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The Automorphism Group of Certain Radical Matrix Rings[1]

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INTRODUCTION

This paper is devoted to the study of automorphisms of matrix radical rings. The area has been under active investigation since the 1950s. Automorphisms of the algebra $NT_n(K)$ of all (lower) niltriangular $n \times n$ matrices over a field K were described by Dubish and Perlis [1, Theorem 5-7]. It is easy to verify that the automorphism group $\operatorname{Aut} R$ of any radical ring R coincides with the intersection of the automorphism group of the adjoint group $G(R)$ and the automorphism group of the associated Lie ring $\Lambda(R)$ of R. The adjoint group of $NT_n(K)$ is isomorphic to the unitriangular group $UT_n(K)$. If K is a finite field, then the group $UT_n(K)$ is a Sylow subgroup of $GL_n(K)$ and its automorphisms were studied in [13, 14, 16, 17]. For arbitrary associative ring K with identity automorphism groups of

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$NT_n(K)$, $G(NT_n(K))$ and $\Lambda(NT_n(K))$ were described in [9; 10, Theorem 1]; see surveys in [2, 15]. This result was extended to all Chevalley groups in [11, 12] and so the problem (1.5) of [6] on unipotent subgroups of Chevalley groups was solved. On the other hand, the question about description of automorphisms of Sylow p-subgroups of Chevalley groups over Z_{p^m} for $m > 1$ [7, Question 12.42] is still open. Let $M_n(J)$ be the ring of all $n \times n$ matrices over an ideal J of K and

$$R_n(K, J) := NT_n(K) + M_n(J).$$

By [4, 11.3.3] Sylow p-subgroups of the group $GL_n(Z_{p^m})$ are isomorphic to the adjoint group of the ring $R_n(Z_{p^m}, (p))$. Note that for any radical ring $R_n(K, J)$ investigations of the question about description of automorphism groups $\mathrm{Aut}\, G(R)$ and $\mathrm{Aut}\, \Lambda(R)$ for $R = R_n(K, J)$ have some additional difficulties. In fact, general results in [9, 10] were found by using close structural connections between the associated Lie ring and the adjoint group of $NT_n(K)$. However, for $R_n(K, J)$, these structural connections do not hold; see [7, Question 10.19; 8].

The aim of the present paper is to describe the automorphism group $\mathrm{Aut}\, R_n(K, J)$ for arbitrary K and quasi-regular ideal J with certain specific properties. Theorems 2.1 and 3.1 establish the structure of the automorphism group $\mathrm{Aut}\, R_n(K, J)$ when J coincides with a one-sided or two-sided annihilator of J^t in K for $t \geq 0$. As a corollary, Proposition 3.3 describes automorphisms of K-algebra $R_n(K, J)$. The order of $\mathrm{Aut}\, R_n(K, J)$ is given in Proposition 3.2 for any finite ring K and J as in Theorem 2.1. In particular, for an arbitrary divisor d of m $(1 \leq d < m)$ we obtain $|\mathrm{Aut}\, R_2(Z_{p^m}, (p^d))| = (p^m - p^{m-1}) \cdot p^{2m}$ and

$$|\mathrm{Aut}\, R_n(Z_{p^m}, (p^d))| = (p^m - p^{m-1})^{n-1} \cdot p^{(2m-d)\cdot C_n^2 + d(n-2)}, \qquad n > 2.$$

1. FUNDAMENTAL AUTOMORPHISMS AND POWERS OF $R_n(K, J)$

Throughout this paper K, J, and J^+ denote an associative ring with identity, an ideal of K, and the additive group of J, respectively. If $\|a_{uv}\|$ is a matrix, then a_{ij} is called the (i, j)-coefficient. We denote by e, the identity matrix, by e_{ij}, the matrix unit of $M_n(K)$ in which the (i, j)-coefficient is equal to 1 and others are zero. We use standard terminology, as in [3, 4].

The following lemma determines "annihilator" automorphisms of an arbitrary ring R. We set $\mathrm{Ann}\, R = \{\alpha \in R \mid \alpha R = R\alpha = 0\}$.

LEMMA 1.1. *Let* $\zeta : R \to \operatorname{Ann} R$ *be an additive map. Then*

(a) *the map* $1 + \zeta : x \to x + \zeta(x)$ *is an endomorphism of the ring* R *if and only if* $\zeta(R^2) = 0$;

(b) *if* $\zeta(R^2) = 0$ *and* $\operatorname{Ann} R \subseteq R^2$, *then* $1 + \zeta$ *is an automorphism of the ring* R.

Proof. (a) It follows from equalities $(x + \zeta(x))(y + \zeta(y)) = xy (x, y \in R)$.

(b) Evidently $\operatorname{Ker}(1 + \zeta) \subseteq \zeta(R) \subseteq \operatorname{Ann} R$ and if $\zeta(R^2) = 0$, then $1 + \zeta$ induces the identity map on R^2. If also $\operatorname{Ann} R \subseteq R^2$, then the map $1 + \zeta$ is an endomorphism of the ring R with zero kernel. It remains to note that inclusions

$$R \subseteq \zeta(R) + (1 + \zeta)R \subseteq R^2 + (1 + \zeta)R \subseteq (1 + \zeta)R \subseteq R$$

are equalities. The lemma is proved.

For an arbitrary associative ring R the adjoint multiplication $\circ$ and the associated Lie multiplication $*$ are defined as

$$\alpha \circ \beta = \alpha + \beta + \alpha\beta, \qquad \alpha * \beta = \alpha\beta - \beta\alpha.$$

An element $\alpha \in R$ is called quasi-regular if there exists an element $\alpha' \in R$ such that $\alpha \circ \alpha' = \alpha' \circ \alpha = 0$. For instance, the quasi-inverse element for a nilpotent element $-\alpha$ is defined as $(-\alpha') = \alpha + \alpha^2 + \alpha^3 + \cdots$. The adjoint conjugation of R by a quasi-regular element

$$\alpha' \circ y \circ \alpha = y + y * \alpha + \alpha'(y * \alpha), \qquad y \in R, \tag{1}$$

gives an "inner" automorphism of the ring R. It coincides with ordinary conjugation of R by the element $e + \alpha$ when the ring R contains identity e. A ring R is called radical if $(R, \circ)$ is a group. Each element α of any radical ring determines an inner automorphism as in (1).

Let R be the ring $R_n(K, J)$. It is a radical ring if and only if J is a quasi-regular ideal of K; i.e., $(J, \circ)$ is a group. The conjugation $\delta^{-1}\alpha\delta$ $(\alpha \in R)$ by an arbitrary invertible diagonal $n \times n$ matrix δ over K determines an automorphism of R which is called "diagonal." An automorphism θ of the ring K determines an automorphism $\|a_{uv}\| \to \|\theta(a_{uv})\|$ of the ring R if and only if the ideal J is θ-invariant. Such an automorphism of R is called a "K-ring" or "ring" automorphism as usual. On the other hand, an automorphism θ of the additive group K^+ determines an automorphism of the ring $R_2(K, J)$ as above if the ideal J is θ-invariant and the relation $(zy)^\theta = z^\theta y^\theta$ is satisfied for $z \in K$, $y \in J$ and for $z \in J$, $y \in K$. This generalization of a K-ring automorphism will be called a (K^+, J)-ring automorphism of $R_2(K, J)$ if $1^\theta = 1$.

Note that the ring R is generated by sets Ke_{i+1i} $(i = 1, 2, \ldots, n - 1)$ and Je_{1n} since $1 \in K$. The following lemmas describe powers R^k and their annihilators in the ring R. We put $J^0 = K$.

LEMMA 1.2. *Let k be a positive integer and $k = sn + t$, $0 \leq t < n$. Then the ideal R^k consists of all matrices $\|a_{uv}\|$ such that the element a_{uv} is placed in the ideal J^s, J^{s+1}, J^{s+2} respectively to cases $t \leq u - v$, $t - n \leq u - v < t$, $u - v < t - n$.*

Proof. It is easy to show by induction on k. (See also [4, 16.1.2; 5, Theorem 3].)

An ideal J is called nilpotent of class m, if m is the smallest positive integer such that $J^m = 0$. As a corollary of Lemma 1.2 we obtain that if J is a nilpotent ideal of K of class m, then the ring R is nilpotent of class mn.

LEMMA 1.3. *The left* (*resp. right*) *annihilator of R^k ($k = sn + t$, $0 \leq t < n$) in the ring R consists of all matrices $\alpha \in R$ such that all elements of the first t columns* (*resp. last $(n - t)$ rows*) *of α are in the left* (*resp. right*) *annihilator of J^{s+1} in K and other elements are placed in the left* (*resp. right*) *annihilator of J^s in K.*

Proof. It is sufficient to note that elements of the first t rows of matrices of R^k are ranged over the ideal J^{s+1}. Remaining elements of the first column of these matrices are ranged over the ideal J^s by Lemma 1.2.

Let $\mathrm{Ann}_K J = \{x \in K \mid xJ = Jx = 0\}$. Then $\mathrm{Ann}\, R = (\mathrm{Ann}_K J)e_{n1}$ by Lemma 1.3. If $n > 2$ or $n = 2$ but $\mathrm{Ann}_K J \subseteq J$, then $\mathrm{Ann}\, R \subseteq R^2$ by Lemma 1.2 and an arbitrary annihilator automorphism of the ring R has the form

$$\|a_{uv}\| \to \|a_{uv}\| + \left(\lambda_n(a_{1n}) + \sum_{i=1}^{n-1} \lambda_i(a_{i+1i})\right)e_{n1} \qquad (\|a_{uv}\| \in R), \tag{2}$$

where additive maps λ_n of J and $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ of K into $\mathrm{Ann}_K J$ satisfy $\lambda_n(J^2) = 0$, $\lambda_i(J) = 0$, $1 \leq i < n$. We denote by $\zeta_i(\lambda)$ $(1 \leq i \leq n)$ an annihilator automorphism (2) of R such that $\lambda_i = \lambda$ and λ_j are zero for all $j \neq i$. It is clear that the annihilator automorphism (2) of R is equal to the product $\zeta_1(\lambda_1)\zeta_2(\lambda_2)\cdots\zeta_n(\lambda_n)$.

Choose an arbitrary homomorphism $\sigma : J^+ \to K^+$ and $\lambda, \mu \in \mathrm{End}(J^+)$. Consider the following map of the set of all elementary matrices

$$\begin{aligned} &ye_{1n} \to ye_{1n} + y^\lambda e_{11} + y^\mu e_{nn} + y^\sigma e_{n1}, \qquad ye_{in} \to ye_{in} + y^\lambda e_{i1}, \\ &ye_{1j} \to ye_{1j} + y^\mu e_{nj}, \qquad 1 < i \leq n, \quad 1 \leq j < n, \quad y \in J. \end{aligned} \tag{3}$$

(We assume that the remaining elementary matrices from R are fixed.) If map (3) determines an automorphism of the ring R, then the invariance under (3) of relations $xe_{i1}ye_{1n} = xye_{in}$, $ye_{1n}xe_{nj} = yxe_{1j}$, and $(ye_{1n})\,(ze_{1n}) = 0$ gives

$$\begin{aligned} &(xy)^\lambda = xy^\lambda, \quad (yx)^\mu = y^\mu x, \quad yz^\mu = -y^\lambda z, \quad (zy)^\sigma = z^\mu y^\lambda, \\ &y^\mu z^\sigma + y^\sigma z^\lambda = yz^\sigma + y^\lambda z^\lambda = y^\sigma z + y^\mu z^\mu = 0, \qquad y, z \in J, \quad x \in K. \end{aligned} \tag{4}$$

On the other hand, if λ, μ, and σ satisfy (4), then map (3) preserves all basic relations

$$xe_{ij}+ye_{ij}=(x+y)e_{ij}, \qquad (xe_{ij})(ye_{jm})=xye_{im}, \qquad (xe_{ij})(ye_{km})=0,\ j\neq k,$$

in the ring R and hence it determines an automorphism of the ring R which will be called almost-annihilator. We denote by $\zeta^{(l)}(\lambda)$ (resp. $\zeta^{(r)}(\mu)$), an automorphism (3) with zero μ, σ (resp. λ, σ). By Lemma 1.3, $\zeta^{(l)}(\lambda)$ is the identity map of R modulo the left annihilator of R.

2. THE AUTOMORPHISM GROUP

We investigate the automorphism group $\operatorname{Aut} R_n(K, J)$ of a radical ring $R_n(K, J)$. Let K be an associative ring with identity, as above, and $K^{\#}$ be the multiplicative group of all invertible elements of K. Denote by $\mathscr{B}$ (resp. $\mathscr{B}'$), the subgroup of $\operatorname{Aut} R_n(K, J)$ which is generated by all annihilator and almost-annihilator (resp. almost-annihilator) automorphisms. Also, we denote by $\mathscr{D}$, $\mathscr{I}$, $\mathscr{A}(K, J)$, and $\mathscr{A}(K^+, J)$, subgroups which form all diagonal, inner, K-ring, and (K^+, J)-ring (for $n = 2$) automorphisms, respectively.

The following theorem is the main result of this section.

THEOREM 2.1. *Let J be an ideal of K such that a one-sided or two-sided annihilator of J^t in K coincides with J for a nonnegative integer t. Then* $\operatorname{Aut} R_n(K, J) = \mathscr{B}\mathscr{I}\mathscr{D}\mathscr{A}(K, J)$ *for $n > 2$. If inclusion*

$$\{c \in K | cJ = Jc = J (\operatorname{mod} J^2)\} \subseteq K^{\#} \tag{5}$$

is satisfied then $\operatorname{Aut} R_2(K, J) = \mathscr{B}'\mathscr{I}\mathscr{D}\mathscr{A}(K^+, J)$.

Let $R = R_n(K, J)$. We require the following lemmas.

LEMMA 2.2. *Let K be an associative ring with identity and $n > 2$. Then each automorphism of the ring $NT_n(K)$ is equal to a product of certain diagonal, inner, K-ring, and annihilator automorphisms of $NT_n(K)$.*

Proof. See [9; 10, Theorem 1].

LEMMA 2.3. *If an ideal J of the ring K coincides with a one-sided or two-sided annihilator of J^t in K for a nonnegative integer t and $n \geq 2$, then the ideal $M_n(J)$ of the ring R is characteristic.*

Proof. If $t = 0$, then $J^t = K$ and $J = 0$ since $1 \in K$. Suppose $t > 0$. All powers of R and also their one-sided annihilators are characteristic in R. The left (resp. right) annihilator of R^{tn} in R is equal to the set of all matrices of R over the left (resp. right) annihilator of J^t in K by Lemma 1.3. The intersection of one-sided annihilators is equal to $M_n(\operatorname{Ann}_K(J^t)) \cap R$. The lemma is proved.

LEMMA 2.4. *Let J be a quasi-regular ideal of K, $n \geq 2$, and let* (5) *hold for $n = 2$. Let ϕ be an automorphism of the ring R and let the ideal $M_n(J)$ be ϕ-invariant. Then there exists a diagonal automorphism δ of R such that the $(i+1, i)$-coefficient of the matrix $e_{i+1i}^{\phi\delta}$ is equal to 1 for all i, $1 \leq i < n$.*

Proof. Denote the $(i+1, i)$-coefficient of matrix e_{i+1i}^{ϕ} by c_i. First, we show that $c_i \in K^{\#}$ for all i, $1 \leq i < n$. If $n = 2$, we obtain $e_{21}^{\phi} \in c_1 e_{21} + M_2(J)$ and

$$R^2 = (e_{21} + M_2(J))^{\phi} R = Je_{21} + c_1 Je_{22} + Je_{11} + M_2(J^2)$$

since R^2 and $M_2(J)$ are ϕ-invariant. It gives $c_1 J + J^2 = J$ and similarly $Jc_1 + J^2 = J$. Consequently, $c_1 \in K^{\#}$ by (5). Suppose $n > 2$. The automorphism ϕ induces an automorphism of the quotient-ring $R/M_n(J)$ which is isomorphic to the ring $NT_n(K/J)$ over the associative ring K/J with identity. By Lemma 2.2 there exist elements $f_i \in K$ and $u_i \in J$ such that $c_i f_i = 1 + u_i$, $i = 1, 2, \ldots, n-1$. Therefore all elements $1 + u_i$ and c_i are invertible in K since the ideal J is quasi-regular.

Choose now the conjugation δ of R by the diagonal matrix diag$(d_1, d_2, \ldots, d_n)$ where $d_1 = 1$ and $d_{i+1} = c_i c_{i-1} \cdots c_2 c_1$, $1 \leq i < n$. Then the $(i+1, i)$-coefficient of matrix $e_{i+1i}^{\phi\delta}$ is equal to 1 for all i as required.

LEMMA 2.5. *Let $n \geq 2$ and let ϕ be an automorphism of a ring $R_n(K, J)$ such that the $(i+1, i)$-coefficient of a matrix e_{i+1i}^{ϕ} is equal to 1 for each i, $1 \leq i < n$. Then $\phi \in \mathcal{B}\mathcal{A}(K, J)\mathcal{I}$ for $n > 2$ and $\phi \in \mathcal{B}'\mathcal{A}(K^+, J)\mathcal{I}$ for $n = 2$.*

Proof. First, we show that there exists an inner automorphism ψ such that each matrix $e_{i+1i}^{\phi\psi} - e_{i+1i}$ has zero ith column. Clearly, for any matrix β the mth column of the matrix βe_{km} is equal to the kth column of β and other columns of βe_{km} are zero. Let $\alpha_t = (e_{tt-1}^{\phi} - e_{tt-1})e_{t-1t}$, $1 < t \leq n$. The matrix α_t is placed in the left ideal Re_{tt} of the ring R and $\alpha_t^2 = 0$. By (1) we get

$$\alpha_t' \circ e_{i+1i}^{\phi} \circ \alpha_t = e_{i+1i}^{\phi} - \alpha_t e_{i+1i}^{\phi} + (e - \alpha_t) e_{i+1i}^{\phi} \alpha_t \in e_{i+1i}^{\phi} - \alpha_t e_{i+1i}^{\phi} + Re_{tt}.$$

Denote by d_j the $(2, j)$-coefficient of the matrix e_{21}^{ϕ}. Since $d_1 = 1$, matrices α_2 and $\alpha_2 e_{21}^{\phi}$ have zero second rows and hence

$$\alpha_2 e_{21}^{\phi} = (e_{21}^{\phi} - e_{21}) e_{12} e_{21}^{\phi} = \sum_{j=1}^{n} (e_{21}^{\phi} - e_{21}) d_j e_{1j},$$

$$(\alpha_2' \circ e_{21}^{\phi} \circ \alpha_2) e_{11} = e_{21}^{\phi} e_{11} - (e_{21}^{\phi} - e_{21}) e_{11} = e_{21}.$$

Consequently, the first column of the matrix $\alpha_2' \circ e_{21}^{\phi} \circ \alpha_2$ is equal to the second column of the identity matrix. Suppose that $1 < i < n$ and each matrix $e_{t+1t}^{\phi} - e_{t+1t}$, $1 \le t < i$, has the zero tth column. The adjoint conjugation of the element α_{i+1} does not change the tth column of such a matrix since the tth column of $\alpha_{i+1}e_{t+1t}^{\phi}$ is zero. On the other hand, the ith column of the matrix $(\alpha'_{i+1} \circ e_{i+1i}^{\phi} \circ \alpha_{i+1}) - e_{i+1i}$ is also zero. Thus, without loss of generality we may assume that the ith column of each matrix $e_{i+1i}^{\phi} - e_{i+1i}$ $(1 \le i < n)$ is zero.

Consider the product $(xe_{km})^{\phi}e_{i+1i}^{\phi}$, $1 \le i < n$. Its ith column is equal to the $(i+1)$st column of the first factor. If $i+1 \ne m$, this product is equal to zero. Therefore, all columns of matrix $(xe_{km})^{\phi}$ $(1 \le k \le n, 1 \le m \le n)$ are zeros except the first and mth columns. In particular, $e_{i+1i}^{\phi} \in e_{i+1i} + Re_{11}$ for $1 < i < n$ and $e_{21}^{\phi} = e_{21}$. Consequently, the first row of each matrix $(xe_{km})^{\phi}$ for $k > 1$ is zero since $e_{21}^{\phi}(xe_{km})^{\phi} = 0$. For $n > 2$ we set $\alpha_1 = -b_3e_{21} - b_4e_{31} - \cdots - b_ne_{n-11}$ where b_{i+1} is the $(i+1, 1)$-coefficient of the matrix e_{i+1i}^{ϕ}. By (1) we obtain

$$\alpha'_1 \circ e_{21} \circ \alpha_1 = e_{21}, \quad \alpha'_1 \circ e_{i+1i}^{\phi} \circ \alpha_1 = e_{i+1i}^{\phi} + e_{i+1i}^{\phi}\alpha_1 = e_{i+1i}^{\phi} - b_{i+1}e_{i+11}$$

for $1 < i < n$. Therefore, without loss of generality we may assume that the $(i+1)$st row of each matrix $e_{i+1i}^{\phi} - e_{i+1i}$ $(1 \le i < n)$ is also zero. Since $e_{i+1i}^{\phi}(xe_{km})^{\phi} = 0$ for $i \ne k$, $1 \le i < n$, we obtain that all rows of a matrix $(xe_{km})^{\phi}$ are zeros except the kth and nth rows. In particular, the restriction of ϕ on $NT_n(K)$ is an automorphism of the ring $NT_n(K)$.

Suppose $n > 2$. By Lemma 2.2 there exist an automorphism θ of the ring K and endomorphisms ϕ_i of the additive group K^+ such that

$$(xe_{i+1i})^{\phi} = x^{\theta}e_{i+1i} + x^{\phi_i}e_{n1}, \tag{6}$$

$$e_{i+1i}^{\phi} = e_{i+1i} + a_ie_{n1}, \qquad a_1 = a_{n-1} = 0 \quad (x \in K,\ 1 \le i < n) \tag{7}$$

for $a_i = 1^{\phi_i}$. Clearly $(xe_{ij})^{\phi} = x^{\theta}e_{ij}$ for $i - j > 1$. The relations $ye_{n1} = e_{nn-1} \cdots e_{32}(ye_{21}) = e_{n1}(ye_{1n})e_{n1}$ are ϕ-invariant for all $y \in J$. Hence the $(1, n)$-coefficient of a matrix $(ye_{1n})^{\phi}$ is equal to y^{θ}. By using (6) and (7) we get

$$(ye_{1n})^{\phi} = y^{\theta}e_{1n} + y^{\lambda}e_{11} + y^{\mu}e_{nn} + y^{\sigma}e_{n1}, \qquad (ye_{in})^{\phi} = y^{\theta}e_{in} + y^{\lambda}e_{i1},$$

$$(ye_{1j})^{\phi} = (ye_{1n})^{\phi}e_{nj} = y^{\theta}e_{1j} + y^{\mu}e_{nj}, 1 \le j < n, \qquad 1 < i \le n, \qquad y \in J, \tag{8}$$

where $\lambda, \mu \in \mathrm{End}(J^+)$ and σ is a homomorphism of J^+ into K^+. Since the set of all $(1, n)$-coefficient of matrices in R^{ϕ} coincides with J^{θ} we obtain the equality $J = J^{\theta}$. Therefore, θ induces a K-ring automorphism of the ring R. Without loss of generality we may assume that θ is the identity map of K. The ϕ-invariance of relations $(Ke_{i+1i})(Je_{1n}) = 0 = (Je_{1n})(Ke_{ii-1})$ gives $(K^{\phi_i})J = 0 = J(K^{\phi_{i-1}})$ for $1 < i < n$. Also we obtain

$$(xJe_{i+1i})^{\phi} = (xe_{i+1i})^{\phi}(Je_{ii})^{\phi} = (xe_{i+1i} + x^{\phi_i}e_{n1})(Je_{ii}), \qquad 1 \le i < n, \quad x \in K.$$

Consequently, $J^{\phi_i} = (K^{\phi_i})J = a_iJ$ and similarly $J^{\phi_i} = J(K^{\phi_i}) = Ja_i$. Taking into account (7) we get that ϕ is a product of the annihilator and almost-annihilator automorphisms as in Section 1.

Assume $n = 2$. Let x^θ be the (2,1)-coefficient of a matrix $(xe_{21})^\phi$ for $x \in K$. As above, we get $1^\theta = 1$ and

$$(xe_{21})^\phi = x^\theta e_{21}(x \in K),$$
$$e_{21}(ye_{12})^\phi e_{21} = [e_{21}(ye_{12})e_{21}]^\phi = y^\theta e_{21}, \quad y \in J.$$

Therefore, (8) is satisfied and θ is an automorphism of the additive group K^+ such that $J^\theta = J$. Finally, relations $(zy)^\theta e_{21} = (ze_{21})^\phi (ye_{11})^\phi = z^\theta y^\theta e_{21}$ show that the relation $(zy)^\theta = z^\theta y^\theta$ is satisfied for $z \in K$, $y \in J$ and similarly for $z \in J$, $y \in K$. Consequently, ϕ is a product of the almost-annihilator and (K^+, J)-ring automorphisms of $R_2(K, J)$. The lemma is proved.

Now Theorem 2.1 follows easily by Lemmas 2.3–2.5.

We consider some cases when the conditions of Theorem 2.1 hold.

(A) Let J be a maximal ideal of K which is nilpotent of a class $t+1 > 1$. Then $\text{Ann}_K(J^t) = J$ since $\text{Ann}_K(J^t)$ is a proper ideal of K which contains J. If K is a local ring, then $K \backslash J = K^\#$ and (5) is satisfied.

(B) Let a be an element of a ring K and $aK = Ka = \text{Ann}_K(a^t)$ for a positive integer t. Let J be the principal ideal (a). Clearly $\text{Ann}_K(J^t) = J$. Suppose J contains one-sided annihilators of a. (For instance, $\text{Ann}_K a = \text{Ann}_K J \subseteq \text{Ann}_K(J^t) = J$ if a is in the center of the ring K.) Then (5) is satisfied. In fact, if $c \in K$ and $cJ + J^2 = J$ then there exist elements $x, y \in K$ such that $(cx + ya - 1)a = 0$ and $cx \in 1 + J \subseteq K^\#$. Therefore there exists a right (similarly, left) inverse of c in K.

(C) Let p be a prime and m be a positive integer. Let $K = M_n(Z_{p^m})$ for $n \geq 1$ or K is a ring of polynomials in commutative or noncommutative indeterminates (of finite or infinite number) over Z_{p^m}. If d is an arbitrary divisor of m, $1 \leq d < m$, and J is the principal ideal p^dK of K, then the case (B) for $t = (m-d)/d$ holds.

EXAMPLE 2.6. Let K_1 be an associative ring with identity which has a nilpotent ideal J_1 of class two. Let K be a direct product (K_1, K_1) of two copies of the ring K_1 and let J be the ideal $(J_1, 0)$ of K. If $\lambda : (a, 0) \to (a, a)(a \in J_1)$ then $\zeta_n(\lambda)$ is an automorphism of the ring $R_n(K, J)(n > 2)$ by Lemma 1.1 and the ideal $M_n(J)$ is not $\zeta_n(\lambda)$-invariant.

Remark 2.7. Let J be an arbitrary quasi-regular ideal of a ring K and $n > 2$. All automorphisms of the ring $R_n(K, J)$ that leave invariant the ideal $M_n(J)$ are described by Lemmas 2.4 and 2.5. In the general case, the subgroup of such automorphisms does not coincide with the automorphism group of the ring $R_n(K, J)$ as the last example shows. However, the authors have no example of a radical ring $R_n(K, J)$ such that the equality $\operatorname{Aut} R_n(K, J) = \mathcal{BIDA}(K, J)$ does not hold.

3. THE STRUCTURE OF THE AUTOMORPHISM GROUP

We investigate the structure of the automorphism group of a radical ring R in Theorem 2.1. As above, $R = R_n(K, J)$. Consider the subgroup series

$$\mathcal{I} \subseteq \mathcal{BI} \subseteq \mathcal{BID} \subseteq \mathcal{BIDA}(K, J). \tag{9}$$

We denote the multiplicative group of all invertible diagonal $n \times n$ matrices over K by $D_n(K)$ as usual. Let $\mathcal{B}'_{\mathcal{I}}$ (resp. $\mathcal{B}_{\mathcal{I}}$) be the subgroup of inner automorphisms that are induced by adjoint conjugations with elements from Ke_{n1} (resp. $\{Ke_{n1} + (\operatorname{Ann}_K J)e_{n2} + (\operatorname{Ann}_K J)e_{n-11}\} \cap R$). Let $\Lambda(K, J)$ (resp. $\Lambda'(K, J)$) be the additive group of all homomorphisms $\lambda : K^+ \to \operatorname{Ann}_K J$ (resp. $\lambda : J^+ \to \operatorname{Ann}_K J$) such that $\lambda(J) = 0$ (resp. $\lambda(J^2) = 0$). We also denote by $\Lambda^{(l)}(K, J)$ the additive group of all K-module homomorphisms of the left K-module J into the left annihilator of J in J. Using (4) it is easy to verify that maps

$$\zeta_i\colon \Lambda(K, J) \to \mathcal{B}(1 \le i < n), \qquad \zeta_n\colon \Lambda'(K, J) \to \mathcal{B},$$
$$\zeta^{(l)}\colon \Lambda^{(l)}(K, J) \to \mathcal{B},$$

(see Section 1) are group monomorphisms.

THEOREM 3.1. *Let $C(K)$ be the center of a ring K, $n \ge 2$, and $C(R) = \operatorname{Ann} R + (J \cap C(K))e$. Let $\operatorname{Ann}_K J \subseteq J$ for $n = 2$. Then,*

(i) *the subgroup series* (9) *is normal in the group $\mathcal{BIDA}(K, J)$ and equalities $(\mathcal{BID}) \cap \mathcal{A}(K, J) = \mathcal{D} \cap \mathcal{A}(K, J)$, $(\mathcal{BI}) \cap \mathcal{D} = \mathcal{I} \cap \mathcal{D}$, and $\mathcal{I} \cap \mathcal{B} = \mathcal{B}_{\mathcal{I}}$ hold*;

(ii) *there exist the isomorphisms*

$$\mathcal{D} \simeq D_n(K)/(K^\# \cap C(K))e, \qquad \mathcal{D} \cap \mathcal{A}(K, J) \simeq K^\#/(K^\# \cap C(K)),$$
$$\mathcal{I} \simeq (R, \circ)/C(R), \qquad \mathcal{I} \cap \mathcal{D} \simeq \left(\sum_{i=1}^{n} Je_{ii}, \circ\right)\Big/(J \cap C(K))e;$$

(iii) *the subgroup $\mathcal{B}$ is a direct product of subgroups $\mathcal{B}'$, $\zeta_i(\Lambda(K, J))$, $1 \le i < n$*;

(iv) *if J is a principal ideal (a) and $aK = Ka$, then*

$$\mathcal{B}' = \mathcal{B}'_{\mathcal{I}} \times \zeta_n(\Lambda'(K, J)) \times \zeta^{(l)}(\Lambda^{(l)}(K, J)).$$

Proof. (i) The subgroup $\mathcal{I}$ is normal in $\operatorname{Aut} R$ since $\operatorname{Aut} R \subseteq \operatorname{Aut}(R, \circ)$ and $\mathcal{I} \trianglelefteq \operatorname{Aut}(R, \circ)$. It is easy to show that $\mathcal{D} \trianglelefteq (\mathcal{D}\mathcal{A}(K, J))$. Similarly, normalizers in $\operatorname{Aut} R$ of subgroups $\zeta_i(\Lambda(K, J))$, $1 \leq i < n$, and $\mathcal{B}'$ contain $\mathcal{D}$ and $\mathcal{A}(K, J)$. By (2) subgroups $\zeta_i(\Lambda(K, J))$ and $\mathcal{B}'$ generate $\mathcal{B}$ so $\mathcal{B}\mathcal{I}$ is a normal subgroup of series (9). Consequently, the subgroup series (9) of the group $\mathcal{B}\mathcal{I}\mathcal{D}\mathcal{A}(K, J)$ is normal. We get $(\mathcal{B}\mathcal{I}) \cap \mathcal{D} = \mathcal{I} \cap \mathcal{D}$ since each intersection $(Ke_{ij}) \cap R$ is $\mathcal{D}$-invariant. Similarly, $(\mathcal{B}\mathcal{I}\mathcal{D}) \cap \mathcal{A}(K, J) = \mathcal{D} \cap \mathcal{A}(K, J)$. Clearly, $\mathcal{B}_{\mathcal{I}} \subseteq \mathcal{B} \cap \mathcal{I}$ for $n > 2$. It is also true for $n = 2$ if J is a quasi-regular ideal such that $\operatorname{Ann}_K J \subseteq J$. Suppose that the adjoint conjugation of R by an element $\alpha \in R$ is equal to an element $\chi \in \mathcal{B}$. By (1) we get $(Ke_{i+1i}) * \alpha \subseteq (e + \alpha)\operatorname{Ann} R = \operatorname{Ann} R$ for $1 \leq i < n$ since $\beta^\chi - \beta \in \operatorname{Ann} R$ for each $\beta \in NT_n(K)$. It follows that $\chi \in \mathcal{B}_{\mathcal{I}}$ and $\mathcal{I} \cap \mathcal{B} = \mathcal{B}_{\mathcal{I}}$.

(ii) The subgroup $\mathcal{I}$ is isomorphic to the quotient-group of the adjoint group of R by its center. The center of the ring R coincides with the center of the adjoint group and it contains $C(R)$. The inverse inclusion is also true since any matrix α in the center of R satisfies relations $\alpha * (Ke_{i+1i}) = \alpha * (Je_{1n}) = 0$, $1 \leq i < n$. Thus, the center of the adjoint group is equal to $C(R)$ and $\mathcal{I} \simeq (R, \circ)/C(R)$.

The intersection $\mathcal{D} \cap \mathcal{A}(K, J)$ coincides with the set of all conjugations of R by matrices from $K^{\#}e$. In fact, if $\theta \in \mathcal{D} \cap \mathcal{A}(K, J)$ and θ coincides with the conjugation of R by a diagonal matrix $\alpha \in D_n(K)$, then all elements of the main diagonal of α pairwise coincide because $e_{i+1i}^\theta = e_{i+1i}$, $1 \leq i < n$. The centralizer of R in $D_n(K)$ coincides with $(K^{\#} \cap C(K))e$. It gives required isomorphisms of $\mathcal{D}$ and $\mathcal{D} \cap \mathcal{A}(K, J)$. Also we get $\mathcal{I} \cap \mathcal{D} \simeq (C(R) + (R \cap (D_n(K) - e)), \circ)/C(R)$. Since $C(R) \cap R \cap (D_n(K) - e) = C(R) \cap (D_n(K) - e) = (J \cap C(K))e$ we obtain the required isomorphism of $\mathcal{I} \cap \mathcal{D}$.

(iii) Note that the subring $NT_n(K)$ of R is $\mathcal{B}$-invariant and each almost-annihilator automorphism of R induces the identity map on $NT_n(K)$. By using (2) we obtain $\mathcal{B} = \mathcal{B}' \times \zeta_1(\Lambda(K, J)) \times \cdots \times \zeta_{n-1}(\Lambda(K, J))$.

(iv) Suppose that $J = aK = Ka$ for some $a \in K$. The decomposition of the subgroup $\mathcal{B}'$ follows easily if we show that subgroups $\zeta_n(\Lambda'(K, J))$, $\zeta^{(l)}(\Lambda^{(l)}(K, J))$, and $\mathcal{B}'_I$ generate the subgroup $\mathcal{B}'$. Choose an arbitrary almost-annihilator automorphism χ of the ring R. It is determined in (3) by means of a homomorphism $\sigma: J^+ \to K^+$ and endomorphisms $\lambda, \mu \in \operatorname{End}(J^+)$ which satisfy (4). In particular, λ and μ are K-module endomorphisms of the left and right K-module J, respectively. By (1) we get

$$(-xe_{n1}) \circ (ae_{1n})^\chi \circ xe_{n1} \in ae_{1n} + (a^\lambda + ax)e_{11} + (a^\mu - xa)e_{nn} + Ke_{n1}$$

for all $x \in K$. The equation $a^\mu - xa = 0$ is solvable in K because $J^\mu \subseteq J = Ka$. Therefore we can account $a^\mu = 0$ up to multiplication of χ by an inner automorphism from $\mathcal{B}'_I$. Hence $J^\mu = (aK)^\mu = a^\mu K = 0$ since μ is a K-module endomorphism of the right K-module J. By (4) we obtain $(J^2)^\sigma = J^\mu J^\lambda = 0 = (J^\mu)^2 = J^\sigma J$ and $J^\lambda J = JJ^\mu = 0 = (J^\lambda)^2 = JJ^\sigma$. Consequently, $\sigma \in \Lambda'(K, J)$, $\lambda \in \Lambda^{(l)}(K, J)$, and $\chi = \zeta_n(\sigma) \cdot \zeta^{(l)}(\lambda)$. The theorem is proved.

We now consider the order $|\mathrm{Aut}\, R_n(K, J)|$ of the automorphism group for any finite ring K (which are within Theorem 2.1). Taking into account Remark 2.7 we define Q_n to be the order of the subgroup of $\mathcal{B}\mathcal{I}\mathcal{D}\mathcal{A}(K, J)$ and Q_2^+ to be the order of $\mathcal{B}\mathcal{I}\mathcal{D}\mathcal{A}(K^+, J)$ for $n = 2$.

PROPOSITION 3.2. *Let K be a finite ring and J be a quasi-regular ideal of K. Suppose* $\mathrm{Ann}_K J \subseteq J$ *for* $n = 2$. *Then* $Q_2^+ = |\mathcal{B}'| \cdot |\mathcal{A}(K^+, J)| \cdot |\mathcal{K}^{\#}| \cdot |J|$ *and*

$$Q_n = (|\mathcal{B}'|/(|K| \cdot |\mathrm{Ann}_K J|^2)) \cdot |\mathcal{A}(K, J)| \cdot (|K^{\#}| \cdot |\Lambda(K, J)|)^{n-1}$$
$$\cdot (|K| \cdot |J|)^{C_n^2}, \qquad n > 2.$$

If $J = (a)$ for $a \in C(K)$, then $|\mathcal{B}'| = |\Lambda'(K, J)| \cdot |K| \cdot |\mathrm{Ann}_J J| \cdot |\mathrm{Ann}_K J|^{-1}$.

Proof. By Theorem 3.1 we get

$$|\mathcal{D}|/|\mathcal{D} \cap \mathcal{A}(K, J)| = |D_n(K)|/|K^{\#}| = |K^{\#}|^{n-1},$$
$$|\mathcal{I}|/|\mathcal{I} \cap \mathcal{D}| = |R|/(|\mathrm{Ann}\, R| \cdot |J|^n)$$
$$= (|K| \cdot |J|)^{C_n^2}/|\mathrm{Ann}_K J|,$$
$$|\mathcal{B}| = |\Lambda(K, J)|^{n-1} \cdot |\mathcal{B}'|,$$
$$|\mathcal{B} \cap \mathcal{I}| = |\mathcal{B}_{\mathcal{I}}| = |K| \cdot |\mathrm{Ann}_K J|,$$

for each $n \geq 2$. Note that the order $|HM|$ of the product of two arbitrary subgroups H, M in an arbitrary group is equal to the product $|H| \cdot |M| \cdot |H \cap M|^{-1}$; see [3, Theorem I.4.7]. Therefore, we obtain the required decomposition of Q_n by Theorem 3.1(i). Suppose $n = 2$ and $\mathrm{Ann}_K J \subseteq J$. Then $\zeta_1(\Lambda(K, J)) \subseteq \mathcal{D}\mathcal{A}(K^+, J)$ and $\mathcal{B}\mathcal{I}\mathcal{D}\mathcal{A}(K^+, J) = \mathcal{B}'\mathcal{I}\mathcal{D}\mathcal{A}(K^+, J)$ as in the proof of Theorem 2.1. We get $\mathcal{B}' \cap \mathcal{I} = \mathcal{B}' \cap \mathcal{B}_{\mathcal{I}} = \mathcal{B}'_{\mathcal{I}}$ and $|\mathcal{B}'_{\mathcal{I}}| = |K|/|\mathrm{Ann}_K J|$. The formula for Q_2^+ follows easily since by 3.1(i) we obtain

$$(\mathcal{B}'\mathcal{I}) \cap \mathcal{D} = \mathcal{I} \cap \mathcal{D},$$
$$(\mathcal{B}'\mathcal{I}\mathcal{D}) \cap \mathcal{A}(K^+, J) = \mathcal{D} \cap \mathcal{A}(K^+, J) = \mathcal{D} \cap \mathcal{A}(K, J).$$

Suppose that $J = aK = Ka$ for some element $a \in K$. Each K-module endomorphism of the left K-module J is uniquely defined by an image of the element a and this image may be an arbitrary element in J. Therefore $|\Lambda^{(l)}(K, J)| = |\mathrm{Ann}_J J|$ for $a \in C(K)$. Using Theorem 3.1(iv) we now obtain the required decomposition of $|\mathcal{B}'|$. This completes the proof.

Using Theorem 2.1 we may describe automorphisms of K-algebras $R_n(K, J)$. Let $\mathscr{A}_{\mathrm{mod}}$ be the automorphism group of the algebra $R_n(K, J)$.

PROPOSITION 3.3. *Let K be a commutative ring and let J be an ideal of K such that $\mathrm{Ann}_K(J^t) = J$ for a positive integer t. Suppose* (5) *is satisfied for $n = 2$. Then $\mathscr{A}_{\mathrm{mod}} = (\mathscr{A}_{\mathrm{mod}} \cap \mathscr{B})\mathscr{I}\mathscr{D}$. If K is a finite ring and J is a principal ideal, then $|\mathscr{A}_{\mathrm{mod}}| = |K^{\#}| \cdot |K| \cdot |J| \cdot |\mathrm{Ann}_K J|$ for $n = 2$ and*

$$|\mathscr{A}_{\mathrm{mod}}| = |K^{\#}|^{n-1} \cdot |\mathrm{Ann}_K J|^{n-2} \cdot (|K| \cdot |J|)^{C_n^2}, \qquad n > 2.$$

Proof. Let $\mathscr{B}_{\mathrm{mod}} = \mathscr{A}_{\mathrm{mod}} \cap \mathscr{B}$ and let $\phi \in \mathscr{A}_{\mathrm{mod}}$. By Theorem 2.1 there exist a K-ring or (K^+, J)-ring automorphism θ of R and an automorphism $\chi \in \mathscr{B}\mathscr{I}\mathscr{D}$ such that $\phi = \chi\theta$. Without loss of generality we may assume that $\chi \in \mathscr{B}$ since $\mathscr{I}\mathscr{D} \subseteq \mathscr{A}_{\mathrm{mod}}$. Similarly $\chi \in \mathscr{B}'$ for $n = 2$ as in Theorem 2.1 so $(xe_{21})^\chi = xe_{21}$ for $n \geq 2$. We get

$$x^\theta e_{21} = (xe_{21})^\theta = (xe_{21})^\phi = x(e_{21}^\phi) = x(e_{21}^\theta) = xe_{21}.$$

Consequently, θ is the identity map, $\chi \in \mathscr{B}_{\mathrm{mod}}$, and the decomposition of $\mathscr{A}_{\mathrm{mod}}$ is proved.

By using Theorem 3.1(iii) we obtain that $\mathscr{B}_{\mathrm{mod}}$ is equal to a direct product of subgroups $\mathscr{B}_{\mathrm{mod}} \cap \mathscr{B}', \mathscr{B}_{\mathrm{mod}} \cap \zeta_i(\Lambda(K, J))$, $1 \leq i < n$. Clearly, an annihilator automorphism $\zeta_i(\lambda)$ (resp. an almost-annihilator automorphism (3)) of R is a K-module if and only if λ (resp. σ) is a K-module homomorphism of the K-module K (resp. J). Therefore, we obtain $|\mathscr{B}_{\mathrm{mod}} \cap \zeta_i(\Lambda(K, J))| = |\mathrm{Ann}_K J|$ $(1 \leq i < n)$ for a finite ring K. Suppose $J = aK$ for some $a \in K$. Then $\mathscr{B}' \cap \mathscr{B}_{\mathrm{mod}}$ is equal to a direct product of subgroups $\mathscr{B}_{\mathrm{mod}} \cap \zeta_n(\Lambda'(K, J))$, $\zeta^{(l)}(\Lambda^{(l)}(K, J))$ and $\mathscr{B}'_{\mathscr{J}}$ by Theorem 3.1(iv). Since $\mathrm{Ann}_K J \subseteq \mathrm{Ann}_K(J^t) = J$ we get equalities.

$$|\mathscr{B}_{\mathrm{mod}} \cap \zeta_n(\Lambda'(K, J))| = |\mathrm{Ann}_K J| = |\mathrm{Ann}_J J| = |\Lambda^{(l)}(K, J)|.$$

Using Theorem 3.1 and Proposition 3.2 we obtain the required formula for $\mathscr{A}_{\mathrm{mod}}$. This completes the proof.

Note that the description of $\mathscr{A}_{\mathrm{mod}}$ was found by Dubish and Perlis [1, Theorem 5-7] for arbitrary field K and $J = 0$. See also [9, Corollary 1]. If $K = Z_{p^m}$, then $\mathscr{A}_{\mathrm{mod}} = \mathrm{Aut}\, R_n(K, J)$. Therefore,

COROLLARY 3.4. *Let $K = Z_{p^m}$ and d be an arbitrary divisor of m such that $1 \leq d < m$. If $J = (p^d)$, then $|\mathrm{Aut}\, R_2(K, J)| = (p^m - p^{m-1}) \cdot p^{2m}$ and*

$$|\mathrm{Aut}\, R_n(K, J)| = (p^m - p^{m-1})^{n-1} \cdot p^{(2m-d)\cdot C_n^2 + d(n-2)}, \qquad n > 2.$$

Proof. It follows from the equality $|K| = |\mathrm{Ann}_K J| \cdot |J|$ and Proposition 3.3.

According to [1] the automorphism group $\mathrm{Aut}\, R$ of an arbitrary associative ring R has a normal subgroup $\mathcal{M}$ of all "monic" automorphisms of R which induce the identity map into quotient-ring R^k/R^{k+1} for all positive integers k. Let $R = R_n(K, J)$, $n > 2$. Clearly $\mathcal{M} \supseteq \mathcal{B}\mathcal{I}$. If $J = 0$, then $\mathcal{M} \cap \mathcal{D} = 1$ (see [1, 9]) and even the group $\mathrm{Aut}\, R$ is equal to the semidirect product of subgrpups $\mathcal{M}$ and $\mathcal{D}\mathcal{A}(K, J)$ [9]. However, the intersection $\mathcal{M} \cap \mathcal{D}$ is nontrivial for each nonzero quasi-regular ideal J by Theorem 3.1(ii).

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