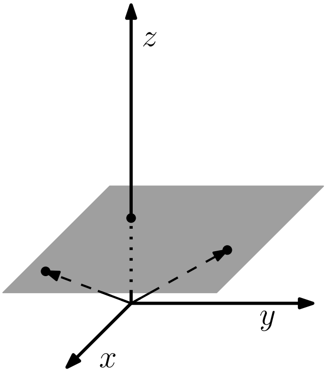
Again each facet must not move into the corresponding mold facet



Representing directions in 3D

The circle of directions for 2D becomes a sphere of directions for 3D; the line of directions for 2D becomes a plane of directions for 3D: take $z = 1$

Which directions represented in the plane does a facet rule out as removal directions?

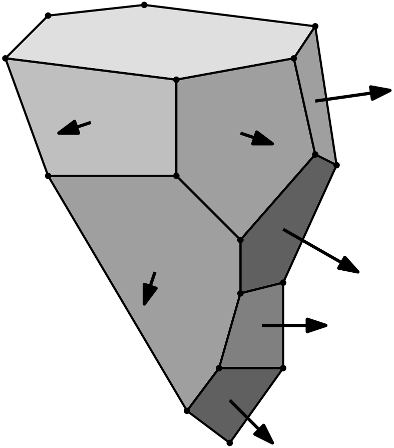


Directions in 3D

Consider the outward normal vectors of all facets

An allowed removal direction must make an angle of at least $\pi/2$ with every facet (except the topmost one)

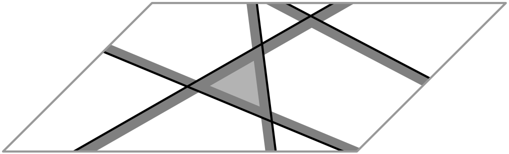
$\Rightarrow$ every facet in 3D makes a half-plane in $z = 1$ invalid



Common intersection of half-planes

We get: common intersection of half-planes in the plane

The problem of deciding castability of a polyhedron with n facets, with a given top facet, where the polyhedron must be removed from the cast by a single translation, can be solved by computing the common intersection of $n-1$ half-planes



Common intersection of half-planes

Half-planes in the plane:

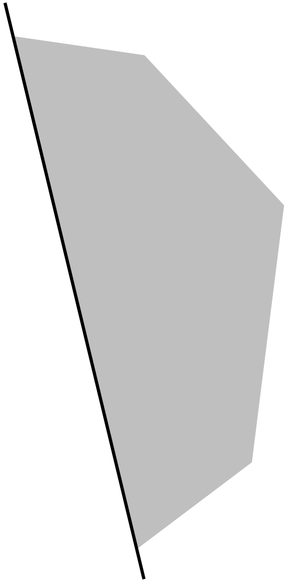
- $y \geq m \cdot x + c$
- $y \leq m \cdot x + c$
- $x \geq c$
- $x \leq c$

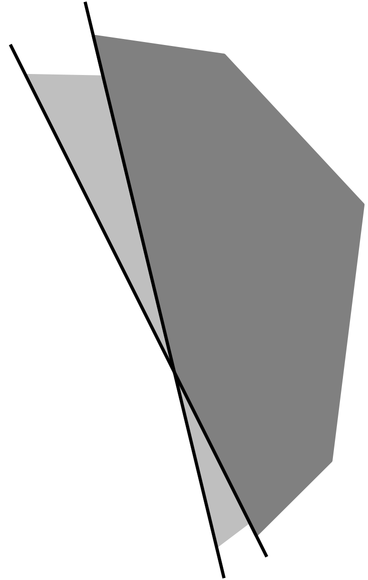
An approach

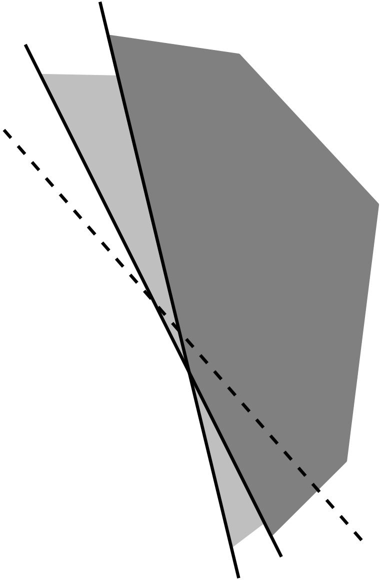
Take the first set:

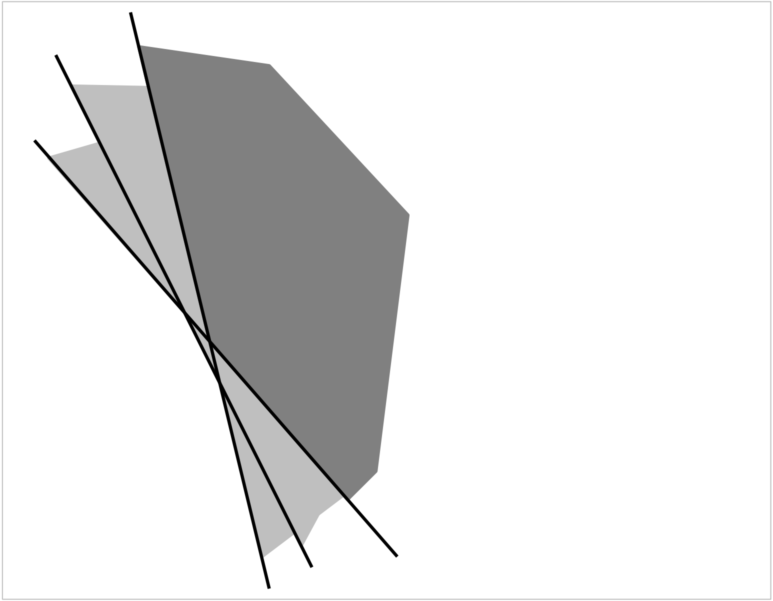
- $y \geq m \cdot x + c$

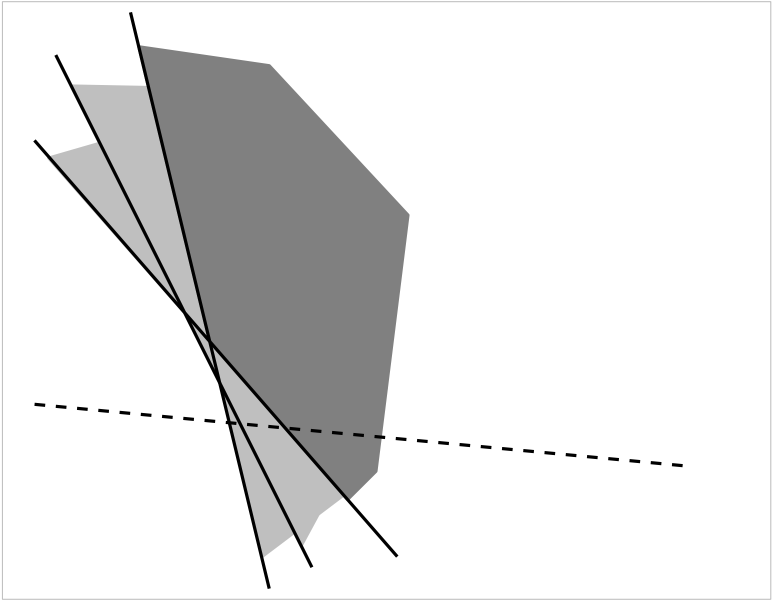
Sort by angle, and add incrementally

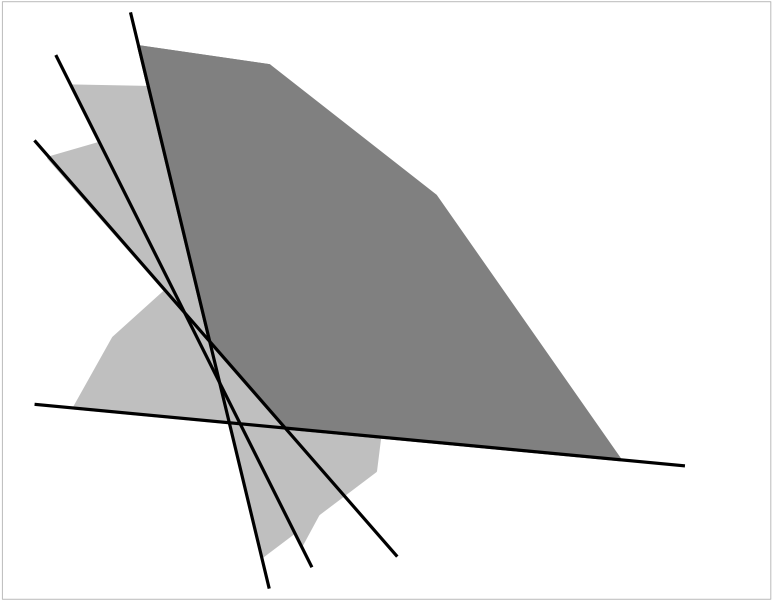


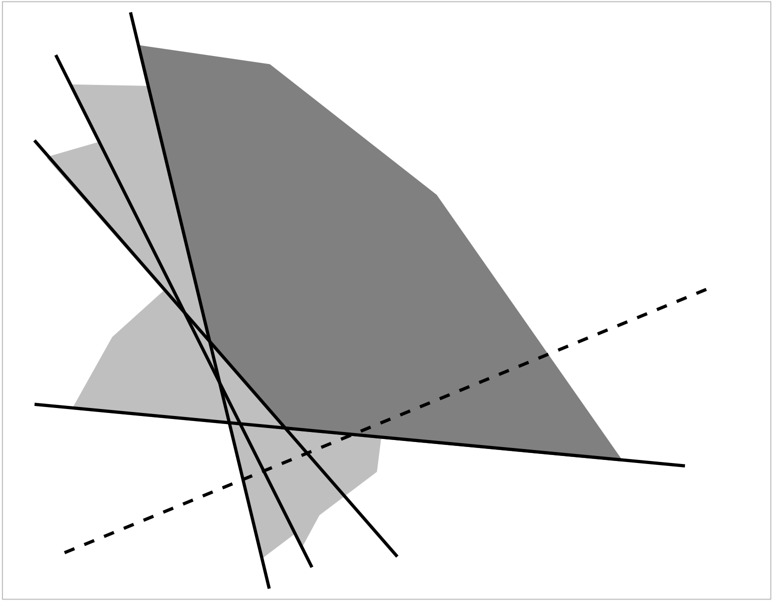


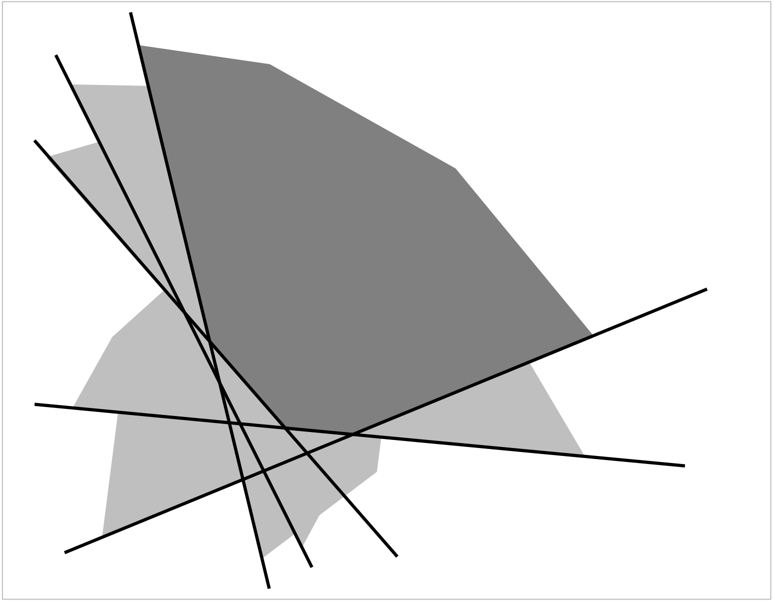


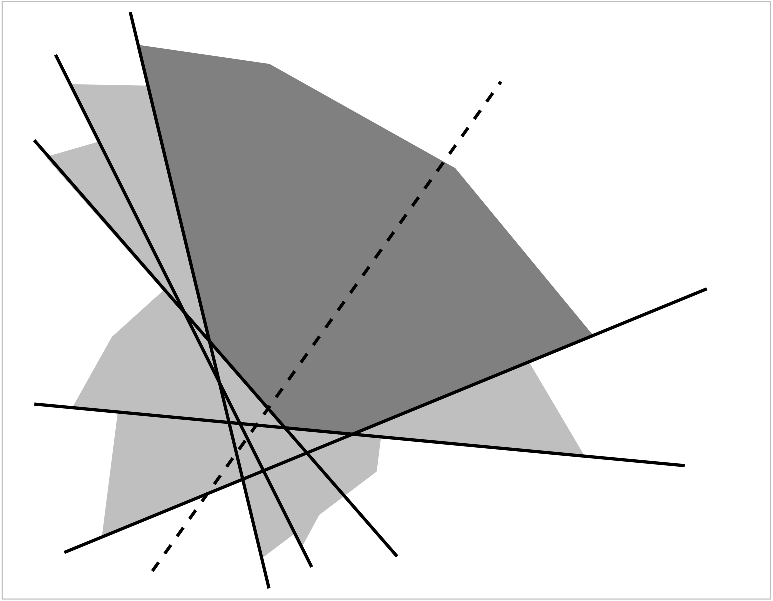








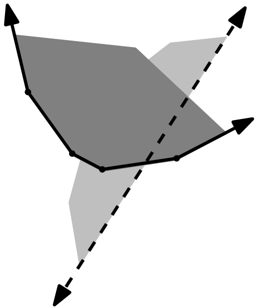




Incremental common intersection

The boundary of the valid region is a polygonal convex chain that is unbounded at both sides

The next half-plane has a steeper bounding line and will always contribute to the next valid region

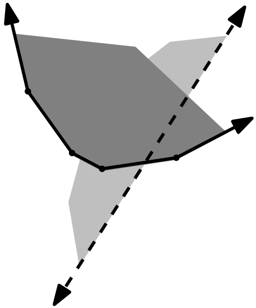


Incremental common intersection

Maintain the contributing bounding lines in increasing angular order

For the new half-plane, remove any no longer contributing bounding lines from the end

Then add the line bounding the new half-plane



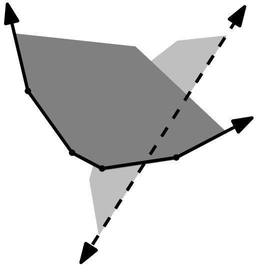
Incremental common intersection

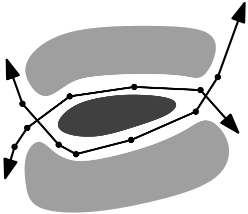
After sorting on angle, this takes only $O(n)$ time

Question: Why?

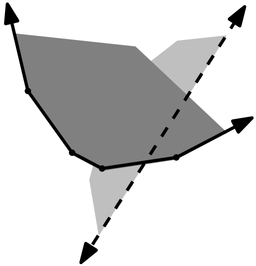
The half-planes bounded from above give a similar chain

Intersecting the two chains is simple with a left-to-right scan



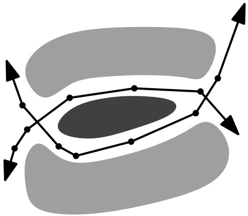


Incremental common intersection



Half-planes with vertical bounding lines can be added by restricting the region even more

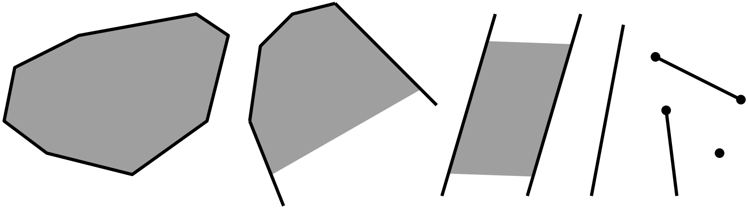
This can also be done in linear time



Result

Theorem: The common intersection of n half-planes in the plane can be computed in $O(n \log n)$ time

The common intersection may be empty, or a convex polygon that can be bounded or unbounded



Back to casting

The common intersection of half-planes cannot be computed faster (we are sorting the lines along the boundary)

The region we compute represents *all mold removal directions* . . .

. . . but to determine castability, we only need one!

We will find the *lowest* point in the common intersection

Notice that half-planes are linear constraints

Minimize y

Subject to

$$
\begin{aligned}
y &\geq m_1 \cdot x + c_1 \\
y &\geq m_2 \cdot x + c_2 \\
&\vdots \\
y &\geq m_i \cdot x + c_i \\
y &\leq m_{i+1} \cdot x + c_{i+1} \\
&\vdots \\
y &\leq m_n \cdot x + c_n
\end{aligned}
$$

Linear programming

Minimize $c_1 \cdot x_1 + \cdots + c_k \cdot x_k$

Subject to

$$\begin{aligned} a_{1,1} \cdot x_1 + \cdots + a_{k,1} \cdot x_k &\leq b_1 \\ a_{1,2} \cdot x_1 + \cdots + a_{k,2} \cdot x_k &\leq b_2 \\ &\vdots \\ a_{1,n} \cdot x_1 + \cdots + a_{k,n} \cdot x_k &\leq b_n \end{aligned}$$

where $a_{1,1}, \ldots, a_{k,n}$, $b_1, \ldots, b_n$, $c_1, \ldots, c_k$ are given coefficients

This is LP with k unknowns (dimensions) and n inequalities

Question: Where are the $\geq$ inequalities?

Terminology

LP with k unknowns (dimensions) and n inequalities: k-dimensional linear programming

The subspace that is the common intersection is the feasible region. If it is empty, the LP is infeasible

The vector $(c_1, \ldots, c_k)^T$ is the objective vector or cost vector

If the LP has solutions with arbitrarily low cost, then the LP is unbounded

Note: The feasible region may be unbounded while the LP is bounded

LP for casting

LP for determining castability of 3D polyhedra is 2-dimensional linear programming with n constraints

We only want to decide feasibility, so we can choose any objective function

We will make it ourselves easy

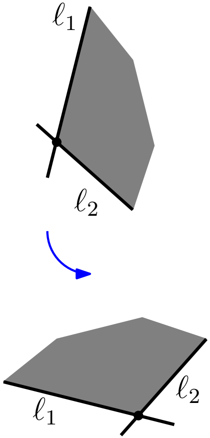
Incremental LP

Let $h_1, \ldots, h_n$ be the constraints and $\ell_1, \ldots, \ell_n$ their bounding lines

Find any two constraints h_1 and h_2 where ℓ_1 and ℓ_2 are non-parallel

Rotate h_1 and h_2 over an angle α around the origin to make $\ell_1 \cap \ell_2$ the optimal solution for the objective function that minimizes y

Rotate all other constraints over α too

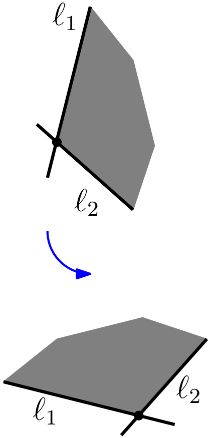


Incremental LP

Solve the LP with the rotated constraints

If the rotated LP is infeasible, then so is the unrotated version

If the rotated LP gives an optimal solution (p_x, p_y), then rotate it over an angle $-\alpha$ around the origin to get the removal direction for the original position of the polyhedron



The algorithm adds the constraints $h_3, \ldots, h_n$ incrementally and maintains the optimum so far

Let $H_i = \{h_1, \ldots, h_i\}$

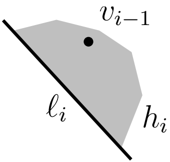
Let v_i be the optimum for H_i (unless we already have infeasibility)

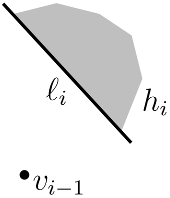
LP for casting

The incremental step: suppose we know v_{i-1} and want to add h_i

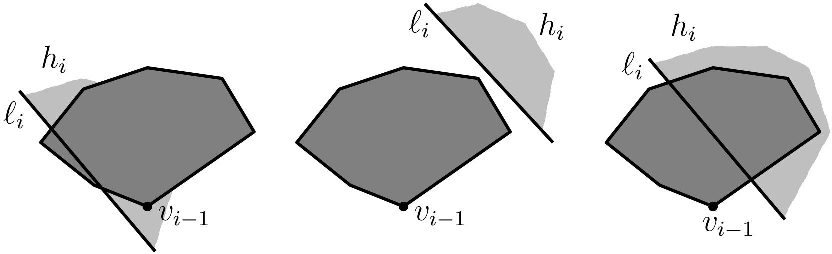
There are two possibilities:

- If $v_{i-1} \in h_i$, then $v_i = v_{i-1}$
- If $v_{i-1} \notin h_i$, then either the LP is infeasible, or v_i lies on ℓ_i





Incremental LP



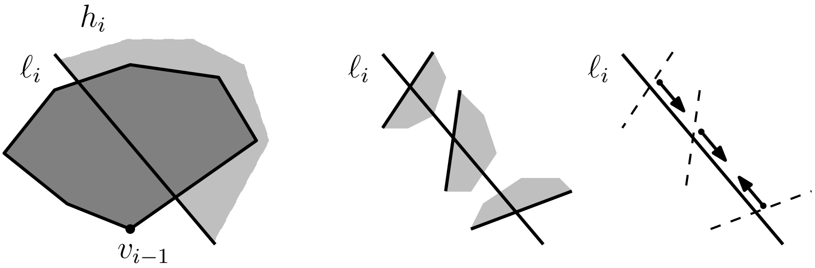
LP for casting

Algorithm LPFORCASTING(H)

1. Let h_1, h_2, and v_2 be as chosen
2. **for** $i \leftarrow 3$ **to** n
3. **do if** $v_{i-1} \in h_i$
4. **then** $v_i \leftarrow v_{i-1}$
5. **else** $v_i \leftarrow$the point p on ℓ_i that minimizes y, subject to the constraints in H_{i-1}.
6. **if** p does not exist
7. **then** Report that the LP is infeasible, and quit.
8. **return** v_n

LP for casting

If $v_{i-1} \notin h_i$, how do we find the point p on ℓ_i?

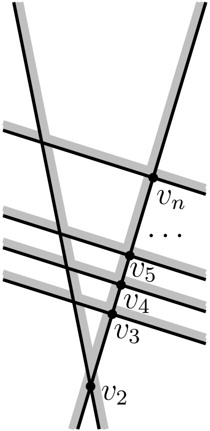


Efficiency

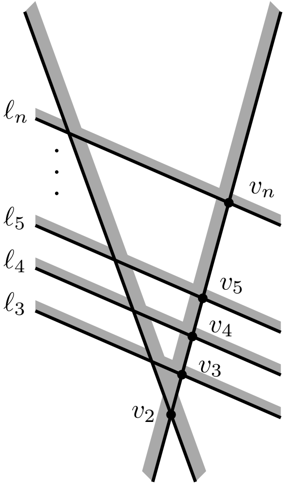
If $v_{i-1} \in h_i$, then the incremental step takes only $O(1)$ time

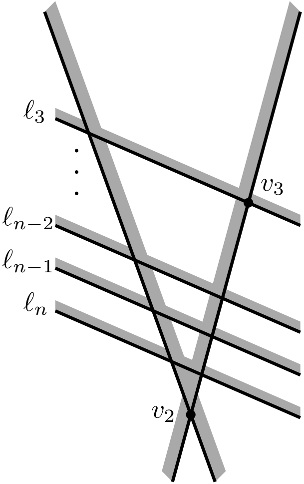
If $v_{i-1} \notin h_i$, then the incremental step takes $O(i)$ time

The LP-for-casting algorithm takes $O(n^2)$ time in the worst case



Efficiency





Algorithm RANDOMIZEDLPFORCASTING(H)

1. Let h_1, h_2, and v_2 be as chosen
2. Let $h_3, h_4, \ldots, h_n$ be in a random order
3. **for** $i \leftarrow 3$ **to** n
4. **do if** $v_{i-1} \in h_i$
5. **then** $v_i \leftarrow v_{i-1}$
6. **else** $v_i \leftarrow$the point p on ℓ_i that minimizes y, subject to the constraints in H_{i-1}.
7. **if** p does not exist
8. **then** Report that the LP is infeasible, and quit.
9. **return** v_n

Putting in random order

The constraints may be given in any order, the algorithm will just reorder them

- Let j be a random integer in $[3,n]$
- Swap h_j and h_n
- Recursively shuffle $h_3,\ldots,h_{n-1}$

Putting in random order takes $O(n)$ time

Expected running time

Every one of the $(n-2)!$ orders is equally likely

The expected time taken by the algorithm is the *average* time over all orders

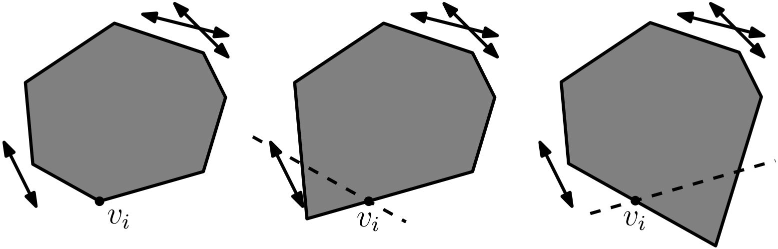
$$\frac{1}{(n-2)!} \cdot \sum_{\Pi \text{ permutation}} \text{time if the random order is } \Pi$$

Expected running time

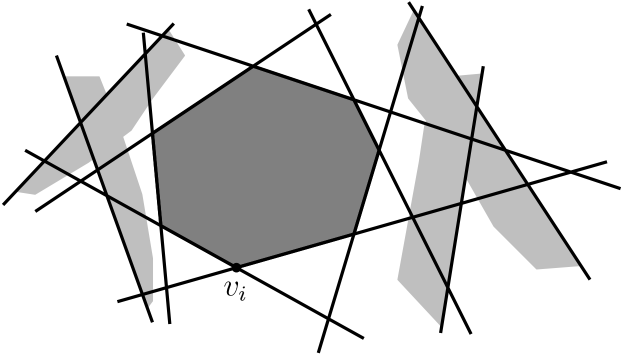
If the order of the constraints $h_3, \ldots, h_n$ is random, what is the probability that $v_{i-1} \in h_i$?

We use backwards analysis: consider the situation *after* h_i *is inserted*, and v_i is computed (either by $v_i = v_{i-1}$, or somewhere on ℓ_i)

Expected running time



Only if one of the dashed lines was ℓ_i, the last step where h_i was added was expensive and took $\Theta(i)$ time



If h_i does not bound the feasible region, or not at v_i, then the addition step was cheap and took $\Theta(1)$ time

Expected running time

There are i half-planes that could have been one of the lines defining v_i, and $i-2$ of these are in random order

Since the order was random, each of the $i-2$ half-planes has the same probability to be the last one added, and only ≤ 2 of these caused the expensive step

- ≤ 2 out of $i-2$ cases: expensive step; $\Theta(i)$ time for i-th addition
- $\geq i-4$ out of $i-2$ cases: cheap step; $\Theta(1)$ time for i-th addition

Expected running time

Expected time for i-th addition at most:

$$\frac{i-4}{i-2} \cdot \Theta(1) + \frac{2}{i-2} \cdot \Theta(i) = \Theta(1)$$

Total running time:

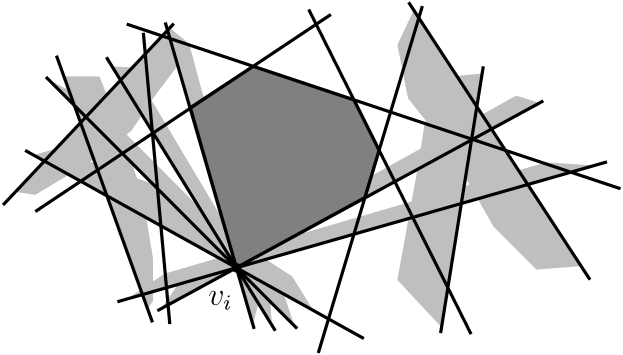
$$\Theta(n) + \sum_{i=3}^{n} \Theta(1) = \Theta(n) \text{ expected time}$$

Degenerate cases

The optimal solution may not be unique, if the feasible region is bounded from below by a horizontal line. How to solve it?

There may be many lines from $\ell_3, \ldots, \ell_i$ passing through v_i; how does this affect the probability of an expensive step?

Degenerate cases



Degenerate cases

In degenerate cases, the probability that the last addition was expensive is even smaller: $1/(i-2)$, or 0

Without any adaptations, the running time holds

Result

Theorem: Castability of a simple polyhedron with n facets, given a top facet, can be decided in $O(n)$ expected time

Theorem: 2-dimensional linear programming with n constraints can be solved in $O(n)$ expected time

Question: What does "expected time" mean? Expectation over what?

Higher dimensions?

Question: Can you imagine whether we can also solve 3-dimensional linear programming efficiently?