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Isomorphisms of the Unitriangular Groups and Associated Lie Rings for the Exceptional Dimensions

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Abstract. Isomorphisms between finitary unitriangular groups and those of associated Lie rings are studied. In this paper we investigate exceptional cases.

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Introduction

For an arbitrary chain Γ and associative ring K with identity 1_K, the ring $\mathrm{NT}(\Gamma, K)$ is generated by elements xe_{ij} $(x \in K,\ i, j \in \Gamma, i > j)$ with the usual rules of the addition and multiplication of elementary matrices; if $|\Gamma| = n < \infty$, then we write $\mathrm{NT}(n, K)$. Let $R = \mathrm{NT}(\Gamma, K)$. The adjoint group of the ring R is isomorphic to the unitriangular group $\mathrm{UT}(\Gamma, K)$. Structural connections between the adjoint group $G(R)$ and associated Lie ring $\Lambda(R)$ of R are investigated in [4], see also [5].

Standard automorphisms and isomorphisms of the rings R, $\Lambda(R)$ and the adjoint group $G(R)$ were distinguished in [2–4], see also [1]. Let $R' = \mathrm{NT}(\Omega, S)$ for a chain Ω and an associative ring S with identity. By [4] and [2], if either $2 < |\Gamma| < \infty$ or K is a ring with no zero-divisors, then every isomorphism between rings R and R' is standard; the same is true for their adjoint groups and associated Lie rings at $|\Gamma| > 4$. It was shown in [3], for $|\Gamma| \leqslant 4$ there exist nonstandard automorphisms of $G(R)$ and $\Lambda(R)$, even if the ring K is commutative. The aim of this paper is to investigate isomorphisms in the exceptional cases.

Note that for $\Gamma = \{1, 2\}$ the ring R has zero multiplication and the adjoint group $G(R)$ is isomorphic to the additive group K^+ of K. It follows that every isomorphism θ of the additive group K^+ determines an isomorphism of the ring R by rule $xe_{21} \to x^\theta e_{21}$ $(x \in K)$. For a finite chain Γ, the adjoint group $G(R)$ and the associated Lie ring $\Lambda(R)$ are nilpotent of the class $|\Gamma| - 1$. Consequently, if either $G(R) \simeq G(R')$ or $\Lambda(R) \simeq \Lambda(R')$, then $|\Omega| = |\Gamma|$. Therefore our task is to investigate isomorphisms between the adjoint groups $G(R)$, $G(R')$ and associated Lie rings $\Lambda(R)$ and $\Lambda(R')$ at $\Omega = \Gamma$ and $|\Gamma| = 3$ or 4.

1. Certain Isomorphisms and the Case $n = 3$

First we need to define certain automorphisms and isomorphisms.

Let K and S be associative rings with identities $R = \mathrm{NT}(n, K)$ and $R_S = \mathrm{NT}(n, S)$. Evidently, every ring isomorphism $\theta\colon K \to S$ determines an isomorphism $\|a_{uv}\| \to \|\theta(a_{uv})\|$ of the ring R onto R_S which is called an "S-ring" or "ring" isomorphism as usual. The central automorphism of the ring R is an automorphism acting like the identity, modulo the center. (See also [1, Lemma 1.1].) Conjugation by unitriangular matrices and by invertible diagonal matrices over K give "inner" and "diagonal" automorphisms of the ring R, respectively. In [2] all isomorphisms between rings R and R_S are described and the following is proved.

LEMMA 1. *Let K and S be associative rings with identities $R = \mathrm{NT}(n, K)$, $R_S = \mathrm{NT}(n, S)$ and $n > 2$. If $R \simeq R_S$, then $K \simeq S$ and every isomorphism of the ring R onto R_S is a product of an inner, diagonal and central automorphisms of R and a S-ring isomorphism.*

By [2], $G(R) \simeq G(R_S)$ or $\Lambda(R) \simeq \Lambda(R_S)$ for $n > 4$ iff the rings K and S are idempotent isomorphic, i.e. there exists some idempotent-ring isomorphism between them. According to [2], "idempotent-ring" isomorphisms between rings are a generalization of ring isomorphisms. However, every idempotent-ring isomorphism of a commutative ring is an isomorphism.

For $n > 4$ every isomorphism of $G(R)$ onto $G(R_S)$ (similarly, $\Lambda(R)$ onto $\Lambda(R_S)$) is a product of a diagonal automorphism, an automorphism of $G(R)$ (resp. $\Lambda(R)$), acting like the identity modulo R^2 and an isomorphism of $G(R)$ (resp. $\Lambda(R)$) which is induced by some idempotent-ring isomorphism of K onto S, see [2]. Automorphisms of $G(R)$ and $\Lambda(R)$ acting like the identity modulo R^2 for all n are described in [3].

On the other hand, for exceptional cases $n = 3, 4$, there exist nonstandard automorphisms of $G(R)$ and $\Lambda(R)$, by [3]. Note that each isomorphism of $G(R)$ or $\Lambda(R)$ is uniquely determined by its actions on elementary matrices xe_{ij} in which the (i, j)-coefficient is equal to x and others are zero. Let K be a commutative ring and $\alpha = ||a_{ij}|| \in \mathrm{GL}(2, K)$. Then the following map:

$$\pi_\alpha^\star\colon\ xe_{i+1i} \to x(a_{i1}e_{21} + a_{i2}e_{32}), \quad xe_{31} \to (\det(\alpha))xe_{31}, \quad i = 1, 2, x \in K,$$

defines an automorphism of the Lie ring $\Lambda(R)$ at $n = 3$. Similarly, the map

$$\pi_\alpha\colon\ xe_{i+1i} \to x(a_{i1}e_{21} + a_{i2}e_{32}) + x^{\psi_i}e_{31}, \quad xe_{31} \to \det(\alpha)xe_{31}, \quad i = 1, 2, x \in K,$$

defines an automorphism of $G(R)$ for maps $\psi_i\colon K \to K$ with the following condition:

$$(x+y)^{\psi_i} = x^{\psi_i} + y^{\psi_i} + a_{i1}a_{i2}xy, \quad x, y \in K,\ i = 1, 2.$$

The following theorem describes isomorphisms for the exceptional case $n = 3$.

THEOREM 2. *Let $R = \mathrm{NT}(3, K)$ and $R_S = \mathrm{NT}(3, S)$. Let K and S be commutative rings. Then every isomorphism of $G(R_S)$ onto $G(R)$ (resp. $\Lambda(R_S)$ onto $\Lambda(R)$) is a product of a ring isomorphism between them, a central automorphism and some automorphism π_α (resp. $\pi_\alpha^\star$) of $G(R)$ (resp. $\Lambda(R)$). In particular, $G(R) \simeq G(R_S)$ or $\Lambda(R) \simeq \Lambda(R_S)$ iff the rings K and S are isomorphic.*

Proof. Recall that the adjoint multiplication $\circ$ and the associated Lie multiplication $*$ of an arbitrary associative ring are defined by

$$\alpha \circ \beta = \alpha + \beta + \alpha\beta, \qquad \alpha * \beta = \alpha\beta - \beta\alpha.$$

Consider an arbitrary isomorphism ϕ of $G(R_S)$ onto $G(R)$ or of $\Lambda(R_S)$ onto $\Lambda(R)$. The action of ϕ on the set of generating elementary matrices

$$(xe_{i+1i})^{\phi} = x^{\phi_{i1}}e_{21} + x^{\phi_{i2}}e_{32} + x^{\phi_i}e_{31} \quad (x \in S), i = 1, 2,$$

defines maps ϕ_{ij}, ϕ_i of the ring S onto K. Since $\phi(Se_{31}) = \phi((R_S)^2) = R^2 = Ke_{31}$ and the operations of addition and adjoint multiplication coincides on R^2, there exists an isomorphism ϕ_0 of the additive group S^+ onto K^+ such that $\phi(xe_{31}) = \phi_0(x)e_{31}$ for all $x \in S$. We obtain $(xy)^{\phi_0} = x^{\phi_{22}}y^{\phi_{11}} - y^{\phi_{12}}x^{\phi_{21}}$ $(x, y \in S)$, since the operations of Lie multiplication in the ring R_S and the commutator in the adjoint group coincide.

Since K and S are commutative rings, we get $(xy)^{\phi_0}d = x^{\phi_0}y^{\phi_0}$ for $d = 1^{\phi_0}$. It follows that $S^{\phi_0} = K$ and d is an invertible element of the ring K. Hence, the map $\theta\colon x \to d^{-1}x^{\phi_0}(x \in S)$ is an isomorphism of the additive group S^+ onto K^+. The equalities $(xy)^{\theta} = d^{-1}(xy)^{\phi_0} = d^{-2}x^{\phi_0}y^{\phi_0} = x^{\theta}y^{\theta}$ show that θ is an isomorphism of the ring S onto K. Therefore, up to multiplication of ϕ by a ring automorphism of the ring R, we may account that $S = K$ and $\phi \in \operatorname{Aut}\Lambda(R)$ or $\phi \in \operatorname{Aut}G(R)$.

The equalities $dx^{\phi_{ij}} = x^{\phi_0}a_{ij}$ $(x \in S)$ are satisfied for $a_{ij} = 1^{\phi_{ij}}$ and $d = \det||a_{ij}||$. Thus, $\alpha = ||a_{ij}|| \in \mathrm{GL}(2, K)$. Because of the choice of ϕ we obtain as in [3] that ϕ is a product of the automorphism π_α or $\pi_\alpha^\star$ (resp. of $G(R)$ or $\Lambda(R)$) and a central automorphism of R. This completes the proof. □

Note that by Theorem 2, if K is an arbitrary domain of characteristic 2, then all isomorphisms of $G(R_S)$ onto $G(R)$ are standard.

2. The Exceptional Case $n = 4$

Let K and S be associative rings with identities, $R = \mathrm{NT}(4, K)$ and $R_S = \mathrm{NT}(4, S)$. In this section we describe all isomorphisms between the adjoint groups and associated Lie rings of the rings R and R_S.

Consider an arbitrary isomorphism ϕ of $G(R_S)$ onto $G(R)$ or of $\Lambda(R_S)$ onto $\Lambda(R)$. It is not difficult to show that R^i and the centralizer $C(R^2) = N_{32} = Ke_{31} + Ke_{32} + Ke_{41} + Ke_{42}$ are characteristic in the adjoint group and associated Lie ring and also $\phi[C(R_S^2)] = C(R^2)$. Therefore there exists an isomorphism θ: $S^+ \to K^+$ such that $(xe_{32})^\phi = x^\theta e_{32} \bmod R^2, x \in S$. The element 1_S^θ is invertible in K, because $e_{32}^\phi \star R = R^2 \bmod R^3$. Consequently, $e_{32}^\phi = e_{32}$, up to multiplication of ϕ by a diagonal, inner and central automorphisms of the ring R. Similarly, we define 2×2 matrix $||c_{ij}||$ over K by the equalities

$$e_{21}^\phi = c_{11}e_{21} + c_{12}e_{43},\ e_{43}^\phi = c_{21}e_{21} + c_{22}e_{43}\ (\bmod N_{32}).$$

By using the existence of the isomorphism ϕ^{-1}, we get that the system of equations

$$d_{i1}c_{11} + d_{i2}c_{21} = \delta_{i1}, \qquad c_{12}d_{i1} + c_{22}d_{i2} = \delta_{i2}, \quad i = 1, 2, \tag{1}$$

can be solved in K for d_{ij}. (δ_{ij} is the Kronecker delta.) The ϕ-invariance of fundamental relations between elementary matrices shows that

$$2c_{i2}Kc_{i1} = 0, \quad i = 1, 2; \tag{2}$$

$$(xy)^\theta c_{11} = x^\theta y^\theta c_{11}, \qquad c_{12}(xy)^\theta = c_{12}y^\theta x^\theta, \quad 1_S^\theta = 1_K,$$
$$(xy)^\theta c_{21} = y^\theta x^\theta c_{21}, \qquad c_{22}(xy)^\theta = c_{22}x^\theta y^\theta \quad (x, y \in S). \tag{3}$$

Now, it is not difficult to verify that the map

$$xe_{21} \to x^\theta c_{11}e_{21} + c_{12}x^\theta e_{43}, \qquad xe_{32} \to x^\theta e_{32},$$
$$xe_{43} \to x^\theta c_{21}e_{21} + c_{22}x^\theta e_{43}, \quad x \in S, \tag{4}$$

define an isomorphism of the Lie ring $\Lambda(R_S)$ onto $\Lambda(R)$ which acts like ϕ, modulo R_S^2. If ϕ is an isomorphism of $G(R_S)$ onto $G(R)$, then we get

$$c_{i2}(z^2 - z)[y^2 - y(d_{1i} + d_{2i})]c_{i1} = 0 \quad (z, y \in K),\ i = 1, 2. \tag{5}$$

In this case, there exists the following isomorphism of the adjoint group $G(R_S)$:

$$xe_{21} \to x^\theta c_{11}e_{21} + c_{12}x^\theta e_{43}, \qquad xe_{43} \to x^\theta c_{21}e_{21} + c_{22}x^\theta e_{43},$$
$$xe_{32} \to x^\theta e_{32} + (x^2 - x)^\theta d_{22}c_{21}e_{31} + c_{12}d_{11}(x^2 - x)^\theta e_{42}, \quad x \in S. \tag{6}$$

Note that if K is a commutative ring, then condition (2) coincides with $2c_{i2}c_{i1} = 0$ ($i = 1, 2$) and the system of equations (1) is consistent. This means that $||c_{ij}||$ is an invertible matrix. In this case θ is a ring isomorphism of K onto S, by (3) and (1). Therefore, $G(R) \simeq G(R_S)$ or $\Lambda(R) \simeq \Lambda(R_S)$ iff the rings K and S are isomorphic. Consequently, we have proved the following theorem:

THEOREM 3. *Let* $R = \mathrm{NT}(4, K)$ *and* $R_S = \mathrm{NT}(4, S)$. *If* ϕ *is an isomorphism of* $\Lambda(R_S)$ *onto* $\Lambda(R)$ (*resp.* $G(R_S)$ *onto* $G(R)$) *then there exist an isomorphism* $\theta\colon S^+ \to K^+$ *and* 2×2 *matrices* $||c_{ij}||$, $||d_{ij}||$ *over* K, *satisfying the conditions* (1)–(3) (*resp.* (1)–(3) *and* (5)), *such that* ϕ *is equal to a product of the isomorphism* (4) (*resp.* (6)), *a diagonal automorphism and an automorphism of* $\Lambda(R)$ (*resp.* $G(R)$), *acting like the identity, modulo* R^2. *In particular,* $G(R) \simeq G(R_S)$ *or* $\Lambda(R) \simeq \Lambda(R_S)$ *for a commutative ring* K *iff the rings* K *and* S *are isomorphic.*

If either the annihilator of the element $2(1_K)$ in K is zero or K is a noncommutative ring without zero-divisors, then for $n = 4$ each isomorphism ϕ of $G(R_S)$ onto $G(R)$ or of $\Lambda(R_S)$ onto $\Lambda(R)$ is standard.

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