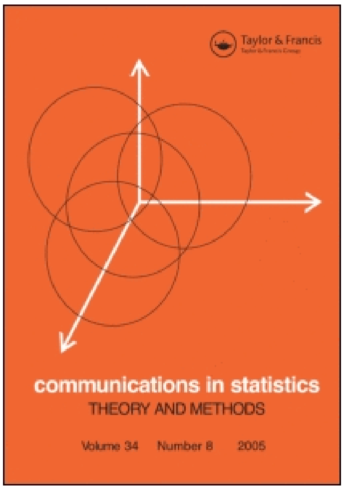

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Family of Estimators of Population Mean Using Two Auxiliary Variables in Stratified Random Sampling

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A general family of estimators, which use the information of two auxiliary variables in the stratified random sampling, is proposed to estimate the population mean of the variable under study. Under stratified random sampling without replacement scheme, the expressions of bias and mean square error (MSE) up to the first- and second-order approximations are derived. The family of estimators in its optimum case is discussed. Also, an empirical study is carried out to show the properties of the proposed estimators.

Keywords Auxiliary information; Efficiency; Mean square error; Ratio estimator; Stratified random sampling.

Mathematics Subject Classification Primary 62D05.

1. Introduction

In sampling theory the use of suitable auxiliary information results in considerable reduction in mean square error of the ratio estimators. For this reason, many authors used the auxiliary information at the estimation stage. Diana (1993) suggested a class of estimators of the population mean using one auxiliary variable in the stratified sampling and examined the Mean Square Error (MSE) of the estimators up to the kth order approximation. Many authors suggested estimators using some known population parameters of an auxiliary variable. Upadhyaya and Singh (1999), Khoshnevisan et al. (2007), etc. suggested estimators in simple random sampling, Kadilar and Cingi (2003) and Shabbir and Gupta (2005) extended these estimators for the stratified random sampling. Singh et al. (2008) suggested class of estimators using power transformation based on the estimators developed by Kadilar and Cingi (2003). Moreover, when two auxiliary variables are available, Singh (1965, 1967) and Perri (2007) suggested some ratio cum product estimators in the simple random sampling and Gupta and Shabbir (2007) used some known

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population parameters of the auxiliary variables in their estimators. Dalabehara and Sahoo (1997) suggested a class of estimators and Dalabehara and Sahoo (1999) developed a regression-type estimator in the stratified random sampling using two auxiliary variables. Singh and Vishwakarma (2008) suggested a family of estimators using transformation in the stratified random sampling. In this study, we suggest some estimators using the auxiliary information in the stratified random sampling when two auxiliary variables are available.

Let us consider a finite population $U = (u_1, u_2, \ldots, u_N)$ of size N and let Y, X, and Z, respectively, be the study and two auxiliary variables associated with each unit $u_j (j = 1, 2, \ldots, N)$ of the population. Assume that the population of size, N, is stratified into L strata with hth stratum containing N_h units, where $h = 1, 2, \ldots, L$ such that $\sum_{h=1}^{L} N_h = N$. A simple random sample of size n_h is drawn without replacement from the hth stratum such that $\sum_{h=1}^{L} n_h = n$. Let (y_{hi}, x_{hi}, z_{hi}) denote the observed values of y, x, and z on the ith unit of the hth stratum, where $i = 1, 2, \ldots, N_h$. Moreover, let $\bar{y}_h = \sum_{i=1}^{n_h} \frac{y_{hi}}{n_h}$, $\bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h$, and $\overline{Y}_h = \sum_{i=1}^{N_h} \frac{Y_{hi}}{N_h}$, $\overline{Y} = \overline{Y}_{st} = \sum_{h=1}^{L} W_h \overline{Y}_h$ be the sample and population means of Y, respectively, where $W_h = \frac{N_h}{N}$ is the stratum weight. Similar expressions for X and Z can also be defined.

To obtain the bias and MSE, let us define $e_0 = (\bar{y}_{st} - \overline{Y})/\overline{Y}$, $e_1 = (\bar{x}_{st} - \overline{X})/\overline{X}$ and $e_2 = (\bar{z}_{st} - \overline{Z})/\overline{Z}$. Using these notations,

$$\begin{aligned} &E(e_0) = E(e_1) = E(e_2) = 0, \\ &V_{rst} = \sum_{h=1}^{L} W_h^{r+s+t} \frac{E\lfloor(\bar{y}_h - \overline{Y}_h)^r (\bar{x}_h - \overline{X}_h)^s (\bar{z}_h - \overline{Z}_h)^t\rfloor}{\overline{Y}^r \overline{X}^s \overline{Z}^t}. \end{aligned} \tag{1}$$

Using (1), we can write

$$E(e_0^2) = \frac{\sum_{h=1}^{L} W_h^2 \lambda_h S_{yh}^2}{\overline{Y}^2} = V_{200}, \quad E(e_1^2) = \frac{\sum_{h=1}^{L} W_h^2 \lambda_h S_{xh}^2}{\overline{X}^2} = V_{020},$$

$$E(e_2^2) = \frac{\sum_{h=1}^{L} W_h^2 \lambda_h S_{zh}^2}{\overline{Z}^2} = V_{002}, \quad E(e_0 e_1) = \frac{\sum_{h=1}^{L} W_h^2 \lambda_h S_{xyh}}{\overline{X}\overline{Y}} = V_{110},$$

$$E(e_1 e_2) = \frac{\sum_{h=1}^{L} W_h^2 \lambda_h S_{xzh}}{\overline{X}\overline{Z}} = V_{011}, \quad E(e_0 e_2) = \frac{\sum_{h=1}^{L} W_h^2 \lambda_h S_{yzh}}{\overline{Y}\overline{Z}} = V_{101},$$

where

$$S_{yh}^2 = \frac{\sum_{i=1}^{N_h} (y_{hi} - \overline{Y}_h)^2}{N_h - 1}, \quad S_{xh}^2 = \frac{\sum_{i=1}^{N_h} (x_{hi} - \overline{X}_h)^2}{N_h - 1},$$

$$S_{xyh} = \frac{\sum_{i=1}^{N_h} (x_{hi} - \overline{X}_h)(y_{hi} - \overline{Y}_h)}{N_h - 1}, \quad S_{zyh} = \frac{\sum_{i=1}^{N_h} (z_{hi} - \overline{Z}_h)(y_{hi} - \overline{Y}_h)}{N_h - 1},$$

$$S_{xzh} = \frac{\sum_{i=1}^{N_h} (x_{hi} - \overline{X}_h)(z_{hi} - \overline{Z}_h)}{N_h - 1}, \quad \lambda_h = \frac{1 - f_h}{n_h}, \quad \text{and} \quad f_h = \frac{n_h}{N_h}.$$

2. Adapted Estimators

When the information on the two auxiliary variables is known, Singh (1965, 1967) proposed some estimators called ratio-cum-product estimators in the simple random sampling to estimate the population mean of the study variable Y. We propose the stratified versions of these estimators that can be expressed as

$$\bar{y}_1 = \bar{y}_{st}\left(\frac{\overline{X}}{\bar{x}_{st}}\right)\left(\frac{\overline{Z}}{\bar{z}_{st}}\right), \quad \bar{y}_2 = \bar{y}_{st}\left(\frac{\bar{x}_{st}}{\overline{X}}\right)\left(\frac{\bar{z}_{st}}{\overline{Z}}\right),$$
$$\bar{y}_3 = \bar{y}_{st}\left(\frac{\overline{X}}{\bar{x}_{st}}\right)\left(\frac{\bar{z}_{st}}{\overline{Z}}\right), \quad \bar{y}_4 = \bar{y}_{st}\left(\frac{\bar{x}_{st}}{\overline{X}}\right)\left(\frac{\overline{Z}}{\bar{z}_{st}}\right). \tag{2}$$

It is worth pointing out that the estimator, $\bar{y}_1$, can be defined as the classical ratio estimator in the stratified random sampling when there are two auxiliary variables.

The MSE equations of these estimators can be written as

$$\mathrm{MSE}(\bar{y}_1) = \overline{Y}^2(V_{200} + V_{020} + V_{002} - 2V_{110} + 2V_{011} - 2V_{101}), \tag{3}$$
$$\mathrm{MSE}(\bar{y}_2) = \overline{Y}^2(V_{200} + V_{020} + V_{002} + 2V_{110} + 2V_{011} + 2V_{101}), \tag{4}$$
$$\mathrm{MSE}(\bar{y}_3) = \overline{Y}^2(V_{020} + V_{200} + V_{002} - 2V_{110} - 2V_{011} + 2V_{101}), \tag{5}$$
$$\mathrm{MSE}(\bar{y}_4) = \overline{Y}^2(V_{200} + V_{020} + V_{002} + 2V_{110} - 2V_{011} - 2V_{101}). \tag{6}$$

Considering the estimators in (2) and motivated by Perri (2007), we adapt a class of combined ratio estimators for estimating the population mean of the study variable using two auxiliary variables as

$$\bar{y}_n = \bar{y}_{st}\left(\frac{\overline{X}}{\bar{x}_{st} + \eta_1(\overline{X} - \bar{x}_{st})}\right)^{\alpha_1}\left(\frac{\overline{Z}}{\bar{z}_{st} + \eta_2(\overline{Z} - \bar{z}_{st})}\right)^{\alpha_2}, \tag{7}$$

where α_1, α_2, η_1, η_2 are suitably chosen scalars such that the mean square error of $\bar{y}_n$ is minimum. Some new estimators, which are generated from (7) for different combinations of α_1, α_2, η_1, and η_2, are given in Table 1.

Expressing the estimator with e's

$$\bar{y}_n = \overline{Y}(1 + e_0)\left[1 + (1 - \eta_1)e_1\right]^{-\alpha_1}\left[1 + (1 - \eta_2)e_2\right]^{-\alpha_2} \tag{8}$$

and to the first-order approximation we can write

$$\bar{y}_n - \overline{Y} = \overline{Y}\left\{e_0 + \frac{\alpha_1(1+\alpha_1)}{2}(1-\eta_1)^2e_1^2 + \frac{\alpha_2(1+\alpha_2)}{2}(1-\eta_2)^2e_2^2 - \alpha_1(1-\eta_1)e_1 \right.$$
$$- \alpha_2(1-\eta_2)e_2 - \alpha_1(1-\eta_1)e_0e_1 - \alpha_2(1-\eta_2)e_2e_0$$
$$\left. + \alpha_1\alpha_2(1-\eta_1)(1-\eta_2)e_1e_2\right\}. \tag{9}$$

Taking expectations of both sides of (9), we get the bias of $\bar{y}_n$ as

$$B(\bar{y}_n) = \overline{Y}\left\{\frac{\alpha_1(1+\alpha_1)}{2}(1-\eta_1)^2V_{020} + \frac{\alpha_2(1+\alpha_2)}{2}(1-\eta_2)^2V_{002} \right.$$
$$\left. + \alpha_1\alpha_2(1-\eta_1)(1-\eta_2)V_{011} - \alpha_1(1-\eta_1)V_{110} - \alpha_2(1-\eta_2)V_{101}\right\}. \tag{10}$$

Table 1
Some members of the family of estimators of $\bar{y}_n$

	α_1	α_2	η_1	η_2
$\bar{y}_{n1} = \bar{y}_{st}\left(\dfrac{\overline{X}_{st}}{\bar{x}_{st} + \eta_1\,(\overline{X}_{st} - \bar{x}_{st})}\right) \times \left(\dfrac{\overline{Z}_{st}}{\bar{z}_{st} + \eta_2(\overline{Z}_{st} - \bar{z}_{st})}\right)$	1	1	$\eta_1^* = 1 - \beta_{yx,z(st)}$	$\eta_2^* = 1 - \beta_{yz,x(st)}$
$\bar{y}_{n2} = \bar{y}_{st}\left(\dfrac{\bar{x}_{st} + \eta_1(\overline{X}_{st} - \bar{x}_{st})}{\overline{X}_{st}}\right) \times \left(\dfrac{\bar{z}_{st} + \eta_2(\overline{Z}_{st} - \bar{z}_{st})}{\overline{Z}_{st}}\right)$	−1	−1	$\eta_1^* = 1 + \beta_{yx,z(st)}$	$\eta_2^* = 1 + \beta_{yz,x(st)}$
$\bar{y}_{n3} = \bar{y}_{st}\left(\dfrac{\overline{X}_{st}}{\bar{x}_{st} + \eta_1(\overline{X}_{st} - \bar{x}_{st})}\right) \times \left(\dfrac{\bar{z}_{st} + \eta_2(\overline{Z}_{st} - \bar{z}_{st})}{\overline{Z}_{st}}\right)$	1	−1	$\eta_1^* = 1 - \beta_{yx,z(st)}$	$\eta_2^* = 1 + \beta_{yz,x(st)}$
$\bar{y}_{n4} = \bar{y}_{st}\left(\dfrac{\bar{x}_{st} + \eta_1(\overline{X}_{st} - \bar{x}_{st})}{\overline{X}_{st}}\right) \times \left(\dfrac{\overline{Z}_{st}}{\bar{z}_{st} + \eta_2(\overline{Z}_{st} - \bar{z}_{st})}\right)$	−1	1	$\eta_1^* = 1 + \beta_{yx,z(st)}$	$\eta_2^* = 1 - \beta_{yz,x(st)}$
$\bar{y}_{n5} = \bar{y}_{st}\left(\dfrac{\overline{X}_{st}}{\bar{x}_{st} + \eta_1(\overline{X}_{st} - \bar{x}_{st})}\right)$ (One auxiliary variable)	1	0	$\eta_1^* = 1 - \dfrac{\sqrt{V_{200}}}{\sqrt{V_{020}}}\rho_{yx(st)} = 1 - \dfrac{\beta_c}{R}$	0
$\bar{y}_{n6} = \bar{y}_{st}\left(\dfrac{\bar{x}_{st} + \eta_1(\overline{X}_{st} - \bar{x}_{st})}{\overline{X}_{st}}\right)$ (One auxiliary variable)	−1	0	$\eta_1^* = 1 + \dfrac{\sqrt{V_{200}}}{\sqrt{V_{020}}}\rho_{yx(st)} = 1 + \dfrac{\beta_c}{R}$	0

Squaring both sides of Eq. (9) and taking expectation on both sides of this equation, we get the MSE $(\bar{y}_n)$ to the first order of approximation as given below:

$$\text{MSE}\,(\bar{y}_n) = \overline{Y}^2\big(t_1^2 V_{020} + V_{200} + t_2^2 V_{002} - 2t_1 V_{110} + 2t_1 t_2 V_{011} - 2t_2 V_{101}\big), \tag{11}$$

where $t_1 = \alpha_1(1 - \eta_1)$ and $t_2 = \alpha_2(1 - \eta_2)$. The MSE $(\bar{y}_n)$ is minimized for

$$\begin{aligned} t_1^* &= \left[\frac{\rho_{yx(st)} - \rho_{yz(st)}\rho_{xz(st)}}{1 - \rho_{xz(st)}^2}\right]\sqrt{\frac{V_{200}}{V_{020}}} = \beta_{yx,z(st)}, \\ t_2^* &= \left[\frac{\rho_{yz(st)} - \rho_{yx(st)}\rho_{xz(st)}}{1 - \rho_{xz(st)}^2}\right]\sqrt{\frac{V_{200}}{V_{002}}} = \beta_{yz,x(st)}, \end{aligned} \tag{12}$$

where t_1^* and t_2^* are, respectively, partial regression coefficients of y on x and of y on z in the stratified random sampling. Here, $\rho_{ab(st)} = \frac{\sum_{h=1}^{L} W_h^2 \lambda_h S_{abh}}{\sqrt{\sum_{h=1}^{L} W_h^2 \lambda_h S_{ah}^2}\sqrt{\sum_{h=1}^{L} W_h^2 \lambda_h S_{bh}^2}}$ is the correlation coefficient in the stratified random sampling across all strata where

ab can be YX, YZ, or XZ. Then, the minimum MSE of the estimator $\bar{y}_n$ is given by

$$\begin{aligned}\mathrm{MSE}_{\min}(\bar{y}_n) &= \overline{Y}^2\left(V_{200} - \frac{V_{110}^2 V_{002} + V_{101}^2 V_{020} - 2V_{101}V_{110}V_{011}}{(V_{020}V_{002} - V_{011}^2)}\right) \\ &= \overline{Y}^2 V_{200}\left(1 - \frac{\rho_{yx(st)}^2 + \rho_{yz(st)}^2 - 2\rho_{yx(st)}\rho_{yz(st)}\rho_{xz(st)}}{(1-\rho_{xz(st)}^2)}\right) \\ &= \overline{Y}^2 V_{200}(1 - R_{y.xz(st)}^2),\end{aligned} \tag{13}$$

where $R_{y\cdot xz(st)}^2 = \frac{\rho_{yx(st)}^2 + \rho_{yz(st)}^2 - 2\rho_{yx(st)}\rho_{yz(st)}\rho_{xz(st)}}{(1-\rho_{xz(st)}^2)}$ is the multiple correlation coefficient of y on x and z in the stratified random sampling. Note that the minimum MSE in (13) is equal to the MSE of combined regression estimator based on two auxiliary variables.

3. Suggested Estimators

Following Khoshnevisan et al. (2007), we can define a general family of estimators for the population mean when the information on the two auxiliary variables is known:

$$\begin{aligned}k = \bar{y}_{st}&\left[\frac{a_{st}\overline{X} + b_{st}}{\alpha_1(a_{st}\bar{x}_{st} + b_{st}) + (1-\alpha_1)(a_{st}\overline{X} + b_{st})}\right]^{g_1} \\ &\times \left[\frac{c_{st}\overline{Z} + d_{st}}{\alpha_2(c_{st}\bar{z}_{st} + d_{st}) + (1-\alpha_2)(c_{st}\overline{Z} + d_{st})}\right]^{g_2},\end{aligned} \tag{14}$$

where α_1, α_2, g_1, and g_2 are suitable constants, $a_{st}(\neq 0)$ and b_{st} are either real numbers or the functions of the known parameters for hth stratum of the auxiliary variable, x, such as standard deviation $S_{x(st)} = \sum_{h=1}^{L} W_h \sigma_{xh}$, coefficient of variation $C_{x(st)} = \sum_{h=1}^{L} W_h C_{xh}$, skewness $\beta_{1(x)st} = \sum_{h=1}^{L} W_h \beta_{1h}(x)$, kurtosis $\beta_{2(x)st} = \sum_{h=1}^{L} W_h \beta_{2h}(x)$. For $c_{st}(\neq 0)$ and d_{st}, we can define similar expressions for the auxiliary variable, z, as a_{st} and b_{st}. Some new estimators, which are generated from (14) for different combinations of a_{st}, b_{st}, c_{st}, d_{st}, α_1, α_2, g_1, and g_2 are given in Tables 2 and 3.

Expressing k in terms of e_i $(i = 0, 1, 2)$, we can write (14) as

$$k = \bar{y}_{st}[1 + u_1\alpha_1 e_1]^{-g_1}[1 + u_2\alpha_2 e_2]^{-g_2}, \tag{15}$$

where $u_1 = \frac{a_{st}\overline{X}}{a_{st}\overline{X} + b_{st}}$ and $u_2 = \frac{c_{st}\overline{Z}}{c_{st}\overline{Z} + d_{st}}$. Suppose $|\alpha_1 \upsilon_1 e_1| < 1$ and $|\alpha_2 \upsilon_2 e_2| < 1$ so that $[1 + u_1\alpha_1 e_1]^{-g_1}$ and $[1 + u_2\alpha_2 e_2]^{-g_2}$ are expandable. Expanding the right-hand side of (15) to the first order of approximation and subtracting $\overline{Y}$ from both sides of the equation, we obtain

$$\begin{aligned}k - \overline{Y} = \overline{Y}\Big[&e_0 - g_1u_1\alpha_1 e_0e_1 - g_2u_2\alpha_2 e_0e_2 + \alpha_1\alpha_2 g_1g_2u_1u_2e_1e_2 - g_1u_1\alpha_1 e_1 \\ &- g_2u_2\alpha_2 e_2 + \frac{g_1(g_1+1)}{2}\alpha_1^2u_1^2e_1^2 + \frac{g_2(g_2+1)}{2}\alpha_2^2u_2^2e_2^2\Big].\end{aligned} \tag{16}$$

Table 2
Some members of the family of estimators of k

	a_{st}	b_{st}	c_{st}	d_{st}	α_1	α_2	g_1	g_2	u_1	u_2
$k_1 = \bar{y}_{st}\left[\frac{\overline{X}+C_{x(st)}}{\bar{x}_{st}+C_{x(st)}}\right]\left[\frac{\overline{Z}+C_{z(st)}}{\bar{z}_{st}+C_{z(st)}}\right]$	1	$C_{x(st)}$	1	$C_{z(st)}$	1	1	1	1	$\frac{\overline{X}}{\overline{X}+C_{x(st)}}$	$\frac{\overline{Z}}{\overline{Z}+C_{z(st)}}$
$k_2 = \bar{y}_{st}\left[\frac{\overline{X}+\beta_{2(x)st}}{\bar{x}_{st}+\beta_{2(x)st}}\right]\left[\frac{\overline{Z}+\beta_{2(z)st}}{\bar{z}_{st}+\beta_{2(z)st}}\right]$	1	$\beta_{2(x)st}$	1	$\beta_{2(z)st}$	1	1	1	1	$\frac{\overline{X}}{\overline{X}+\beta_{2(x)st}}$	$\frac{\overline{Z}}{\overline{Z}+\beta_{2(z)st}}$
$k_3 = \bar{y}_{st}\left[\frac{\beta_{2(x)st}\overline{X}+C_{x(st)}}{\beta_{2(x)st}\bar{x}_{st}+C_{x(st)}}\right]\left[\frac{\beta_{2(z)st}\overline{Z}+C_{z(st)}}{\beta_{2(z)st}\bar{z}_{st}+C_{z(st)}}\right]$	$\beta_{2(x)st}$	$C_{x(st)}$	$\beta_{2(z)st}$	$C_{z(st)}$	1	1	1	1	$\frac{\beta_{2(x)st}\overline{X}}{\beta_{2(x)st}\overline{X}+C_{x(st)}}$	$\frac{\beta_{2(z)st}\overline{Z}}{\beta_{2(z)st}\overline{Z}+C_{z(st)}}$
$k_4 = \bar{y}_{st}\left[\frac{C_{x(st)}\overline{X}+\beta_{2(x)st}}{C_{x(st)}\bar{x}_{st}+\beta_{2(x)st}}\right]\left[\frac{C_{z(st)}\overline{Z}+\beta_{2(z)st}}{C_{z(st)}\bar{z}_{st}+\beta_{2(z)st}}\right]$	$C_{x(st)}$	$\beta_{2(x)st}$	$C_{z(st)}$	$\beta_{2(z)st}$	1	1	1	1	$\frac{C_{x(st)}\overline{X}}{C_{x(st)}\overline{X}+\beta_{2(x)st}}$	$\frac{C_{z(st)}\overline{Z}}{C_{z(st)}\overline{Z}+\beta_{2(z)st}}$
$\bar{y}_1 = \bar{y}_{st}\left(\frac{\overline{X}}{\bar{x}_{st}}\right)\left(\frac{\overline{Z}}{\bar{z}_{st}}\right)$	1	0	1	0	1	1	1	1	1	1
$\bar{y}_2 = \bar{y}_{st}\left(\frac{\bar{x}_{st}}{\overline{X}}\right)\left(\frac{\bar{z}_{st}}{\overline{Z}}\right)$	1	0	1	0	1	1	−1	−1	1	1
$\bar{y}_3 = \bar{y}_{st}\left(\frac{\overline{X}}{\bar{x}_{st}}\right)\left(\frac{\bar{z}_{st}}{\overline{Z}}\right)$	1	0	1	0	1	1	1	−1	1	1
$\bar{y}_4 = \bar{y}_{st}\left(\frac{\bar{x}_{st}}{\overline{X}}\right)\left(\frac{\overline{Z}}{\bar{z}_{st}}\right)$	1	0	1	0	1	1	−1	1	1	1

Table 3
Some members of the family of estimators of k

Estimator	a_{st}	b_{st}	c_{st}	d_{st}	α_1	α_2	g_1	g_2	Estimator	a_{st}	b_{st}	c_{st}	d_{st}	α_1	α_2	g_1	g_2
k_5	1	$C_{x(st)}$	1	$C_{z(st)}$	α_1	α_2	-1	1	k_{23}	$\beta_{2(x)st}$	$C_{(xst)}$	$\beta_{2(z)st}$	$C_{z(st)}$	α_1	α_2	-2	2
k_6	1	$\beta_{2(x)st}$	1	$\beta_{2(z)st}$	α_1	α_2	-1	1	k_{24}	$C_{x(st)}$	$\beta_{2(x)st}$	$C_{z(st)}$	$\beta_{2(z)st}$	α_1	α_2	-2	2
k_7	$\beta_{2(x)st}$	$C_{x(st)}$	$\beta_{2(z)st}$	$C_{z(st)}$	α_1	α_2	-1	1	k_{25}	1	$C_{x(st)}$	1	$C_{z(st)}$	α_1	α_2	2	-2
k_8	$C_{x(st)}$	$\beta_{2(x)st}$	$C_{z(st)}$	$\beta_{2(z)st}$	α_1	α_2	-1	1	k_{26}	1	$\beta_{2(x)st}$	1	$\beta_{2(z)st}$	α_1	α_2	2	-2
k_9	1	$C_{x(st)}$	1	$C_{z(st)}$	α_1	α_2	1	-1	k_{27}	$\beta_{2(x)st}$	$C_{x(st)}$	$\beta_{2(z)st}$	$C_{z(st)}$	α_1	α_2	2	-2
k_{10}	1	$\beta_{2(x)st}$	1	$\beta_{2(z)st}$	α_1	α_2	1	-1	k_{28}	$C_{x(st)}$	$\beta_{2(x)st}$	$C_{z(st)}$	$\beta_{2(z)st}$	α_1	α_2	2	-2
k_{11}	$\beta_{2(x)st}$	$C_{x(st)}$	$\beta_{2(z)st}$	$C_{z(st)}$	α_1	α_2	1	-1	k_{29}	1	$C_{x(st)}$	1	$C_{z(st)}$	α_1	α_2	2	2
k_{12}	$C_{x(st)}$	$\beta_{2(x)st}$	$C_{z(st)}$	$\beta_{2(z)st}$	α_1	α_2	1	-1	k_{30}	1	$\beta_{2(x)st}$	1	$\beta_{2(z)st}$	α_1	α_2	2	2
k_{13}	1	$C_{x(st)}$	1	$C_{z(st)}$	α_1	α_2	1	1	k_{31}	$\beta_{2(x)st}$	$C_{x(st)}$	$\beta_{2(z)st}$	$C_{z(st)}$	α_1	α_2	2	2
k_{14}	1	$\beta_{2(x)st}$	1	$\beta_{2(z)st}$	α_1	α_2	1	1	k_{32}	$C_{x(st)}$	$\beta_{2(x)st}$	$C_{z(st)}$	$\beta_{2(z)st}$	α_1	α_2	2	2
k_{15}	$\beta_{2(x)st}$	$C_{x(st)}$	$\beta_{2(z)st}$	$C_{z(st)}$	α_1	α_2	1	1	k_{33}	1	$C_{x(st)}$	1	$C_{z(st)}$	α_1	α_2	-2	-2
k_{16}	$C_{x(st)}$	$\beta_{2(x)st}$	$C_{z(st)}$	$\beta_{2(z)st}$	α_1	α_2	1	1	k_{34}	1	$\beta_{2(x)st}$	1	$\beta_{2(z)st}$	α_1	α_2	-2	-2
k_{17}	1	$C_{x(st)}$	1	$C_{z(st)}$	α_1	α_2	-1	-1	k_{35}	$\beta_{2(x)st}$	$C_{x(st)}$	$\beta_{2(z)st}$	$C_{z(st)}$	α_1	α_2	-2	-2
k_{18}	1	$\beta_{2(x)st}$	1	$\beta_{2(z)st}$	α_1	α_2	-1	-1	k_{36}	$C_{x(st)}$	$\beta_{2(x)st}$	$C_{z(st)}$	$\beta_{2(z)st}$	α_1	α_2	-2	-2
k_{19}	$\beta_{2(x)st}$	$C_{x(st)}$	$\beta_{2(z)st}$	$C_{z(st)}$	α_1	α_2	-1	-1	k_{37}	1	$C_{x(st)}$	1	$C_{z(st)}$	1	1	g_1	g_2
k_{20}	$C_{x(st)}$	$\beta_{2(x)st}$	$C_{z(st)}$	$\beta_{2(z)st}$	α_1	α_2	-1	-1	k_{38}	1	$\beta_{2(x)st}$	1	$\beta_{2(z)st}$	1	1	g_1	g_2
k_{21}	1	$C_{x(st)}$	1	$C_{z(st)}$	α_1	α_2	-2	2	k_{39}	$\beta_{2(x)st}$	$C_{x(st)}$	$\beta_{2(z)st}$	$C_{z(st)}$	1	1	g_1	g_2
k_{22}	1	$\beta_{2(x)st}$	1	$\beta_{2(z)st}$	α_1	α_2	-2	2	k_{40}	$C_{x(st)}$	$\beta_{2(x)st}$	$C_{z(st)}$	$\beta_{2(z)st}$	1	1	g_1	g_2

Using Eq. (16), bias, and MSE of the estimator k are, respectively, given by

$$B(k) = \overline{Y}\bigg[-g_1u_1\alpha_1V_{110} - g_2u_2\alpha_2V_{101} + \alpha_1\alpha_2g_1g_2u_1u_2V_{011} + \frac{g_1(g_1+1)}{2}\alpha_1^2u_1^2V_{020} + \frac{g_2(g_2+1)}{2}\alpha_2^2u_2^2V_{002}\bigg] \quad (17)$$

$$\text{MSE}(k) = \overline{Y}^2\big(V_{200} + g_1^2\alpha_1^2u_1^2V_{020} + g_2^2\alpha_2^2u_2^2V_{002} - 2g_1\alpha_1u_1V_{110} - 2g_2\alpha_2u_2V_{101} + 2g_1g_2\alpha_1\alpha_2u_1u_2V_{011}\big). \quad (18)$$

Now we can take $l_1 = g_1\alpha_1u_1$ and $l_2 = g_2\alpha_2u_2$, so we can rewrite (18) as

$$\text{MSE}(k) = \overline{Y}^2(V_{200} + l_1^2V_{020} + l_2^2V_{002} - 2l_1V_{110} - 2l_2V_{101} + 2l_1l_2V_{011}). \quad (19)$$

Note that optimum values of l_1 and l_2 are obtained as $l_1^* = \beta_{yx,z(st)}$ and $l_2^* = \beta_{yz,x(st)}$, respectively. Using these optimal values, the minimum MSE of the estimator k is

$$\text{MSE}_{\min}(k) = \overline{Y}^2V_{200}(1 - R^2_{y\cdot xz(st)}) \quad (20)$$

which is equal to the minimum MSE of $\bar{y}_n$ estimator.

Now we modify the proposed family of estimator k as

$$k^* = \bar{y}_{st}\left[\frac{A_{st}^* + b_{st}}{\alpha_1(a_{st}^* + b_{st}) + (1-\alpha_1)(A_{st}^* + b_{st})}\right]^{g_1}\left[\frac{C_{st}^* + d_{st}}{\alpha_2(c_{st}^* + d_{st}) + (1-\alpha_2)(C_{st}^* + d_{st})}\right]^{g_2} \quad (21)$$

and here we define $A_{st}^* = \sum_{h=1}^{L} W_h\overline{X}_hA_h$ and $a_{st}^* = \sum_{h=1}^{L} W_h\bar{x}_hA_h$, where A_h may be some population information of the first auxiliary variable for hth stratum such as S_{xh}, coefficient of variation C_{xh}, skewness $\beta_{1h}(x)$, kurtosis $\beta_{2h}(x)$, correlation coefficient $\rho_{h(xy)}$. Similarly, we can define $C_{st}^* = \sum_{h=1}^{L} W_h\overline{Z}_hB_h$ and $c_{st}^* = \sum_{h=1}^{L} W_h\bar{z}_hB_h$, where B_h may be some population information of the second auxiliary variable for the stratum h such as S_{zh}, coefficient of variation C_{zh}, skewness $\beta_{1h}(z)$, kurtosis $\beta_{2h}(z)$, correlation coefficient $\rho_{h(zy)}$. In sampling literature, Kadilar and Cingi (2003), Shabbir and Gupta (2005), and Singh et al. (2008) used similar population information in the stratified random sampling when there is one auxiliary variable. There are some new estimators, such as k_1^*, k_2^*, and k_3^*, generated from (21) and $\bar{y}_{R(\alpha_{st})}$, $\bar{y}_{R(\delta_{st})}$ suggested by Singh et al. (2008) in Table 4.

We can rewrite k^* in terms of e's as

$$k^* = \overline{Y}(1 + e_0)[1 + e_1^*\omega_1\alpha_1]^{-g_1}[1 + e_2^*\omega_2\alpha_2]^{-g_2}, \quad (22)$$

where $\omega_1 = \frac{A_{st}^*}{A_{st}^* + b_{st}}$, $\omega_2 = \frac{C_{st}^*}{C_{st}^* + d_{st}}$, $e_1^* = \frac{a_{st}^* - A_{st}^*}{A_{st}^*} = \frac{\sum_{h=1}^{L} W_hA_h(\bar{x}_h - \overline{X}_h)}{A_{st}^*}$, and $e_2^* = \frac{c_{st}^* - C_{st}^*}{C_{st}^*} = \frac{\sum_{h=1}^{L} W_hB_h(\bar{z}_h - \overline{Z}_h)}{C_{st}^*}$. It is clear that $E(e_1^*) = E(e_2^*) = 0$.

Expanding (22) to the first order approximation we can get

$$k^* - \overline{Y} = \overline{Y}\bigg(e_0 - g_1\omega_1\alpha_1e_1^* - g_2\omega_2\alpha_2e_2^* - g_1\omega_1\alpha_1e_0e_1^* - g_2\omega_2\alpha_2e_2^*e_0 + g_1g_2\omega_1\omega_2\alpha_1\alpha_2e_1^*e_2^* + \frac{g_1(g_1+1)}{2}\omega_1^2\alpha_1^2e_1^{*2} + \frac{g_2(g_2+1)}{2}\omega_2^2\alpha_2^2e_2^{*2}\bigg) \quad (23)$$

Table 4
Some members of the family of estimators of k^*

	A_h	b_{st}	B_h	d_{st}	α_1	α_2	g_1	g_2	$\psi_1 = \frac{\overline{X}_{st}}{\sum_{h=1}^{L} W_h \overline{X}_h A_h + b_{st}}$	$\psi_2 = \frac{\overline{Z}_{st}}{\sum_{h=1}^{L} W_h \overline{Z}_h B_h + d_{st}}$
$k_1^* = \bar{y}_{st}\left[\frac{\sum_{h=1}^{L} W_h C_{xh} + \sum_{h=1}^{L} W_h \overline{X}_h \beta_{2h}(x)}{\sum_{h=1}^{L} W_h C_{xh} + \sum_{h=1}^{L} W_h \bar{x}_h \beta_{2h}(x)}\right] \times \left[\frac{\sum_{h=1}^{L} W_h C_{zh} + \sum_{h=1}^{L} W_h \overline{Z}_h \beta_{2h}(z)}{\sum_{h=1}^{L} W_h C_{zh} + \sum_{h=1}^{L} W_h \bar{z}_h \beta_{2h}(z)}\right]$	$\beta_{2(x)h}$	$C_{x(st)}$	$\beta_{2(z)h}$	$C_{z(st)}$	1	1	1	1	$\frac{\overline{X}_{st}}{\sum_{h=1}^{L} W_h \overline{X}_h \beta_{2(x)h} + C_{x(st)}}$	$\frac{\overline{Z}_{st}}{\sum_{h=1}^{L} W_h \overline{Z}_h \beta_{2(z)h} + C_{z(st)}}$
$k_2^* = \bar{y}_{st}\left[\frac{\sum_{h=1}^{L} W_h \beta_{2h}(x) + \sum_{h=1}^{L} W_h \overline{X}_h C_{xh}}{\sum_{h=1}^{L} W_h \beta_{2h}(x) + \sum_{h=1}^{L} W_h \bar{x}_h C_{xh}}\right] \left[\frac{\sum_{h=1}^{L} W_h \beta_{2h}(z) + \sum_{h=1}^{L} W_h \overline{Z}_h C_{zh}}{\sum_{h=1}^{L} W_h \beta_{2h}(z) + \sum_{h=1}^{L} W_h \bar{z}_h C_{zh}}\right]$	C_{xh}	$\beta_{2(x)st}$	C_{zh}	$\beta_{2(z)st}$	1	1	1	1	$\frac{\overline{X}_{st}}{\sum_{h=1}^{L} W_h \overline{X}_h C_{xh} + \beta_{2(x)st}}$	$\frac{\overline{Z}_{st}}{\sum_{h=1}^{L} W_h \overline{Z}_h C_{zh} + \beta_{2(z)st}}$
$k_3^* = \bar{y}_{st}\left[\frac{\sum_{h=1}^{L} W_h \beta_{1h}(x) + \sum_{h=1}^{L} W_h \overline{X}_h C_{xh}}{\sum_{h=1}^{L} W_h \beta_{1h}(x) + \sum_{h=1}^{L} W_h \bar{x}_h C_{xh}}\right] \times \left[\frac{\sum_{h=1}^{L} W_h \beta_{1h}(z) + \sum_{h=1}^{L} W_h \overline{Z}_h C_{zh}}{\sum_{h=1}^{L} W_h \beta_{1h}(z) + \sum_{h=1}^{L} W_h \bar{z}_h C_{zh}}\right]$	C_{xh}	$\beta_{1(x)st}$	C_{zh}	$\beta_{1(z)st}$	1	1	1	1	$\frac{\overline{X}_{st}}{\sum_{h=1}^{L} W_h \overline{X}_h C_{xh} + \beta_{1(x)st}}$	$\frac{\overline{Z}_{st}}{\sum_{h=1}^{L} W_h \overline{Z}_h C_{zh} + \beta_{1(z)st}}$
$\bar{y}_{R(\alpha_{st})} = \bar{y}_{st}\left[\frac{\sum_{h=1}^{L} W_h C_{xh} + \sum_{h=1}^{L} W_h \overline{X}_h \beta_{2h}(x)}{\sum_{h=1}^{L} W_h C_{xh} + \sum_{h=1}^{L} W_h \bar{x}_h \beta_{2h}(x)}\right]^{g_1}$ Singh et al. (2008) (One auxiliary variable)	$\beta_{2(x)h}$	$C_{x(st)}$	–	–	1	1	g_1	0	$\frac{\overline{X}_{st}}{\sum_{h=1}^{L} W_h \overline{X}_h \beta_{2(x)h} + C_{x(st)}}$	–
$\bar{y}_{R(\delta_{st})} = \bar{y}_{st}\left[\frac{\sum_{h=1}^{L} W_h \beta_{2h}(x) + \sum_{h=1}^{L} W_h \overline{X}_h C_{xh}}{\sum_{h=1}^{L} W_h \beta_{2h}(x) + \sum_{h=1}^{L} W_h \bar{x}_h C_{xh}}\right]^{g_1}$ Singh et al. (2008) (One auxiliary variable)	C_{xh}	$\beta_{2(x)st}$	–	–	1	1	g_1	0	$\frac{\overline{X}_{st}}{\sum_{h=1}^{L} W_h \overline{X}_h C_{xh} + \beta_{2(x)st}}$	–

and squaring (23) to the first-order approximation

$$(k^* - \overline{Y})^2 = \overline{Y}^2\big(g_1^2\omega_1^2\alpha_1^2 e_1^{*2} + e_0^2 + g_2^2\omega_2^2\alpha_2^2 e_2^{*2} - 2g_1\omega_1\alpha_1 e_1^* e_0 + 2g_1\omega_1\alpha_1 g_2\omega_2\alpha_2 e_1^* e_2^* - 2g_2\omega_2\alpha_2 e_0 e_2^*\big). \tag{24}$$

By taking the expectation of (24), we obtain the MSE as

$$\text{MSE}\,(k^*) = \overline{Y}^2\big(g_1^2\alpha_1^2\psi_1^2 V_{020}^* + V_{200}^* + g_2^2\alpha_2^2\psi_2^2 V_{002}^* - 2g_1\alpha_1\psi_1 V_{110}^* - 2g_2\alpha_2\psi_2 V_{101}^* + 2g_1\alpha_1 g_2\alpha_2\psi_1\psi_2 V_{011}^*\big), \tag{25}$$

where $\psi_1 = \frac{\overline{X}_{st}}{A_{st}^* + b_{st}}$, $\psi_2 = \frac{\overline{Z}_{st}}{C_{st}^* + d_{st}}$, $V_{r,s,t}^* = \sum_{h=1}^{L} W_h^{r+s+t} A_h^s B_h^t \frac{E\lfloor(\bar{y}_h - \overline{Y}_h)^r(\bar{x}_h - \overline{X}_h)^s(\bar{z}_h - \overline{Z}_h)^t\rfloor}{\overline{Y}^r\overline{X}^s\overline{Z}^t}$.

We would like to remark that the MSEs of the estimators proposed by Kadilar and Cingi (2003) and Singh et al. (2008) can be obtained from (25) for one auxiliary variable.

Let $r_1 = g_1\alpha_1\psi_1$ and $r_2 = g_2\alpha_2\psi_2$ we can rewrite (25) as

$$\text{MSE}\,(k^*) = \overline{Y}^2\big[r_1^2 V_{020}^* + V_{200}^* + r_2^2 V_{002}^* - 2r_1 V_{110}^* - 2r_2 V_{101}^* + 2r_1 r_2 V_{011}^*\big]. \tag{26}$$

Optimum values of r_1 and r_2 are $r_1^* = \frac{V_{110}^* V_{002}^* - V_{101}^* V_{011}^*}{(V_{020}^* V_{002}^* - V_{011}^{*2})}$, $r_2^* = \frac{V_{101}^* V_{020}^* - V_{110}^* V_{011}^*}{(V_{002}^* V_{020}^* - V_{011}^{*2})}$ and by this way the minimum MSE of the estimator k^* is obtained as

$$\text{MSE}_{\min}\,(k^*) = \overline{Y}^2\left(V_{200}^* - \frac{V_{002}^* V_{110}^{*2} + V_{101}^{*2} V_{020}^* - 2V_{110}^* V_{101}^* V_{011}^*}{V_{002}^* V_{020}^* - V_{011}^{*2}}\right). \tag{27}$$

Note that the minimum MSE of the estimator k does not depend on the selection of A_h and B_h, whereas the minimum MSE of the estimator k^* changes according to the values of A_h and B_h. Based on this information, we can compare the efficiencies of the proposed estimators between each other using (13), (20), and (27), as follows:

$$\begin{gathered} \text{MSE}_{\min}\,(\bar{y}_n) = \text{MSE}_{\min}\,(k) < \text{MSE}_{\min}\,(k^*), \\ \tau^* < \tau, \end{gathered} \tag{28}$$

where $\tau^* = \frac{V_{002}^* V_{110}^{*2} + V_{101}^{*2} V_{020}^* - 2V_{110}^* V_{101}^* V_{011}^*}{V_{002}^* V_{020}^* - V_{011}^{*2}}$ and $\tau = \frac{V_{002} V_{110}^2 + V_{101}^2 V_{020} - 2V_{110} V_{101} V_{011}}{V_{002} V_{020} - V_{011}^2}$.

When the condition (28) is satisfied, the proposed estimator k and the adapted estimator $\bar{y}_n$ are more efficient than the modification of the proposed estimator k^*.

Finally, we can also compare the proposed estimator k^* with the stratified adapted of Singh (1965, 1967) estimator, $\bar{y}_1$, by using (3) and (27), as follows:

$$\begin{gathered} \text{MSE}_{\min}\,(k^*) < \text{MSE}\,(\bar{y}_1), \\ (-\tau^*) < \xi \end{gathered} \tag{29}$$

where $\xi = (V_{020} + V_{002} - 2V_{110} + 2V_{011} - 2V_{101})$. When the condition (29) is satisfied, the proposed estimator k^* is more efficient than the stratified adapted of Singh (1965, 1967) estimator, $\bar{y}_1$.

4. Second-Order of Approximation

Adapted and suggested estimators have some parameters and using these various parameters, many estimators can be generated in two ways. In the first way, we define each parameter as independent from other parameters. We generate the estimators given in Table 2 in this way. However, in the second way, we consider two parameters are dependent with other parameters for auxiliary variables. We generate the estimators given in Tables 1 and 3 in this way.

As mentioned in Sec. 3, when we generate some subclasses in the second way up to the first order of approximation from k^*, various A_h and B_h transformations of the auxiliary variables can affect the value of $\mathrm{MSE}_{\min}(k^*)$ and, therefore, we get different MSE values for these subclasses. However, when we generate some subclasses in the same way from adaptive $\bar{y}_n$ and suggest the estimators, k, the minimum MSE of these estimators are equal to the MSE of the combined regression estimator. For this reason, to find the most efficient estimator among $\bar{y}_n$ and k, we decide to find their MSE equations up to the second order of approximation.

Under the second order of approximation, the terms $(1+(1-\eta_1)e_1)^{-\alpha_1}$ and $(1+(1-\eta_2)e_2)^{-\alpha_2}$ in (8) are expanded, respectively, as:

$$\left(1+(1-\eta_1)e_1\right)^{-\alpha_1} = 1-\alpha_1(1-\eta_1)e_1+\frac{\alpha_1(\alpha_1+1)}{2}(1-\eta_1)^2e_1^2+\cdots \tag{30}$$

$$\left(1+(1-\eta_2)e_2\right)^{-\alpha_2} = 1-\alpha_2(1-\eta_2)e_2+\frac{\alpha_2(\alpha_2+1)}{2}(1-\eta_2)^2e_2^2+\cdots \tag{31}$$

Let us define the coefficients of e_1^i and e_2^i in (30) and (31), respectively, as follows:

$$a_i = (-(1-\eta_1))^i\frac{\alpha_1(\alpha_1+1)\cdots(\alpha_1+i-1)}{i!} \tag{32}$$

$$b_i = (-(1-\eta_2))^i\frac{\alpha_2(\alpha_2+1)\cdots(\alpha_2+i-1)}{i!}. \tag{33}$$

Using these notations, up to the second order of approximation we can write

$$\begin{aligned}(\bar{y}_n-\overline{Y}_{st})^2 = \overline{Y}^2\big\{ & e_0^2+a_1^2(e_1^2+e_0^2e_1^2)+a_2^2e_1^4+b_1^2(e_2^2+e_0^2e_2^2)+a_1^2b_1^2e_1^2e_2^2+b_2^2e_2^4\\ &+2a_1(e_1e_0+e_0^2e_1)+2a_1a_2(e_1^3+2e_0e_1^3)+2a_2(e_0e_1^2+e_0^2e_1^2)+2a_3e_1^3e_0\\ &+2a_1a_3e_1^4+2a_1^2e_0e_1^2+2b_1(e_2e_0+e_0^2e_2)+2b_1b_2(e_2^3+2e_0e_2^3)\\ &+2b_2(e_0^2e_2^2+e_2^2e_0)+2b_3e_2^3e_0+2b_1b_3e_2^4+2b_1^2e_0e_2^2\\ &+2a_1b_1(e_1e_2+2e_0^2e_1e_2+3e_0e_1e_2)+2a_1^2b_1(e_1^2e_2+2e_0e_1^2e_2)\\ &+2a_1b_1^2(e_1e_2^2+2e_0e_1e_2^2)+2a_1b_2(3e_0e_1e_2^2+e_1e_2^2)\\ &+2a_2b_1(e_1^2e_2+3e_0e_1^2e_2)+4a_1b_1b_2e_1e_2^3+4a_1a_2b_1e_1^3e_2+2a_2b_2e_1^2e_2^2\\ &+2a_1^2b_2e_1^2e_2^2+2a_2b_1^2e_1^2e_2^2+2a_1b_3e_1e_2^3+2a_3b_1e_1^3e_2\big\}.\end{aligned} \tag{34}$$

By taking the expectation of (34), we obtain the second order MSE as

$$\begin{aligned}\mathrm{MSE}_{II}(\bar{y}_n) = \overline{Y}^2\big\{ & V_{200}+a_1^2(V_{020}+V_{220})+a_2^2V_{040}+b_1^2(V_{002}+V_{202})+a_1^2b_1^2V_{022}+b_2^2V_{004}\\ &+2a_1(V_{110}+V_{210})+2a_1a_2(V_{030}+2V_{130})+2a_2(V_{120}+V_{220})+2a_3V_{130}\end{aligned}$$

$$+ 2a_1a_3V_{040} + 2a_1^2V_{120} + 2b_1(V_{101} + V_{201}) + 2b_1b_2(V_{003} + 2V_{103})$$
$$+ 2b_2(V_{202} + V_{102}) + 2b_3V_{103} + 2b_1b_3V_{004} + 2b_1^2V_{102}$$
$$+ 2a_1b_1(V_{011} + 2V_{211} + 3V_{111}) + 2a_1^2b_1(V_{021} + 2V_{121})$$
$$+ 2a_1b_1^2(V_{012} + 2V_{112}) + 2a_1b_2(3V_{112} + V_{012}) + 2a_2b_1(V_{021} + 3V_{121})$$
$$+ 4a_1b_1b_2V_{013} + 4a_1a_2b_1V_{031} + 2a_2b_2V_{022} + 2a_1^2b_2V_{022} + 2a_2b_1^2V_{022}$$
$$+ 2a_1b_3V_{013} + 2a_3b_1V_{031}\} \quad (35)$$

or we can rewrite (35) as

$$\begin{aligned}\mathrm{MSE}_{II}(\bar{y}_n) = \mathrm{MSE}_I(\bar{y}_N) + \overline{Y}^2\{&a_1^2V_{220} + a_2^2V_{040} + b_1^2V_{202} + b_2^2V_{004} + a_1^2b_1^2V_{022}\\ &+ 2a_1V_{210} + 2a_1a_2(V_{030} + 2V_{130}) + 2a_2(V_{120} + V_{220})\\ &+ 2a_3V_{130} + 2a_1a_3V_{040} + 2a_1^2V_{120}\\ &+ 2b_1V_{201} + 2b_1b_2(V_{003} + 2V_{103}) + 2b_2(V_{202} + V_{102})\\ &+ 2b_3V_{103} + 2b_1b_3V_{004} + 2b_1^2V_{102} + 2a_1b_1(2V_{211} + 3V_{111})\\ &+ 2a_1^2b_1(V_{021} + 2V_{121}) + 2a_1b_1^2(V_{012} + 2V_{112})\\ &+ 2a_1b_2(3V_{112} + V_{012}) + 2a_2b_1(V_{021} + 3V_{121})\\ &+ 4a_1b_1b_2V_{013} + 4a_1a_2b_1V_{031}\\ &+ 2a_2b_2V_{022} + 2a_1^2b_2V_{022} + 2a_2b_1^2V_{022} + 2a_1b_3V_{013} + 2a_3b_1V_{031}\}. \quad (36)\end{aligned}$$

Similarly, to find the MSE of k up to the second-order of approximation, the terms $[1 + u_1\alpha_1e_1]^{-g_1}$ and $[1 + u_2\alpha_2e_2]^{-g_2}$ in (15) are expandable, respectively, as

$$[1 + u_1\alpha_1e_1]^{-g_1} = 1 - g_1u_1\alpha_1e_1 + \frac{g_1(g_1+1)}{2}\alpha_1^2u_1^2e_1^2 + \cdots \quad (37)$$

$$[1 + u_2\alpha_2e_2]^{-g_2} = 1 - g_2u_2\alpha_2e_2 + \frac{g_2(g_2+1)}{2}\alpha_2^2u_2^2e_2^2 + \cdots \quad (38)$$

Let us define the coefficient of e_1^i and e_2^i in (37) and (38), respectively, as follows:

$$c_i = (-u_1\alpha_1)^i\frac{g_1(g_1+1)\cdots(g_1+i-1)}{i!} \quad (39)$$

$$d_i = (-u_2\alpha_2)^i\frac{g_2(g_2+1)\cdots(g_2+i-1)}{i!} \quad (40)$$

Up to the second-order of approximation we can write

$$\begin{aligned}(k - \overline{Y})^2 = \overline{Y}^2\{&e_0^2 + c_1^2(e_1^2 + e_0^2e_1^2) + c_2^2e_1^4 + d_1^2(e_2^2 + e_0^2e_2^2) + c_1^2d_1^2e_1^2e_2^2 + d_2^2e_2^4\\ &+ 2c_1(e_0^2e_1 + e_0e_1) + 2c_1c_2(e_1^3 + 2e_0e_1^3) + 2c_2(e_0^2e_1^2 + e_1^2e_0) + 2c_3e_0e_1^3\\ &+ 2c_1c_3e_1^4 + 2c_1^2e_0e_1^2 + 2d_1(e_0^2e_2 + e_2e_0) + 2d_1d_2(e_2^3 + 2e_0e_2^3)\\ &+ 2d_2(e_0^2e_2^2 + e_2^2e_0) + 2d_3e_2^3e_0 + 2d_1d_3e_2^4 + 2d_1^2e_0e_2^2\\ &+ 2c_1d_1(e_1e_2 + 2e_0^2e_1e_2 + 3e_0e_1e_2) + 2c_1^2d_1(e_1^2e_2 + 2e_0e_1^2e_2)\\ &+ 2c_1d_1^2(e_1e_2^2 + 2e_0e_1e_2^2) + 2c_1d_2(e_1e_2^2 + 3e_0e_1e_2^2)\end{aligned}$$

$$+ 2c_2d_1(e_1^2e_2 + 3e_0e_1^2e_2) + 4c_1d_1d_2e_1e_2^3 + 4c_1c_2d_1e_2e_1^3 + 2c_2d_2e_1^2e_2^2 + 2c_1^2d_2e_1^2e_2^2 + 2c_2d_1^2e_2^2e_1^2 + 2c_1d_3e_1e_2^3 + 2c_3d_1e_1^3e_2\big\}. \quad (41)$$

By taking the expectation of (41), we obtain the second-order MSE as

$$\begin{aligned}\mathrm{MSE}_{II}(k) = \overline{Y}^2\big\{&V_{200} + c_1^2(V_{020} + V_{220}) + c_2^2V_{040} + d_1^2(V_{002} + V_{202}) + c_1^2d_1^2V_{022} + d_2^2V_{004}\\ &+ 2c_1(V_{210} + V_{110}) + 2c_1c_2(V_{030} + 2V_{130}) + 2c_2(V_{220} + V_{120}) + 2c_3V_{130}\\ &+ 2c_1c_3V_{040} + 2c_1^2V_{120} + 2d_1(V_{201} + V_{101}) + 2d_1d_2(V_{003} + 2V_{103})\\ &+ 2d_2(V_{202} + V_{102}) + 2d_3V_{103} + 2d_1d_3V_{004} + 2d_1^2V_{102}\\ &+ 2c_1d_1(V_{011} + 2V_{211} + 3V_{111}) + 2c_1^2d_1(V_{021} + 2V_{121})\\ &+ 2c_1d_1^2(V_{012} + 2V_{112}) + 2c_1d_2(V_{012} + 3V_{112}) + 2c_2d_1(V_{021} + 3V_{121})\\ &+ 4c_1d_1d_2V_{013} + 4c_1c_2d_1V_{031} + 2c_2d_2V_{022} + 2c_1^2d_2V_{022}\\ &+ 2c_2d_1^2V_{022} + 2c_1d_3V_{013} + 2c_3d_1V_{031}\big\}\end{aligned} \quad (42)$$

or we can rewrite (42) as

$$\begin{aligned}\mathrm{MSE}_{II}(k) = \mathrm{MSE}_{I}(k) + \overline{Y}^2\big\{&c_1^2V_{220} + c_2^2V_{040} + d_1^2V_{202} + d_2^2V_{004} + c_1^2d_1^2V_{022}\\ &+ 2c_1V_{210} + 2c_1c_2(V_{030} + 2V_{130}) + 2c_2(V_{220} + V_{120})\\ &+ 2c_3V_{130} + 2c_1c_3V_{040} + 2c_1^2V_{120}\\ &+ 2d_1V_{201} + 2d_1d_2(V_{003} + 2V_{103}) + 2d_2(V_{202} + V_{102})\\ &+ 2d_3V_{103} + 2d_1d_3V_{004} + 2d_1^2V_{102} + 2c_1d_1(2V_{211} + 3V_{111})\\ &+ 2c_1^2d_1(V_{021} + 2V_{121}) + 2c_1d_1^2(V_{012} + 2V_{112})\\ &+ 2c_1d_2(V_{012} + 3V_{112}) + 2c_2d_1(V_{021} + 3V_{121})\\ &+ 4c_1d_1d_2V_{013} + 4c_1c_2d_1V_{031} + 2c_2d_2V_{022}\\ &+ 2c_1^2d_2V_{022} + 2c_2d_1^2V_{022} + 2c_1d_3V_{013} + 2c_3d_1V_{031}\big\}.\end{aligned} \quad (43)$$

To obtain the MSE equations in (35) and (42) for the second-order expressions, we have used some results given in Sukhatme et al. (1984) and Nath (1968) (see Appendix).

5. Optimum Values and Their Estimates

It should be mentioned that adapted and suggested estimators require that population information of the auxiliary variables should be known. However, in applications, to have this information is sometimes impossible. Therefore, in such cases, it is advised to use consistent estimates for the population parameters in order to apply such estimators in practice. Upadhyaya and Singh (2006) also suggested that unless we have prior information about the population parameters, these parameters can be estimated from the sample. Similarly, we can prove that use of these estimation values do not change the minimum MSE up to the first order of approximation but change only the bias term.

It is obvious that the selection of an efficient estimator from families of estimators $\bar{y}_n$ and k depends on the values of $\beta_{yx,z(st)}$ and $\beta_{yz,x(st)}$. Now, in (7), we consider α_1, and α_2 are, respectively, equal to values of consistent estimates $\hat{\alpha}_1$ and $\hat{\alpha}_2$ which are obtained from the sample and η_1, η_2 are conveniently chosen scalars as

$$\hat{\bar{y}}_n = \bar{y}_{st}\left(\frac{\overline{X}}{\bar{x}_{st} + \eta_1(\overline{X} - \bar{x}_{st})}\right)^{\hat{\alpha}_1}\left(\frac{\overline{Z}}{\bar{z}_{st} + \eta_2(\overline{Z} - \bar{z}_{st})}\right)^{\hat{\alpha}_2}. \tag{44}$$

In this case, we define $e_3 = (\hat{\alpha}_1 - \alpha_1)/\alpha_1$ and $e_4 = (\hat{\alpha}_2 - \alpha_2)/\alpha_2$ such that $E(e_3) = E(e_4) = 0$. We can write (44) in terms of e's and expand as

$$\hat{\bar{y}}_n = \overline{Y}(1 + e_0)\left[1 - \alpha_1(1 + e_3)(1 - \eta_1)e_1 + \frac{\alpha_1(1 + e_3)(\alpha_1(1 + e_3) + 1)}{2}(1 - \eta_1)^2 e_1^2 + \cdots\right]$$
$$\times\left[1 - \alpha_2(1 + e_4)(1 - \eta_2)e_2 + \frac{\alpha_2(1 + e_4)(\alpha_2(1 + e_4) + 1)}{2}(1 - \eta_2)^2 e_2^2 + \cdots\right]. \tag{45}$$

Expanding (45) up to the first order of approximation, subtracting $\overline{Y}$ from both sides and squaring we have

$$(\hat{\bar{y}}_n - \overline{Y})^2 = \overline{Y}^2\{\alpha_1^2(1 - \eta_1)^2 e_1^2 + \alpha_2^2(1 - \eta_2)^2 e_2^2 + e_0^2 + 2\alpha_1\alpha_2(1 - \eta_1)(1 - \eta_2)e_1 e_2$$
$$- 2\alpha_1(1 - \eta_1)e_1 e_0 - 2\alpha_2(1 - \eta_2)e_2 e_0\}. \tag{46}$$

Then taking the expectation of (46) we get the same MSE of $\bar{y}_n$ given in (11).

Now, in (7), we can also consider η_1 and η_2 are, respectively, equal to the values of consistent estimates $\hat{\eta}_1$ and $\hat{\eta}_2$ which are obtained from the sample and α_1, α_2 are conveniently chosen scalars as

$$\hat{\bar{y}}_n = \bar{y}_{st}\left(\frac{\overline{X}}{\bar{x}_{st} + \hat{\eta}_1(\overline{X} - \bar{x}_{st})}\right)^{\alpha_1}\left(\frac{\overline{Z}}{\bar{z}_{st} + \hat{\eta}_2(\overline{Z} - \bar{z}_{st})}\right)^{\alpha_2}. \tag{47}$$

In this case, when we use the same notations as $e_3 = (\hat{\eta}_1 - \eta_1)/\eta_1$ and $e_4 = (\hat{\eta}_2 - \eta_2)/\eta_2$ it is clearly seen that we again obtain (46). Therefore, we can easily say that we obtain MSE of $\bar{y}_n$ given in (11) although we use the estimates of population parameters.

This proof is also valid for families of estimators k and k^*.

6. Numerical Example

To illustrate the efficiency of suggested estimators in the application, we consider the data concerning the number of teachers as the study variable (y) and number of students (x), number of classes (z) in both primary and secondary schools as auxiliary variables for 923 districts at 6 regions (as 1:Marmara, 2:Agean, 3:Mediterranean, 4:Central Anatolia, 5:Black Sea, 6:East and Southeast Anatolia) in Turkey in 2007 (source: The Turkish Republic Ministry of Education). The summary statistics of the data are given in Table 5. We used Neyman allocation for allocating the samples to different strata (Cochran, 1977).

The MSE values of the stratified adaptive estimators ($\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$, and $\bar{y}_4$) and the members of suggested family estimators (k_1, k_2, k_3, k_4, k_1^*, k_2^*, and k_3^*) have been

Table 5
Data statistics

$N_1 = 127$	$N_2 = 117$	$N_3 = 103$
$N_4 = 170$	$N_5 = 205$	$N_6 = 201$
$n_1 = 31$	$n_2 = 21$	$n_3 = 29$
$n_4 = 38$	$n_5 = 22$	$n_6 = 39$
$S_{y1} = 883.835$	$S_{y2} = 644.922$	$S_{y3} = 1033.467$
$S_{y4} = 810.585$	$S_{y5} = 403.654$	$S_{y6} = 711.723$
$\overline{Y}_1 = 703.74$	$\overline{Y}_2 = 413$	$\overline{Y}_3 = 573.17$
$\overline{Y}_4 = 424.66$	$\overline{Y}_5 = 267.03$	$\overline{Y}_6 = 393.84$
$S_{x1} = 30486.751$	$S_{x2} = 15180.769$	$S_{x3} = 27549.697$
$S_{x4} = 18218.931$	$S_{x5} = 8497.776$	$S_{x6} = 23094.141$
$\overline{X}_1 = 20804.59$	$\overline{X}_2 = 9211.79$	$\overline{X}_3 = 14309.30$
$\overline{X}_4 = 9478.85$	$\overline{X}_5 = 5569.95$	$\overline{X}_6 = 12997.59$
$S_{xy1} = 25237153.52$	$S_{xy2} = 9747942.85$	$S_{xy3} = 28294397.04$
$S_{xy4} = 14523885.53$	$S_{xy5} = 3393591.75$	$S_{xy6} = 15864573.97$
$\rho_{xy1} = 0.936$	$\rho_{xy2} = 0.996$	$\rho_{xy3} = 0.994$
$\rho_{xy4} = 0.983$	$\rho_{xy5} = 0.989$	$\rho_{xy6} = 0.965$
$\beta_2(x_1) = 4.593$	$\beta_2(x_2) = 18.543$	$\beta_2(x_3) = 15.446$
$\beta_2(x_4) = 10.162$	$\beta_2(x_5) = 21.947$	$\beta_2(x_6) = 23.114$
$\beta_2(y_1) = 2.158$	$\beta_2(y_2) = 16.392$	$\beta_2(y_3) = 14.979$
$\beta_2(y_4) = 12.167$	$\beta_2(y_5) = 21.088$	$\beta_2(y_6) = 20.254$
$w_1 = 0.138$	$w_2 = 0.127$	$w_3 = 0.112$
$w_4 = 0.184$	$w_5 = 0.222$	$w_6 = 0.218$
$\beta_1(x_1) = 2.164$	$\beta_1(x_2) = 3.867$	$\beta_1(x_3) = 3.748$
$\beta_1(x_4) = 3.121$	$\beta_1(x_5) = 4.084$	$\beta_1(x_6) = 4.411$
$S_{z1} = 555.5816$	$S_{z2} = 365.4576$	$S_{z3} = 612.9509281$
$S_{z4} = 458.0282$	$S_{z5} = 260.8511$	$S_{z6} = 397.0481$
$\overline{Z}_1 = 498.28$	$\overline{Z}_2 = 318.33$	$\overline{Z}_3 = 431.36$
$\overline{Z}_4 = 311.32$	$\overline{Z}_5 = 227.20$	$\overline{Z}_6 = 313.71$
$S_{yz1} = 480688.2$	$S_{yz2} = 230092.8$	$S_{yz3} = 623019.3$
$S_{yz4} = 364943.4$	$S_{yz5} = 101539.1$	$S_{yz6} = 277696.1$
$S_{xz1} = 15914648$	$S_{xz2} = 5379190$	$S_{xz3} = 16490674.56$
$S_{xz4} = 8041254$	$S_{xz5} = 2144057$	$S_{xz6} = 8857729$
$\rho_{yz1} = 0.978914$	$\rho_{yz2} = 0.976245$	$\rho_{yz3} = 0.983511$
$\rho_{yz4} = 0.982958$	$\rho_{yz5} = 0.964342$	$\rho_{yz6} = 0.982689$
$\beta_2(z_1) = 2.314926$	$\beta_2(z_2) = 11.19093$	$\beta_2(z_3) = 10.78635$
$\beta_2(z_4) = 8.624111$	$\beta_2(z_5) = 9.720886$	$\beta_2(z_6) = 14.40696$

obtained using (3)–(6), (19), and (26), respectively. These values are given in Table 6. From Table 6, we observe that $\bar{y}_3$ is the most efficient estimator for this data set. Note that all correlation coefficients are positively high so product estimators are not suitable to be applied. For this reason, the MSE of $\bar{y}_2$ is very big.

Table 6
MSE values of estimators

Estimators	MSE	Estimators	MSE	Estimators	MSE
k_1	1603.523	$\bar{y}_1$	1614.375	k_1^*	3180.882
k_2	1531.692	$\bar{y}_2$	17346.603	k_2^*	1606.305
k_3	1613.289	$\bar{y}_3$	1489.853	k_3^*	1653.794
k_4	1549.284	$\bar{y}_4$	3735.386		

We would also like to remark that the value of the $MSE_{min}(\bar{y}_n) = MSE_{min}(k)$ is obtained as 78.093 by (13). In addition, we would like to remind that the minimum MSE of estimator k^* changes according to choice of A_h and B_h. For this data set, if we use kurtosis, coefficient of variation, and skewness, we get the minimum values of MSE as 353.929, 80.754, and 139.025, respectively. Therefore, choosing A_h and B_h as the coefficient of variation in the estimator give the best result. However, the estimator k^* is less efficient than the estimators $\bar{y}_n$ and k for this data set, because the condition (28) is satisfied as $(\tau = 0.01129) > (\tau^* = 0.00985, \tau^* = 0.01128, \tau^* = 0.01097)$ according to the usage of kurtosis, coefficient of variation, and skewness, respectively.

Similarly, as the condition (29) is satisfied, the proposed k^* estimators are more efficient than the classical ratio estimator $\bar{y}_1$ using two auxiliary variables. In this data set, we have obtained the value of $\xi = -0.00323$ that is bigger than the values of $-\tau^*$. Therefore, the condition (29) is satisfied and, hence, the minimum values of the MSE (353.929, 80.754, and 139.025) are all smaller than the MSE of $\bar{y}_1$ that is equal to 1614.375.

When we use one auxiliary variable in the estimator such as $\bar{y}_{n5}$ and $\bar{y}_{n6}$, the minimum MSE of $\bar{y}_n$ is equal to the MSE of the combined regression estimator in the stratified random sampling which is 194.2832. When we use one auxiliary information in family of estimators k^*such as $\bar{y}_{R(\alpha_{st})}$ and $\bar{y}_{R(\delta_{st})}$ in Table 4, we get the minimum MSE of this estimator, respectively, as 430.220 and 184.507. Using A_h as coefficient of variation gives better result than other auxiliary information.

Up to the first order of approximation we get the minimum MSE that is equal to the MSE of the combined regression for the members of $\bar{y}_n$ and k among adapted and suggested estimators. Therefore, to find the most efficient estimator, we generate some members from families of estimators $\bar{y}_n$ and k given in Tables 1 and 3, respectively. The MSE values of these estimators, up to the second order of approximation have been computed using (35) and (42). These MSE values and the optimum values of the parameters are given in Tables 7 and 8. These optimum values have been obtained numerically using 'fminsearch' in Matlab. From Table 7, we observe that k_{21}, k_{22}, k_{23}, and k_{24} are the most efficient estimators among members of k for this data set. When we generate some estimators from k, we see that various transformations of auxiliary information do not affect the minimum MSE for the parameters α_1 and α_2, whereas this result is not true for the parameters g_1 and g_2 as they have negligible affect on the minimum MSE. From Table 8, we observe that $\bar{y}_{n1}$ is the most efficient estimator among members of $\bar{y}_n$.

Table 7
MSE of members of the family of estimators of k (second order)

Estimator	α_1^*	α_2^*	MSE II
k_5	−0.279530529355939	0.90212901168308	76.6374
k_6	−0.279867684334168	0.925278737590788	76.6374
k_7	−0.279487103040297	0.899065859010617	76.6374
k_8	−0.279708095648431	0.919811380821664	76.6374
k_9	0.311846442034744	−0.850354781955005	79.5843
k_{10}	0.312238680187503	−0.872168752130243	79.5843
k_{11}	0.311738383908655	−0.847556840232841	79.5843
k_{12}	0.312022600611856	−0.867074768037556	79.5843
k_{13}	0.289741631803247	0.890318432582972	76.2605
k_{14}	0.290096195013314	0.91315439278094	76.2605
k_{15}	0.289646745209769	0.887382598478207	76.2605
k_{16}	0.289888226391647	0.90778922020242	76.2605
k_{17}	−0.292970409479164	−0.871690030811558	81.1898
k_{18}	−0.293392172299289	−0.89395560708767	81.1898
k_{19}	−0.293379396786048	−0.868060703979399	81.1898
k_{20}	−0.293217897792675	−0.888702708510339	81.1898
k_{21}	−0.166361209238613	0.415902061095708	**73.0070**
k_{22}	−0.166548199134137	0.426558805374278	**73.0070**
k_{23}	−0.166350509324151	0.414434277495063	**73.0070**
k_{24}	−0.166489248090624	0.424012058211518	**73.0070**
k_{25}	0.150529192813828	−0.433601445660134	79.3600
k_{26}	0.150740391112601	−0.444705409664005	79.3600
k_{27}	0.15048562906852	−0.432182681261255	79.3600
k_{28}	0.150647107713412	−0.442097167398893	79.3600
k_{29}	0.143678900152989	0.445729253741683	76.7525
k_{30}	0.144851500354654	0.455718967503257	76.7525
k_{31}	0.143651422311461	0.444228378296056	76.7525
k_{32}	0.144678473737327	0.453148516040082	76.9128
k_{33}	−0.144711888276134	−0.440745641319829	79.2536
k_{34}	−0.14489916784232	−0.452004483517902	79.2536
k_{35}	−0.144679754043758	−0.439269043379338	79.2536
k_{36}	−0.144839017788132	−0.449317802856227	79.2536
	g_1^*	g_2^*	MSE II
k_{37}	0.298366851929565	0.889401764952618	75.9106
k_{38}	0.297986736579188	0.912907824493279	75.9691
k_{39}	0.298322893362842	0.886426893849436	75.9119
k_{40}	0.298623611432529	0.886344728825602	75.9120

Table 8
MSE of members of the family of estimators of $\bar{y}_n$ (second order)

	α_1	α_2	$(1-\eta_1)^*$	$(1-\eta_2)^*$	MSEs II
$\bar{y}_{n1}$	1	1	0.289681316683106	0.88696721526703	**76.2605**
$\bar{y}_{n2}$	−1	−1	−0.292926883772075	−0.868423348296856	81.1898
$\bar{y}_{n3}$	1	−1	0.311813653890672	−0.847141476276657	79.5843
$\bar{y}_{n4}$	−1	1	−0.279443907929654	0.898781771817806	76.6374

7. Conclusion

In this article, we suggested some families of estimators in the stratified random sampling using two auxiliary variables when some population parameters are known and unknown. These families of estimators include some other estimators proposed in literature such as Singh et al. (2008), Kadilar and Cingi (2003), etc., and from these families many new estimators can also be obtained. We examine the effect of various transformations of auxiliary information on the families of estimators. We also study the second order of approximation on the proposed families of estimators and we see that the MSE values of the k_5-k_8, $k_{13}-k_{16}$, $k_{21}-k_{24}$, $k_{29}-k_{32}$, $k_{37}-k_{40}$, $\bar{y}_{n1}$, and $\bar{y}_{n4}$ decrease according to first order of approximation. From Tables 6 and 8, we can infer that using the optimum values of (α_1^*, α_2^*) given in Table 7, decrease the MSE dramatically.

Appendix

$$V_{rst} = \sum_{h=1}^{L} W_h^{r+s+t} \frac{E\lfloor(\bar{y}_h-\overline{Y}_h)^r(\bar{x}_h-\overline{X}_h)^s(\bar{z}_h-\overline{Z}_h)^t\rfloor}{\overline{Y}^r\overline{X}^s\overline{Z}^t}$$

$$C_{rst(h)} = \frac{1}{N_h}\sum_{i=1}^{N_h}\left[(\bar{y}_{hi}-\overline{Y}_h)^r(\bar{x}_{hi}-\overline{X}_h)^s(\bar{z}_{hi}-\overline{Z}_h)^t\right]$$

$$E(e_0^2e_1) = V_{210} = \sum_{h=1}^{L} W_h^3\frac{k_{1(h)}C_{210(h)}}{\overline{Y}^2\overline{X}} \quad E(e_0^2e_2) = V_{201} = \sum_{h=1}^{L} W_h^3\frac{k_{1(h)}C_{201(h)}}{\overline{Y}^2\overline{Z}}$$

$$E(e_1^2e_2) = V_{021} = \sum_{h=1}^{L} W_h^3\frac{k_{1(h)}C_{021(h)}}{\overline{X}^2\overline{Z}} \quad E(e_0e_2^2) = V_{102} = \sum_{h=1}^{L} W_h^3\frac{k_{1(h)}C_{102(h)}}{\overline{Y}\overline{Z}^2}$$

$$E(e_0e_1^2) = V_{120} = \sum_{h=1}^{L} W_h^3\frac{k_{1(h)}C_{120(h)}}{\overline{Y}\overline{X}^2} \quad E(e_1e_2^2) = V_{012} = \sum_{h=1}^{L} W_h^3\frac{k_{1(h)}C_{012(h)}}{\overline{X}\overline{Z}^2}$$

$$E(e_1^3e_2) = V_{031} = \sum_{h=1}^{L} W_h^4\frac{k_{2(h)}C_{031(h)}+3k_{3(h)}C_{011(h)}C_{020(h)}}{\overline{X}^3\overline{Z}}$$

$$E(e_1e_2^3) = V_{013} = \sum_{h=1}^{L} W_h^4\frac{k_{2(h)}C_{013(h)}+3k_{3(h)}C_{011(h)}C_{002(h)}}{\overline{X}\overline{Z}^3}$$

$$E(e_0e_1^3) = V_{130} = \sum_{h=1}^{L} W_h^4\frac{k_{2(h)}C_{130(h)}+3k_{3(h)}C_{110(h)}C_{020(h)}}{\overline{Y}\overline{X}^3}$$

$$E(e_0e_2^3) = V_{103} = \sum_{h=1}^{L} W_h^4 \frac{k_{2(h)}C_{103(h)} + 3k_{3(h)}C_{101(h)}C_{002(h)}}{\overline{Y}\overline{Z}^3}$$

$$E(e_1^2e_2^2) = V_{022} = \sum_{h=1}^{L} W_h^4 \frac{k_{2(h)}C_{022(h)} + k_{3(h)}(C_{020(h)}C_{002(h)} + 2C_{011(h)}^2)}{\overline{X}^2\overline{Z}^2}$$

$$E(e_0^2e_2^2) = V_{202} = \sum_{h=1}^{L} W_h^4 \frac{k_{2(h)}C_{202(h)} + k_{3(h)}(C_{200(h)}C_{002(h)} + 2C_{101(h)}^2)}{\overline{Y}^2\overline{Z}^2}$$

$$E(e_0^2e_1^2) = V_{220} = \sum_{h=1}^{L} W_h^4 \frac{k_{2(h)}C_{220(h)} + k_{3(h)}(C_{200(h)}C_{020(h)} + 2C_{110(h)}^2)}{\overline{Y}^2\overline{X}^2}$$

$$E(e_1^3) = V_{030} = \sum_{h=1}^{L} W_h^3 \frac{k_{1(h)}C_{030(h)}}{\overline{X}^3} \quad E(e_2^3) = V_{003} = \sum_{h=1}^{L} W_h^3 \frac{k_{1(h)}C_{003(h)}}{\overline{Z}^3}$$

$$E(e_0e_1e_2) = V_{111} = \sum_{h=1}^{L} W_h^3 \frac{k_{1(h)}C_{111(h)}}{\overline{Y}\overline{X}\overline{Z}}$$

$$E(e_1^4) = V_{040} = \sum_{h=1}^{L} W_h^4 \frac{k_{2(h)}C_{040(h)} + 3k_{3(h)}C_{020(h)}^2}{\overline{X}^4},$$

$$E(e_2^4) = V_{004} = \sum_{h=1}^{L} W_h^4 \frac{k_{2(h)}C_{004(h)} + 3k_{3(h)}C_{002(h)}^2}{\overline{Z}^4}$$

$$E(e_0e_1^2e_2) = V_{121} = \sum_{h=1}^{L} W_h^4 \frac{k_{2(h)}C_{121(h)} + k_{3(h)}(C_{020(h)}C_{101(h)} + 2C_{110(h)}C_{011(h)})}{\overline{Y}\overline{X}^2\overline{Z}}$$

$$E(e_0e_1e_2^2) = V_{112} = \sum_{h=1}^{L} W_h^4 \frac{k_{2(h)}C_{112(h)} + k_{3(h)}(C_{002(h)}C_{110(h)} + 2C_{101(h)}C_{011(h)})}{\overline{Y}\overline{X}\overline{Z}^2}$$

$$E(e_0^2e_1e_2) = V_{211} = \sum_{h=1}^{L} W_h^4 \frac{k_{2(h)}C_{211(h)} + k_{3(h)}(C_{200(h)}C_{011(h)} + 2C_{110(h)}C_{101(h)})}{\overline{Y}^2\overline{X}\overline{Z}}$$

where

$$k_{1(h)} = \frac{1}{n^2}\frac{(N_h - n_h)(N_h - 2n_h)}{(N_h - 1)(N_h - 2)}$$

$$k_{2(h)} = \frac{(N_h - n_h)}{n^3}\frac{(N_h + 1)N_h - 6n_h(N_h - n_h)}{(N_h - 1)(N_h - 2)(N_h - 3)}$$

$$k_{3(h)} = \frac{(N_h - n_h)}{n^3}\frac{N_h(N_h - n_h - 1)(n_h - 1)}{(N_h - 1)(N_h - 2)(N_h - 3)}.$$

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