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Isomorphisms of Certain Locally Nilpotent Finitary Groups and Associated Rings ⋆

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Abstract. For any chain Γ the ring $\mathrm{NT}(\Gamma, K)$ of all finitary Γ-matrices $\|a_{ij}\|_{i,j\in\Gamma}$ over an associative ring K with zeros on and above the main diagonal is locally nilpotent and hence radical. If $R' = \mathrm{NT}(\Gamma', K')$, $R = \mathrm{NT}(\Gamma, K)$ and either $|\Gamma| < \infty$ or K is a ring with no zero-divisors, then isomorphisms between rings R and R', their adjoint groups and associated Lie rings are described.

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Introduction

Let K be an associative ring with identity and Γ be a chain (or linearly ordered set) by the order relation $\leqslant$. The finitary unitriangular group $\mathrm{UT}(\Gamma, K)$ is always locally nilpotent, however it coincides with the commutator subgroup for any dense chain Γ and also it does not satisfy the normalizer condition for every infinite chain Γ, see [13] and [21]. The aim of this paper is to investigate isomorphisms of the groups $\mathrm{UT}(\Gamma, K)$ and the associated rings.

The ring $\mathrm{NT}(\Gamma, K)$ is generated by elements xe_{ij} $(x \in K, i, j \in \Gamma, i > j)$ with the usual rules of the addition and multiplication of elementary matrices. Set $R = \mathrm{NT}(\Gamma, K)$. In [11–13] the unitriangular group $\mathrm{UT}(\Gamma, K)$ is considered as the adjoint group $G(R)$ of the ring R. (The map $\alpha \to e + \alpha$ $(\alpha \in R)$ for the identity Γ-matrix e is a well-known isomorphism between them.) Automorphisms of $G(R)$ and those of the associated Lie ring $\Lambda(R)$ of R are described if either $|\Gamma| < \infty$ or K is arbitrary ring with no zero-divisors, see [10, 12, 13]. In Remark 2.7 we define a standard isomorphism of $G(R)$ (similarly, $\Lambda(R)$) as a product of some main isomorphisms. Let f be an idempotent in the center of K and S be a ring. Any isomorphism θ of the additive group K^+ of K onto S^+ with $\theta(1_K) = 1_S$ inducing an isomorphism of the ideal fK and an anti-isomorphism of $(1_K - f)K$ will be

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called an f-isomorphism or idempotent isomorphism of the ring K. It determines an 'idempotent S-ring isomorphism' of $G(R)$ (resp. $\Lambda(R)$) by

$$xe_{ij} \mapsto \theta(xf)e_{ij} - \theta(x - xf)e_{j'i'} \quad (x \in K, i, j \in \Gamma, i > j) \tag{1}$$

for any anti-automorphism $'$ of the chain Γ, see [10] and [12]. Rings K and S are called idempotent isomorphic, if there exists some idempotent isomorphism between them. Evidently, every ring isomorphism $\theta\colon K \to S$ and chain isomorphism (or isometry) $'\colon \Gamma \to \Omega$ induce, respectively, a 'S-ring isomorphism' $xe_{ij} \to x^\theta e_{ij}$ and 'Ω-chain isomorphism' $xe_{ij} \to xe_{i'j'}$ $(x \in K, i, j \in \Gamma, i > j)$ of the ring R. Also, the opposite isomorphism, a central (which acts like the identity, modulo the center), hypercentral of height $\leqslant 3$ and triangular automorphisms of $G(R)$ (resp. $\Lambda(R)$) will be use in the paper. We now formulate the main result.

MAIN THEOREM. *Let K and S be associative rings with identities, Γ and Ω be chains, $R = \mathrm{NT}(\Gamma, K)$, $R_S = \mathrm{NT}(\Omega, S)$ and $|\Gamma| > 2$. Then the following hold:*

(i) *If $R \simeq R_S$ and either $|\Gamma| < \infty$ or K is a ring with no zero-divisors, then $K \simeq S$, $\Gamma \simeq \Omega$ and every isomorphism of the ring R onto R_S is a product of a triangular and central automorphisms of R, an Ω-chain and S-ring isomorphisms.*

(ii) *If K is a ring with no zero-divisors,* $|\Gamma| > 4$ *and there exists an isomorphism ψ of $G(R)$ onto $G(R_S)$ or of $\Lambda(R)$ onto $\Lambda(R_S)$, then ψ is a standard isomorphism and either $\Gamma \simeq \Omega$, $K \simeq S$ or $\Gamma \simeq \Omega^{\mathrm{op}}$, $K \simeq S^{\mathrm{op}}$.*

(iii) *For* $4 < |\Gamma| < \infty$ *the adjoint groups $G(R)$ and $G(R_S)$ or associated Lie rings $\Lambda(R)$ and $\Lambda(R_S)$ are isomorphic if and only if rings K and S are idempotent isomorphic and* $|\Omega| = |\Gamma|$; *every isomorphism between them is standard.*

Note that for $\Gamma = \{1, 2\}$ rings R and $\Lambda(R)$ have zero multiplication, $G(R) \simeq R^+ \simeq K^+$ and every isomorphism θ of the additive group K^+ determines an isomorphism of the ring R by rule $xe_{21} \to x^\theta e_{21}$ $(x \in K)$. By Remark 3.4, for $|\Gamma| = 3, 4$ there exist nonstandard automorphisms of $G(R)$ and $\Lambda(R)$.

In the case $\Gamma = \Omega$ and $K = S$ main theorem describes $\mathrm{Aut}\, R$, $\mathrm{Aut}\, G(R)$ and $\mathrm{Aut}\, \Lambda(R)$. Automorphisms and isomorphisms of classical linear groups have been under active research for a long time, see surveys [20, 2, 4, 1, 17]. The unitriangular group $\mathrm{UT}(\Gamma, K)$ (it is a π-group for $\mathrm{char}\, K \neq 0$ and $\pi = \pi(\mathrm{char}\, K)$) of any finite degree $|\Gamma|$ over any finite field is a Sylow subgroup of the general linear group and its automorphisms were studied in [22, 23, 10, 12, 19, 24, 18]. Main results of [12, 13] are obtained by using close structural connections between $G(R)$ and $\Lambda(R)$, see also [11], [8, §1] and [25]. Methods and results of [12] were extended to all Chevalley groups in [14, 15] and to certain radical matrix rings in [9, 16]. In the present paper this approach is also used to investigate of isomorphisms of $G(R)$ and $\Lambda(R)$ for $R = \mathrm{NT}(\Gamma, K)$.

1. Automorphisms and the Central Series

In this section we introduce certain automorphisms and terminology. Henceforth, K denotes an associative ring with identity 1_K and Γ a chain of order > 2. Denote by j^+, the direct successor of j in Γ, i.e. the first element of the subset $\{y \in \Gamma \mid j < y\}$; if it does not exist, then we set $j^+ = j$. Similarly, by using the direct predecessor of j, the element j^- is also defined. If $j < i$, then we write $j \lhd i$ for $i = j^+$, and otherwise $j \ll i$. By [7, p. 209], a proper subset X of Γ is said to be an initial segment if for every $x \in X$ and $y \in \Gamma$ with $y < x$ we have $y \in X$. We denote by $[i, j]$, the segment $\{k \in \Gamma \mid i \leqslant k \leqslant j\}$, by p and q, the first and the last element of Γ (if they exist), respectively. For Γ-matrices $\|a_{ij}\|$ with $i, j \in \Gamma$ we use the standard matrix notation.

Recall that the adjoint multiplication $\circ$ and the associated Lie multiplication $*$ of an arbitrary associative ring are defined by $\alpha \circ \beta = \alpha + \beta + \alpha\beta$ and $\alpha * \beta = \alpha\beta - \beta\alpha$, respectively. Let $R = \mathrm{NT}(\Gamma, K)$ and $\Lambda(R)$ be the associated Lie ring of R. Since the ring R is locally nilpotent, $(e - \beta)^{-1} = e + \beta + \beta^2 + \beta^3 + \cdots$ for all $\beta \in R$. Thus, $(R, \circ)$ is a group (the adjoint group of R) which we denote by $G(R)$. The adjoint conjugation by an element $-\beta$ of R coincides with ordinary conjugation

$$(e - \beta)\alpha(e - \beta)^{-1} = \alpha + (\alpha * \beta)(e - \beta)^{-1}, \quad \alpha \in R.$$

This gives an inner automorphism of R. We now consider its generalization. It is easy to see that all triangular Γ-matrices $\|a_{ij}\|$ (with $a_{ij} = 0$ for $i < j$) over K having a finite number of nonzero elements in each row and column form a ring with identity with respect to the usual matrix addition and multiplication. For arbitrary invertible Γ-matrix γ of this ring the conjugation $\alpha \to \gamma\alpha\gamma^{-1}$ ($\alpha \in R$) is a 'triangular' automorphism of the ring R or a 'diagonal' automorphism, if γ is a diagonal Γ-matrix. If $\gamma - e$ has zero main diagonal, then the conjugation by γ is an automorphism which acts like an inner automorphism of R on each finite set and, therefore, it is a 'locally inner' automorphism of R in according to [3].

We now describe the central series and the central automorphisms.

LEMMA 1.1. *The center of the ring R is nonzero if and only if the chain Γ contains the first element p and the last element q. If $p, q \in \Gamma$ and m is a positive integer, then mth hypercenter of $G(R)$ and $\Lambda(R)$ are generated by the sets Ke_{ij} $(j < i)$ with $|[p, j]| + |[i, q]| \leqslant m + 1$.*

Proof. It is clear for a finite chain Γ. In general case we assume that the center of R contains nonzero element α. Then there exists a finite subchain Γ_1 of Γ satisfying $\alpha \in \mathrm{NT}(\Gamma_1, K)$. For every such subchain the element α is in the center of $\mathrm{NT}(\Gamma_1, K)$ which has form Ke_{ij}. Evidently, it is possible only if $p, q \in \Gamma$ and equalities $i = q$ and $j = p$ are satisfied. In particular, the center of R is zero if and only if the chain Γ has no first or last element. By induction on m we obtain the second statement of the lemma. The lemma is proved. □

Let $\Gamma_0 = \{j \in \Gamma \mid j^+ \neq j\}$. Clearly, $\Gamma_0 = \emptyset$ if and only if the chain Γ is dense.

LEMMA 1.2. *The subgroup of the central automorphisms of the ring R is non-identity if and only if $p, q \in \Gamma$ and $\Gamma_0 \neq \emptyset$. It has a decomposition in the Cartesian product of $|\Gamma_0|$ subgroups which are isomorphic to the additive group* $(\mathrm{End}(K^+))^+$.

Proof. Suppose that $p, q \in \Gamma$ and, therefore, the center of the ring R coincides with Ke_{qp}. It coincides also with the annihilator $\{\alpha \in R \mid \alpha R = R\alpha = 0\}$ of R. Consequently, the restriction on R^2 of an arbitrary central automorphism σ of R is identity. For every $j \in \Gamma_0$ the equalities $(\sigma - 1)(xe_{j^+,j}) = \sigma_j(x)e_{qp}$ $(x \in K)$ defines an additive map $\sigma_j \colon K \to K$. It is clear that $(\sigma + \mu)_j = \sigma_j + \mu_j$. Conversely, for every collection of endomorphisms σ_j of K^+ with $j \in \Gamma_0$ the map

$$\alpha \to \alpha + \sum_{j \in \Gamma_0} \sigma_j(a_{j^+,j})e_{qp} \quad (\alpha = \|a_{km}\| \in R)$$

determines an automorphism of R by [9, Lemma 1.1]. This completes the proof. □

Consider hypercentral automorphisms of $G(R)$ and $\Lambda(R)$. Let m be a positive integer and mth hypercenter does not coincide with R. According to [14, 15], an automorphism is called hypercentral of height $\leqslant m$ if it acts like the identity, modulo the mth hypercenter. By [12, 13], in the conditions of Main Theorem on K and Γ, every hypercentral automorphism of $\Lambda(R)$ is a product of certain inner and central automorphisms and the following automorphisms:

$$\begin{array}{ll} xe_{kp} \to (e_{kp} + ae_{qk})x \quad (x \in K) \quad \text{and} \\ xe_{qt} \to x(e_{qt} + be_{tp}) \quad (x \in K), \end{array} \tag{2}$$

for $p \lhd k < q$, $a(K * K) = 0$ and $p < t \lhd q$, $(K * K)b = 0$, respectively (the remaining elements xe_{uv} are fixed);

$$\begin{array}{ll} xe_{kp} \to (e_{kp} + ce_{qm})x, \qquad xe_{mp} \to (e_{mp} + ce_{qk})x \quad (x \in K) \\ \text{and} \quad xe_{qt} \to x(e_{qt} + de_{sp}), \qquad xe_{qs} \to x(e_{qs} + de_{tp}) \quad (x \in K), \end{array} \tag{3}$$

for $p \lhd k \lhd m < q$, $2c = c(K * K) = 0$ and $p < s \lhd t \lhd q$, $2d = (K * K)d = 0$, respectively. Also, the same is true for the adjoint group $G(R)$, however, the main hypercentral automorphisms of $G(R)$ act like an automorphism (2), modulo the center, or (3), modulo second hypercenter, and elements a, b, c and d of K have additional restrictions.

The following lemma describes the lower central series.

LEMMA 1.3. *The lower central series of $G(R)$ and of $\Lambda(R)$ coincide with the series $R \supset R^2 \supset R^3 \supset \cdots$. The intersection of all members of this series is generated by the sets Ke_{ij} $(j < i)$ with an infinite segment $[j, i]$. In particular, the equality $R = R^2$ is satisfied if and only if the chain Γ is dense.*

Proof. Straightforward. We note only that the inclusion $e_{ij} \in R^2$ for $i, j \in \Gamma$ is satisfied if and only if there exists $k \in \Gamma$ such that $j < k < i$. □

2. Certain Ideals and Isomorphisms

Let $G(R_S) \simeq G(R)$ or $\Lambda(R_S) \simeq \Lambda(R)$ for the ring $R_S = \mathrm{NT}(\Omega, S)$ with some chain Ω and associative ring S with identity. The adjoint group and the associated Lie ring of R are generated by subsets $Ke_{ij},\ i, j \in \Gamma,\ j < i$. The normal closure of Ke_{ij} in $G(R)$ and the minimal Lie ideal in $\Lambda(R)$ containing Ke_{ij} coincide with

$$N_{ij} = \langle Ke_{uv} \mid v < u,\ i \leqslant u,\ v \leqslant j\rangle \quad (i, j \in \Gamma).$$

The analoguos ideal of the ring R_S for $k, m \in \Omega$ is denoted by $N_{km}(S)$. Its ψ-image relative to any isomorphism ψ of $G(R_S)$ onto $G(R)$ or of $\Lambda(R_S)$ onto $\Lambda(R)$ is an ideal of the Lie ring $\Lambda(R)$. It shows

LEMMA 2.1. *A subset of R is a normal subgroup of the adjoint group $G(R)$ if and only if it is an ideal of the associated Lie ring $\Lambda(R)$.*

Proof. It is proved in [13, Corollary 1] even for arbitrary associative ring K with $K = K^2$, in particular, for a ring K with identity. (See also [8, §1].) □

Denote by $\widetilde{\Gamma}$, the set of initial segments of Γ. Putting $\overline{T} = \Gamma \setminus T$ we set

$$N_{VT} = \langle Ke_{km} \mid\ k \in V,\ m \in T,\ m < k\rangle, \quad N_T = N_{\overline{T}T},\ \overline{V}, T \in \widetilde{\Gamma}.$$

LEMMA 2.2. *Let $R = \mathrm{NT}(\Gamma, K)$, $R_S = \mathrm{NT}(\Omega, S)$ and ψ be an isomorphism of $G(R_S)$ onto $G(R)$ or of $\Lambda(R_S)$ onto $\Lambda(R)$. Let $[N_T(S)]^\psi = N_{T^{‘}}$, $T \in \widetilde{\Omega}$, for some bijective map ‘ of $\widetilde{\Omega}$ onto $\widetilde{\Gamma}$. Then there exists an isomorphism or an anti-isomorphism ‘ of the chain Ω onto Γ such that the ideal $[N_{ij}(S)]^\psi$ coincides, respectively, with $N_{i^{‘}j^{‘}}$ or $N_{j^{‘}i^{‘}}$ for all $i, j \in \Omega$, $j < i$. If ψ is an isomorphism of the ring R, then ‘ is an isomorphism of the chain Ω.*

Proof. Choose arbitrary segments $V, T, L \in \widetilde{\Omega}$ with $V \subset T \subset L$. The ψ-invariance of the inclusion $N_T(S) \supset N_V(S) \cap N_L(S)$ gives $N_{T^{‘}} \supset N_{V^{‘}} \cap N_{L^{‘}}$ and, therefore, $V^{‘} \subset T^{‘} \subset L^{‘}$ or $V^{‘} \supset T^{‘} \supset L^{‘}$, see Figure 1.

The relation $\subset$ for segments determines a linear ordering of $\widetilde{\Omega}$ and $\widetilde{\Gamma}$. Fixing two of three segments V, T, L and varying the third one in $\widetilde{\Omega}$, we easily deduce that ‘ is an isomorphism or an anti-isomorphism of the chain $\widetilde{\Omega}$ onto $\widetilde{\Gamma}$. If ψ is an isomorphism of the ring R, then the ψ-invariance of relations $N_T(S)N_L(S) = 0$ and $N_L(S)N_T(S) = N_T(S) \cap N_L(S)$ for $T \subset L$ shows that ‘ is an isomorphism of the chain $\widetilde{\Omega}$ onto $\widetilde{\Gamma}$. The segment $\{p\}$ for $p \in \Gamma$ is the first element of $\widetilde{\Gamma}$. Therefore there exists the first or last element r of Ω such that $[N_{rr}(S)]^\psi = N_{pp}$ and we set $r^{‘} = p$. Similarly, we define inverse image of the last element of Γ.

Note that the centralizer $C(N_{\overline{V}T})$ of $N_{\overline{V}T}$ in R can be written in the form

$$C(N_{\overline{V}T}) = N_{\overline{T}V}, \quad T, V \in \widetilde{\Gamma}.$$

Let T be the initial segment of Γ with the last element i which is not first or last in Γ. Then there exists the predecessor L of T in $\widetilde{\Gamma}$ with $\overline{L} \cap T = \{i\}$ and

$$N_{ii} = N_L \cup N_T = N_{\overline{L}T} = C(N_L \cap N_T), \qquad C(N_{ii}) = N_L \cap N_T = N_{\overline{T}L}.$$

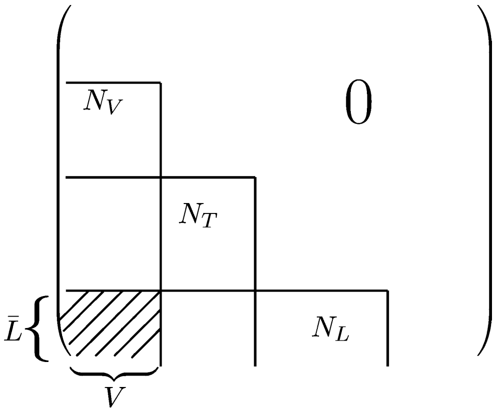

Figure 1.

Also, the intersection of inverse images of $\overline{L}$ and T in Ω consist of unique element m and we set $m` = i$. Evidently, ` is an isomorphism or an anti-isomorphism of the chain Ω onto Γ and $[N_{mm}(S)]^{\psi} = N_{m`m`}$ for all $m \in \Omega$. It remains to note that $N_{ij} = N_{ii} \cap N_{jj}$ for $j < i$. The lemma is proved. □

LEMMA 2.3. *Let $R_S = \mathrm{NT}(\Gamma, S)$ and ψ be an isomorphism of $G(R_S)$ onto $G(R)$ or of $\Lambda(R_S)$ onto $\Lambda(R)$. If $\psi[N_{ij}(S)] = N_{ij}$ for all $i, j \in \Gamma,\ j < i$, then ψ is an isomorphism of the ring R_S which is equal to a product of some K-ring isomorphism of R_S onto R, triangular and central automorphisms of R.*

Proof. The ideals N_{ij} and $N_{ij}(S)$ with $j < i$ have zero mulltiplication and, therefore, ψ is additive on $N_{ij}(S)$. Consequently, ψ preserves the relations

$$xe_{ij} \circ ye_{ij} = xe_{ij} + ye_{ij} = (x + y)e_{ij} \quad (i, j \in \Gamma,\ x, y \in S)$$

of R_S. Choose arbitrary matrices $\alpha \in N_{ij}(S)$ and $\beta \in N_{km}(S),\ m < k$. Evidently, the annihilator of the intersection $N_{ij} \cap N_{km}$ in R contains N_{ij} and N_{km}; the same is true for R_S. Therefore the commutator $[\alpha, \beta]$ in the adjoint group $G(R_S)$ coincides with the Lie product $\alpha * \beta$. Also, they coincide with $\alpha\beta$ for $k \leqslant i$ (because we have $N_{km}N_{ij} = 0$ in R and the same in R_S) and with $-\beta\alpha$ for $i \leqslant k$. It follows the equality $\alpha^{\psi}\beta^{\psi} = (\alpha\beta)^{\psi}$, in particular, $(xe_{ij})^{\psi}(ye_{km})^{\psi} = [(xe_{ij})(ye_{km})]^{\psi}$. Since ψ preserves all basic relations between elementary matrices of the ring R_S, it is an isomorphism of R_S onto the ring R.

Let Q_{ij} be the ideal $R * N_{ij}$ of the ring R; similarly, the ideal $Q_{ij}(S)$ of R_S is defined. Since $\psi[Q_{ij}(S)] = Q_{ij}$, the equalities $(xe_{ij})^{\psi} = x^{\sigma_{ij}}e_{ij} \pmod{Q_{ij}},\ x \in S$, define some isomorphisms σ_{ij} $(i, j \in \Gamma,\ j < i)$ of S^+ onto K^+. The ψ-invariance of relations $xye_{ik} = (xe_{ij})(ye_{jk}) \pmod{Q_{ik}}$ of R_S implies that $(xy)^{\sigma_{ik}} = x^{\sigma_{ij}}y^{\sigma_{jk}}$. In particular, $K = K^{\sigma_{ik}} = d_{ij}K = Kd_{jk}$ for $d_{ij} = (1_S)^{\sigma_{ij}}$ $(j < i)$ and hence all elements d_{ij} of K are invertible. Fixing $m \in \Gamma$ we may assume, up to multiplication of ψ by a diagonal automorphism of R, that $d_{im} = d_{mk} = 1_K$. It follows that all elements d_{ij} coincide with 1_K. Therefore all maps σ_{ij} coincide between them, and also σ_{ij} is an isomorphism of the ring S which induces a K-ring isomorphism τ of the ring R_S. Then $\tau^{-1}\psi$ is an automorphism of the ring R having the identity restriction on every ideal N_{ij}, modulo Q_{ij}. By (e) in the proof of Theorem 3 [13], such automorphism is a product of some locally inner and central automorphisms of R. The lemma is proved. □

We now need a description of maximal Abelian ideals of $\Lambda(R)$. Since $C(N_T) = C(N_{\overline{T}T}) = N_T$, the ideal N_T of R for any $T \in \widetilde{\Gamma}$ (in particular, N_{ij} for $j \lhd i$) is maximal Abelian.

Let (K, K) be the ring of all 1×2 matrices over K with the multiplication $(a, b)(c, d) = (bc, da)$ and usual addition. (This ring is simple for any simple ring K.) Every maximal commutative subset of the ring (K, K), different from $(K, 0)$ and $(0, K)$, can be written in the form $\{(x, x^\nu) \mid x \in F\}$ for some nonzero additive subgroup F of K and an isomorphism ν of F into K^+ with $x^\nu y = y^\nu x$ $(x, y \in F)$. If $p, q \in \Gamma$, then

$$C(N_{ii}) + \{xe_{ip} + x^\nu e_{qi} \mid x \in F\}, \quad p < i < q, \tag{4}$$

is a maximal abelian ideal of R by [11, Lemma 8]. Analogously, if maps λ, μ and an isomorphism ν of an additive subgroup F of K into K^+ satisfy the conditition $y^\nu xz + z^\nu xy = 0,\ x \in K,\ y, z \in F$ (in particular, $2K = 0$), then

$$\begin{aligned} C(N_{mk}) + \{ye_{mp} + (y^\lambda + xz)e_{kp} + (y^\mu - z^\nu x)e_{qk} + y^\nu e_{qm} \mid \\ x \in K,\ y, z \in F\} \end{aligned} \tag{5}$$

is an Abelian ideal of the Lie ring $\Lambda(R)$ for $p < k \lhd m < q$ by [11, Lemma 10].

LEMMA 2.4. *Let K be a ring with no zero-divisors. Then every maximal Abelian ideal of the ring R coincides with N_T for $T \in \widetilde{\Gamma}$ or with* (4). *A maximal Abelian ideal of the Lie ring $\Lambda(R)$ is either an ideal of the ring R or $|\Gamma| > 3,\ 2K = 0$ and it has the form* (5).

Proof. It had been shown in [11, §3] and [13, Theorem 2]. □

COROLLARY 2.5. *The ideals N_T $(T \in \widetilde{\Gamma})$ of the ring R exhaust all maximal Abelian ideals of R with zero multiplication.*

LEMMA 2.6. *Let* $R = \mathrm{NT}(\Gamma, K),\ R_S = \mathrm{NT}(\Omega, S)$ *and ψ be an isomorphism of $G(R_S)$ onto $G(R)$ (or of $\Lambda(R_S)$ onto $\Lambda(R)$). If K is a ring with no zero-divisors and $|\Gamma| > 4$, then $[N_T(S)]^\psi = N_{T^\text{`}}^\chi$ $(T \in \widetilde{\Omega})$ for some bijective map ‘ of $\widetilde{\Omega}$ onto $\widetilde{\Gamma}$ and hypercenter automorphism χ of height $\leqslant 3$ of $G(R)$ (resp. $\Lambda(R)$).*

Proof. If the center of R is zero, then $[N_T(S)]^\psi = N_{T^\text{`}}$ $(T \in \widetilde{\Omega})$ for some bijective map ‘: $\widetilde{\Omega} \to \widetilde{\Gamma}$, by Lemmas 1.1 and 2.4. Therefore we may assume that the center of R is nonzero and hence the chain Γ contains first and last elements by Lemma 1.1; of course, the same is true for the ring R_S.

The mth hypercenter of R is denoted by Z_m. Let $\{r, t\}$ be the subset of Ω consisting of the first and last elements of Ω. Then $[N_{rr}(S) + N_{tt}(S)]^\psi = N_{pp} + N_{qq}$ because the Lie ideal $N_{pp} + N_{qq}$ of R (similarly, $N_{rr}(S) + N_{tt}(S)$ of R_S) which is generated by two maximal Abelian ideals with the intersection coinciding with the center is unique, by Lemma 2.4. Putting $B = [N_{rr}(S)]^\psi$ we may assume that every (i, p)-projection of B ($i \in \Gamma$, $p < i$) is nonzero. By Lemma 2.4, it follows the equality $B = N_{pp}$, modulo $Z_3 \cap (N_{pp} + N_{qq})$. If $N_{pp} + Z_2$ does not contain B, then $Z_3 \neq Z_2$ and B has the form (5) with an isomorphism ν of an additive subgroup F of K into K^+ and $p \lhd k \lhd m < q$. Since ψ induces an isomorphism of the quotient-ring R_S/R_S^2 and $|\Gamma| > 4$, we get $F = K$, $y^\nu = cy$ ($y \in K$) and $2c = c(K * K) = 0$ for $c = 1^\nu$. Therefore hypercentral automorphism (3) of $\Lambda(R)$ is determined. If ψ is an isomorphism of $G(R)$, then the additional condition $c(x^2 - x)(y^2 - y) = 0$ ($x, y \in K$) is satisfied; conversely, for such element c of K the map

$$xe_{kp} \to (e_{kp} + ce_{qm})x, \qquad xe_{mk} \to xe_{mk} + c(x^2 - x)e_{qk},$$
$$xe_{mp} \to cxe_{mp} + cxe_{qk} + cx^2 e_{qp} \quad (x \in K)$$

is a hypercentral automorphism of $G(R)$ [12, §1]. Up to multiplication of ψ by a hypercenter automophism of height $\leqslant 3$, we obtain $B = N_{pp}$, modulo Z_2. Also, up to multiplication of ψ by a hypercenter automophism of height $\leqslant 2$ (of the form (2) for $\Lambda(R)$), we obtain $[N_{rr}(S)]^\psi = N_{pp}$ and similarly, $[N_{tt}(S)]^\psi = N_{qq}$. Consequently, by Lemma 2.4 there exist some bijective map ‘ of $\widetilde{\Omega}$ onto $\widetilde{\Gamma}$ such that the equality $[N_T(S)]^\psi = N_{T^{\prime}}$ are satisfied for all $T \in \widetilde{\Omega}$. The lemma is proved. □

Remark 2.7. Every chain Ω is anti-isomorphic to the ‘opposite’ chain Ω^{op}, i.e. the set Ω by the order relation which is opposite to one in the chain Ω. Denote by S^{op}, the opposite ring of S, see Exercise 17 in [5, p. 122]. It is clear that the map $xe_{ij} \to -xe_{ji}$ ($x \in S, i, j \in \Omega, j < i$) admits extension to an isomorphism of $G(R_S)$ onto $G[\mathrm{NT}(\Omega^{\mathrm{op}}, S^{\mathrm{op}})]$ and to an isomorphism of $\Lambda(R_S)$ onto $\Lambda[\mathrm{NT}(\Omega^{\mathrm{op}}, S^{\mathrm{op}})]$. This isomorphism is said to be opposite. Set $(\Omega_1, S_1) = (\Omega, S)$ for $\Omega \simeq \Gamma$ and otherwise $(\Omega_1, S_1) = (\Omega^{\mathrm{op}}, S^{\mathrm{op}})$. An isomorphism of $G(R)$ (similarly, $\Lambda(R)$) is said to be standard if it is a product of some hypercentral of height $\leqslant 3$ and triangular automorphisms and an idempotent S_1-ring isomorphism of $G(R)$ (resp. $\Lambda(R)$), an Ω_1-chain isomorphism of the ring $\mathrm{NT}(\Gamma, S_1)$ and, finally, an isomorphism τ of $G[\mathrm{NT}(\Omega_1, S_1)]$ onto $G(R_S)$ (resp. $\Lambda[\mathrm{NT}(\Omega_1, S_1)]$ onto $\Lambda(R_S)$), which is the identity map for $\Omega \simeq \Gamma$ and otherwise it is the opposite isomorphism.

EXAMPLE 2.8. Let $\Gamma = [a, b)$ and $\Omega = (c, d]$ be nonempty subchains of the usual chain of rational or real integers. Evidently that the chains Γ and Ω are anti-isomorphic but not isomorphic. Therefore there exists a standard isomorphism between the adjoint groups $G(R)$ and $G(R_S)$ or between Lie rings $\Lambda(R)$ and $\Lambda(R_S)$ if and only if $K \simeq S^{\mathrm{op}}$.

We now consider the case of a ring K with no zero-divisors in the main theorem.

THEOREM 2.9. *Let* $R = \mathrm{NT}(\Gamma, K)$, $R_S = \mathrm{NT}(\Omega, S)$, *$K$ be a ring with no zero-divisors and* $|\Gamma| > 4$. *If there exists an isomorphism* ψ *of* $G(R)$ *onto* $G(R_S)$ *or of* $\Lambda(R)$ *onto* $\Lambda(R_S)$, *then* $\Gamma \simeq \Omega$, $K \simeq S$ *or* $\Gamma \simeq \Omega^{\mathrm{op}}$, $K \simeq S^{\mathrm{op}}$, *and the isomorphism* ψ *is standard.*

Proof. Assume that there exists an isomorphism ψ of $G(R)$ onto $G(R_S)$ (resp. of $\Lambda(R)$ onto $\Lambda(R_S)$) and, therefore, $\Gamma \simeq \Omega$ or $\Gamma \simeq \Omega^{\mathrm{op}}$ by Lemmas 2.6 and 2.2. Let (Ω_1, S_1) and τ be as in Remark 2.7, so $\Gamma \simeq \Omega_1$. By Lemmas 2.6 and 2.2, there exists Ω_1-chain isomorphism σ of the ring $\mathrm{NT}(\Gamma, S_1)$ such that the ideal $\sigma\tau\psi^{-1}[N_{ij}(S_1)]$ $(i, j \in \Gamma,\ j < i)$ of $\Lambda(R)$ coincides with the image of N_{ij} relative to some hypercentral automorphism of $G(R)$ (resp. of $\Lambda(R)$). By Lemma 2.3, $\sigma\tau\psi^{-1} = \theta^{-1}\lambda^{-1}\chi^{-1}$ for a hypercentral automorphism χ of heihgt $\leqslant 3$, a triangular automorphism λ and a S_1-ring isomorphism θ of $G(R)$ (resp. $\Lambda(R)$). To complete the proof it remains to note that the following diagram is commutative:

$$\begin{array}{ccccc}
G(R) & \xrightarrow{\chi} & G(R) & \xrightarrow{\lambda} & G(R) \\
\Big\downarrow{\scriptstyle\psi} & & & & \Big\downarrow{\scriptstyle\theta} \\
G(R_S) & \xleftarrow{\tau} & G[\mathrm{NT}(\Omega_1, S_1)] & \xleftarrow{\sigma} & G[\mathrm{NT}(\Gamma, S_1)]
\end{array}$$

□

3. The Proof of the Main Theorem

Firstly, in this section we will prove the following theorem.

THEOREM 3.1. *Let* $R = \mathrm{NT}(\Gamma, K)$, $R_S = \mathrm{NT}(\Omega, S)$ *and* Γ *be a finite chain of order* > 4. *The adjoint groups* $G(R)$ *and* $G(R_S)$ *or associated Lie rings* $\Lambda(R)$ *and* $\Lambda(R_S)$ *are isomorphic if and only if rings K and S are idempotent isomorphic and* $|\Omega| = |\Gamma|$; *every isomorphism between them is standard.*

We need the following characterization of the one-sided Peirce decompositions.

LEMMA 3.2. *Let K be an associative ring with identity. Let* $K = A_1 + A_2 = B_1 + B_2$ *and* $A_i B_i = 0$ *for some subsets* A_i, B_i *of* K, $i = 1, 2$. *Then there exists an idempotent f of K such that* $A_1 = Kf$, $A_2 = K(1 - f)$, $B_1 = (1 - f)K$ *and* $B_2 = fK$. *If* $Kf = fK$, *then an idempotent f is in the center of K.*

Proof (see [12, Lemma 4]). By hypothesis, $1 = f_1 + f_2 = g_1 + g_2$ for some elements $f_i \in A_i$, $g_i \in B_i$. They satisfy the equations

$$f_1 = f_1(g_1 + g_2) = f_1 g_2 = (f_1 + f_2)g_2 = g_2, \qquad f_2 = g_1,$$
$$f_1 - f_1^2 = f_1(1 - f_1) = f_1 f_2 = f_1 g_1 = 0.$$

Consequently, f_1 is an idempotent and $A_i = A_i(g_1 + g_2) = A_i(1 - g_i) = A_i f_i$ for $i = 1, 2$. Since the sum $Kf_1 + K(1 - f_1) = K$ is direct [4, Sect. 3.7], we get $A_i = Kf_i$. Similarly, $B_i = (1 - f_i)K$, $i = 1, 2$. Thus, the first assertion of the lemma holds for $f = f_1$. Let $fK = Kf$ and $x \in K$. Then $yf = fx$, $xf = fz$ for some elements $y, z \in K$. Hence, $fx = (yf)f = fxf = f(fz) = xf$, i.e., f is a central idempotent. This proves the lemma. □

LEMMA 3.3. *Let* Γ *be a finite chain* $\{1, 2, \dots, n\}$ *and* ' *be an anti-automorphism of* Γ. *If* $1 \leqslant j < n,\ 1 < i \leqslant n$, *then* $N_{j',j} = C(R^j)$, *the ideal* N_{ij} *coincides with the intersection of the left annihilator of* R^j *and the right annihilator of* R^{n-i} *in* R *and also*

$$C(R^{n-j}) = C(R^{n-j}) \cap R^2 + C(R^{n-j-1}) + N_{j+1j} + N_{j',j'-1},$$
$$n - j \leqslant j < n. \tag{6}$$

Proof. The power R^j is additively generated by the sets Ke_{uv} with $u - v \geqslant j$. It is easy to see that the left annihilator of R^j and the right annihilator of R^{n-i} in R coincide with N_{2j} and N_{in-1}, respectively. It is clear that $N_{ij} = N_{in-1} \cap N_{2j}$. The formula of the centralizer $C(N_{\overline{V}T}) = N_{\overline{T}V}$ from Section 2 gives the equality $C(N_{ij}) = N_{j+1,i-1}$. Evidently, the map $m \to n + 1 - m,\ 1 \leqslant m \leqslant n$, is the unique anti-automorphism of the chain Γ. It follows the equalities $N_{j',j} = C(N_{j+1,n-j}) = C(R^j)$ and (6). The lemma is proved. □

Proof of Theorem 3.1. Let R be a ring $\mathrm{NT}(\Gamma, K)$ with a finite chain Γ and $R_S = \mathrm{NT}(\Omega, S)$. If $G(R) \simeq G(R_S)$ or $\Lambda(R_S) \simeq \Lambda(R)$, then $|\Omega| = |\Gamma|$ by Lemma 1.3; for a finite chain Γ it means that $\Omega \simeq \Gamma$. We now may assume (with using a chain isomorphism) that $\Omega = \Gamma$ and Γ coincides with the usual chain $\{1, 2, \dots, n\}$. In this case we may write $R = \mathrm{NT}(n, K)$ and $R_S = \mathrm{NT}(n, S)$, as usual.

Investigate arbitrary isomorphism ψ of $G(R_S)$ onto $G(R)$ or of $\Lambda(R_S)$ onto $\Lambda(R)$. Let $n > 4$ and $H^{(i)} = N_{i+1i}(S)^{\psi},\ 1 \leqslant i < n$. By (6), the sum of $(i+1, i)$-projections of $H^{(i)}$ and $H^{(n-i)}$ coincides with K. If $n = 2m$, then $m' = m + 1$ and by Lemma 3.3, $H^{(m)} = [C(R_S^m)]^{\psi} = C(R^m) = N_{m+1m}$. Choose arbitrary m with $n - m < m < n$. Since $H^{(i)}$ is an (maximal) abelian ideal of $\Lambda(R)$, we have $H^{(i)} \star (H^{(i)} \cap Ke_{m1}) = 0$ for $i = m, n - m$. Denote by $H^{(i)}_{uv}$, the (u, v)-projection of $H^{(i)}$, i.e., the set of all (u, v)-coefficients of matrices in $H^{(i)}$. It is not diffucult to show that $H^{(i)}_{m',m'-1} e_{m1} \subset H^{(i)}$. Therefore we get

$$H^{(i)}_{m+1m} H^{(i)}_{m',m'-1} = 0, \qquad K = H^{(m)}_{i+1i} + H^{(n-m)}_{i+1i}, \quad i = m, n - m.$$

Consequently, by Lemma 3.2, $H^{(i)}_{m+1m} = Kf_i$ and $H^{(i)}_{m',m'-1} = f_{n-i}K\,(i = m, n - m)$ for an idempotent f_m of K and $f_{n-m} = 1_K - f_m$. Using that $H^{(i)}$ is an Abelian ideal of $\Lambda(R)$, for $m > n - m + 1\ (= m')$ we get

$$H^{(i)} \cap Ke_{m1} = f_{n-i}Ke_{m1}, \qquad H^{(i)} \cap Ke_{nm'} = Kf_m e_{nm'}$$

and hence $Kf_mK \subseteq f_mK$; similarly, $Kf_mK \subseteq Kf_m$. Therefore, $Kf_m = f_mK$ and f_m is a central idempotent by Lemma 3.2. Since the Lie product $H^{(m+1)} * H^{(n-m)}$ is congruent to zero modulo R^3 for $n-m < m < n-1$, we have that $f_m(1-f_{m+1}) = f_{m+1}(1-f_m) = 0$ and $f_m = f_{m+1}$. Consequently, we have proved the existence of an idempotent f in the center of the ring K such that

$$\psi(xe_{i+1i}) = \mu_i(x)e_{i+1i} - \nu_i(x)e_{i',i'-1} (\mathrm{mod}\, R^2), \quad x \in S,\ 1 \leqslant i < n,$$

for some homomorphisms $\mu_i\colon S^+ \to fK$ and $\nu_i\colon S^+ \to (1_K - f)K$ of the additive group S^+. Also we get

$$(N_{i+1i}(S))^\psi = H^{(i)} = fN_{i+1i} + (1-f)N_{i',i'-1}, \quad i = 2, 3, \dots, n-2.$$

For $i = 1$ and $i = n-1$ these equations are also valid modulo R^2.

Considering the product $H^i * H^{(i-1)}$ $(1 < i < n)$, we obtain, modulo R^3,

$$fKe_{i+1i-1} + (1_K - f)Ke_{i'+1,i'-1} = \psi(e_{i+1i}) * H^{(i-1)} = H^{(i)} * \psi(e_{ii-1})$$

because the commutation [,] in the adjoint group of R and Lie multiplication $*$ of R coincides, modulo R^3. Now, it is easy to verify that the elements of the form $\mu_i(1_S) + \nu_j(1_S)$ are invertible in the ring K. Therefore, up to multiplication by a diagonal automorphism of the ring R, the isomorphism ψ satisfies the additional condition $\psi(e_{i+1i}) = fe_{i+1i} - (1_K - f)e_{i',i'-1}$, modulo R^2, for $1 \leqslant i < n$ and $\nu_1 = \nu_2 = \cdots = \nu_n - 1$. Since $(R_S^2)^\psi = R^2$ and $fK \cap (1_K - f)K = 0$, the equality $\mu_1(a) + \nu_1(a) = 0$ $(a \in S)$ gives $a = 0$. Therefore $\mu_1 + \nu_1$ is an isomorphism of the additive group S^+ onto K^+.

It is easy to verify that for any g-isomorphism σ of the ring S onto K with an idempotent g in the center of S the map σ^{-1} is an g^σ-isomorphism of the ring K onto S. Let $\theta = (\mu_1 + \nu_1)^{-1}$ and $f^\theta = g$. Since $fK \cap (1-f)K = 0$ we have $f = \theta^{-1}(g) = \mu_1(g) + \nu_1(g) = \mu_1(g)$ and $\nu_1(g) = 0$. Therefore,

$$\begin{aligned}
&\theta^{-1}(g^2) = \mu_1(g)^2 + \nu_1(g)^2 = f^2 = f = \mu_1(g) = \theta^{-1}(g),\\
&\theta^{-1}(gS) = \mu_1(gS) + \nu_1'(gS) = \mu_1(g)fK\\
&\qquad = f^2K = Kf^2 = \mu_1(Sg) = \theta^{-1}(Sg).
\end{aligned}$$

So g is an idempotent in the center of the ring S, by Lemma 3.2, and θ^{-1} is a g-isomorphism of the ring S onto K. Consequently, θ is an f-isomorphism of the ring K onto S.

Denote by τ, the idempotent S-ring isomorphism (1) of $G(R)$ (or $\Lambda(R)$, according to the choice of ψ), which is induced by the f-isomorphism θ. Let $\pi = \psi\tau$. Then $\pi \in \operatorname{Aut} G(R)$ (resp. $\pi \in \operatorname{Aut}\Lambda(R)$). Also for all $y \in K$ and $i > j$ we get

$$\begin{aligned}
\pi(ye_{ij}) &= \psi[(yf)^\theta e_{ij} - (y - yf)^\theta e_{j'i'}]\\
&= [(yf)^\theta g]^{\theta^{-1}} e_{ij} - [(yf)^\theta(1-g)]^{\theta^{-1}} e_{j'i'} -\\
&\quad - [(y-yf)^\theta(g)]e_{j'i'} + [(y-yf)^\theta(1-g)]^{\theta^{-1}} e_{ij}\\
&= [f^2 + (1-f)^2]ye_{ij} - [(1-f)yf + (y-yf)f]e_{j'i'}\\
&= ye_{ij}\ (\mathrm{mod}\, R^2).
\end{aligned}$$

Thus, the automorphism π acts like the identity, modulo R^2; by Lemma 13 of [12] it is a product of some hypercentral of height $\leqslant 3$ and inner automorphisms. Therefore ψ is a product of some idempotent K-ring isomorphism of $G(R_S)$ (resp. $\Lambda(R_S)$), hypercentral and triangular automorphisms of $G(R_S)$ (resp. $\Lambda(R_S)$), i.e., ψ is a standard isomorphism. Theorem 3.1 is proved. □

Remark 3.4. In particular case of $\Omega = \Gamma$ and $S = K$ Theorem 2.9 and Theorem 3.1 give the description of $\operatorname{Aut} G(R)$ and $\operatorname{Aut} \Lambda(R)$. For $n = |\Gamma| = 3, 4$ the statement of Theorem 3.1 is not true. In fact, choose a commutative ring K, a matrix $\|a_{ij}\| \in \mathrm{SL}(2, K)$ with $2a_{i1}a_{i2} = 0$ $(i = 1, 2)$ at $n = 4$ and the anti-automorphism ‘: $j \to n + 1 - j$ of Γ. Then the map

$$xe_{21} \to x(a_{11}e_{21} + a_{12}e_{1'2'}), \qquad xe_{1'2'} \to x(a_{21}e_{21} + a_{22}e_{1'2'}), \quad x \in K,$$

in generating subsets Ke_{j+1j} of the Lie ring $\Lambda(R)$ at $n = 3, 4$ (for $n = 4$ it is the identity map of the remaining subset Ke_{32}) defines its an isomorphism. Evidently, for standardness of this isomorphism it is necessary that the matrix $\|a_{ij}\|$ (and the group $\mathrm{SL}(2, K)$) is generated by transvections, see [12, §3].

Proof of the Main Theorem. Theorems 2.9 and 3.1 prove statements (ii) and (iii) of the main theorem. Consider arbitrary isomorphism ψ of the ring R_S onto R. Up to multiplication of ψ by a chain isomorphism, we may assume, by Corollary 2.5, Lemma 2.2 and (for a finite chain) Lemma 3.3, that $\Gamma = \Omega$ and $\psi[N_{ij}(S)] = N_{ij}$ for all $i, j \in \Gamma$, $j < i$. Then, by Lemma 2.3, ψ is a product of a K-ring isomorphism of R_S, triangular and central automorphisms of R. This completes the proof of the main theorem. □

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