Algebra Colloquium **14**:4 (2007) 625–630

Algebra Colloquium

Direct Summands of ⊕-Supplemented Modules

Nil Orhan **Derya Keskin Tütüncü**
Department of Mathematics, University of Hacettepe
06532 Beytepe, Ankara, Turkey
E-mail: nilorhan@hacettepe.edu.tr keskin@hacettepe.edu.tr

Rachid Tribak
Département de Mathématiques, Faculté des Sciences de Tétouan
B.P 21.21. Tétouan, Morocco
E-mail: tribak12@yahoo.com

Received 28 September 2005
Revised 24 January 2006

Communicated by Nanqing Ding

Abstract. A module M is called $\oplus$-supplemented if every submodule of M has a supplement that is a direct summand of M. It is shown that if M is a $\oplus$-supplemented module and $r(M)$ is a coclosed submodule of M for a left preradical r, then $r(M)$ is a direct summand of M, and both $r(M)$ and $M/r(M)$ are $\oplus$-supplemented.

2000 Mathematics Subject Classification: 16D80, 16D99, 13C13

Keywords: $\oplus$-supplemented module, supplement submodule, left preradical

1 Introduction

Throughout this paper, R denotes an associative ring with identity and all modules will be unitary right R-modules. Let M be an R-module. A submodule A of M is said to be *small* in M (denoted by $A \ll M$) if for any submodule B of M, $M = A+B$ implies $M = B$. A module M is called *hollow* if every proper submodule of M is small in M. Let N and L be submodules of M. N is called a *supplement* of L in M if it is minimal with respect to the property $M = N + L$, equivalently, $M = N + L$ and $N \cap L \ll N$. N is called a *supplement submodule* of M if N is a supplement of some submodule of M. If every submodule of M has a supplement in M, we call M *supplemented*. The module M is *amply supplemented* if for any submodules A, B of M with $M = A + B$, there exists a supplement P of A such that $P \leq B$. The module M is called *weakly supplemented* if for every submodule N of M, there exists a submodule K of M with $M = N + K$ and $N \cap K \ll M$. Clearly, we have the following hierarchy: amply supplemented ⇒ supplemented ⇒ weakly supplemented.

Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then B is called a *co-essential submodule* of A in M. The submodule A of M is called *coclosed* if A has no proper co-essential submodule in M. We will call B an *s-closure* of A in M if B is a co-essential submodule of A in M and B is coclosed in M. The module M is called *lifting* (see [4, 6]) if M is amply supplemented and every supplement submodule of M is a direct summand. As a generalization of lifting modules, Mohamed and Müller [6] called an R-module M *$\oplus$-supplemented* if every submodule of M has a supplement that is a direct summand of M. In recent years, $\oplus$-supplemented modules have been extensively studied (see [2, 3, 5]). It is shown in [2, Theorem 1.4] that a finite direct sum of $\oplus$-supplemented modules is $\oplus$-supplemented. It is well known that any direct summand of a lifting module is a lifting module (see [6, Lemma 4.7]). In [6, p. 107] Mohamed and Müller wondered whether the $\oplus$-supplemented property is inherited by direct summands. This question is still unanswered. It is not our purpose to solve it. We will touch on only some special cases. In particular, we prove the following theorem:

Let R be a ring, r a left preradical for the category of right R-modules and M an R-module such that $r(M)$ has a unique s-closure in M and every direct summand of $r(M)$ has an s-closure in M. Then M is $\oplus$-supplemented if and only if $M = M_1 \oplus M_2$ is a direct sum of $\oplus$-supplemented modules M_1 and M_2 such that $r(M_1) \ll M_1$ and $r(M_2) = M_2$.

2 Results

To prove the main result of this paper, we will need the following properties of a left preradical r in the category of right R-modules:

(i) $r(M)$ is a submodule of M for every right R-module M.

(ii) $r(M_1 \oplus M_2) = r(M_1) \oplus r(M_2)$ for all right R-modules M_1 and M_2.

(iii) $\varphi(r(M)) \subseteq r(M')$ for every homomorphism $\varphi : M \to M'$ of right R-modules M and M'.

For the definition and basic properties of left preradicals, we refer the reader to [8].

Lemma 2.1. *Let R be a ring and r be a left preradical in the category of right R-modules. If M is a $\oplus$-supplemented R-module, then $M = M_1 \oplus M_2$ such that $r(M_1) \ll M_1$ and $r(M_2) = M_2$.*

Proof. Since $r(M) \leq M$, there exists a decomposition $M = M_1 \oplus M_2$ such that $M = M_1 + r(M)$ and $r(M_1) = M_1 \cap r(M) \ll M_1$. Now $M = M_1 + r(M_1) + r(M_2)$ $= M_1 \oplus r(M_2)$ implies that $r(M_2) = M_2$. □

Lemma 2.2. *Let $M = M_1 \oplus M_2$ be a weakly supplemented module. Let N and K be submodules of M such that $N \leq M_1$ and $K \leq M_2$. If E is an s-closure of N in M and F is an s-closure of K in M, then $E \oplus F$ is an s-closure of $N \oplus K$ in M.*

Proof. Let X be a submodule of M with $E+F \leq X$ and

$$\frac{N+K}{E+F} + \frac{X}{E+F} = \frac{M}{E+F}.$$

Since E is an s-closure of N in M, we get $K+E+X = M$. As F is an s-closure of K in M, we have $E+F+X = M$, and hence $X = M$. Therefore, $E+F$ is a co-essential submodule of $N+K$ in M. On the other hand, since E and F are coclosed, there exist submodules $E' \leq M_1$ and $F' \leq M_2$ such that E is a supplement of E' in M_1 and F is a supplement of F' in M_2 (see [4, Lemma 1.1]). Moreover, we have $(E+F)\cap(E'+F') \leq (E+F+E')\cap F' + (E+F+F')\cap E' \leq (M_1+F)\cap F' + (E+M_2)\cap E' \leq F\cap F' + E\cap E' \ll E+F$. Hence, $E+F$ is a supplement submodule of M. Thus, $E+F$ is coclosed in M. It follows that $E+F$ is an s-closure of $N \oplus K$ in M. □

The proof of the next result is exactly the same as that of [3, Proposition 2.5].

Proposition 2.3. *Let M be a nonzero module and let N be a submodule of M such that for each decomposition $M = M_1 \oplus M_2$ of M, we have $N = N\cap M_1 \oplus N\cap M_2$. If M is ⊕-supplemented, then M/N is ⊕-supplemented. If moreover N is a direct summand of M, then N is also ⊕-supplemented.*

Proof. Let L be a submodule of M with $N \leq L$. Since M is ⊕-supplemented, there exist submodules K and K' of M such that $M = K \oplus K'$ and K is a supplement of L in M. By [10, Lemma 1.2(d)], $(K+N)/N$ is a supplement of L/N in M/N. On the other hand, we have $(K+N)\cap(K'+N) = (K \oplus N\cap K')\cap(N\cap K \oplus K') = N\cap K \oplus N\cap K' = N$. Thus,

$$M = \Big[\frac{K+N}{N}\Big] \oplus \Big[\frac{K'+N}{N}\Big].$$

Therefore, M/N is ⊕-supplemented.

Now suppose that N is a direct summand of M. Let U be a submodule of N. Since M is ⊕-supplemented, there exist submodules V and V' of M such that $M = V \oplus V'$, $M = U + V$, and $U \cap V \ll V$. Thus, $N = U + V\cap N$. But by hypothesis, $N = V\cap N \oplus V'\cap N$. Hence, $V \cap N$ is a direct summand of N. Moreover, $U\cap(V\cap N) = U\cap V \ll V$. Then $U\cap(V\cap N) \ll V\cap N$ by [10, Lemma 1.1(b)]. Therefore, $V\cap N$ is a supplement of U in N and it is a direct summand of N. Thus, N is ⊕-supplemented. □

In the statements of Theorem 2.5 and Corollaries 2.4, 2.6 and 2.7, r denotes a left preradical for the category of right R-modules for a fixed ring R.

Corollary 2.4. *Let M be a nonzero ⊕-supplemented R-module. Then $M/r(M)$ is ⊕-supplemented. If moreover $r(M)$ is a direct summand of M, then $r(M)$ is also ⊕-supplemented.*

Proof. By Proposition 2.3. □

Now we can give the main result of this paper.

Theorem 2.5. *Let M be an R-module such that $r(M)$ has a unique s-closure in M and every direct summand of $r(M)$ has an s-closure in M. Then M is $\oplus$-supplemented if and only if $M = M_1 \oplus M_2$ is a direct sum of $\oplus$-supplemented modules M_1 and M_2 such that $r(M_1) \ll M_1$ and $r(M_2) = M_2$.*

Proof. The sufficiency follows from [2, Theorem 1.4]. Conversely, suppose that M is $\oplus$-supplemented. By Lemma 2.1, $M = M_1 \oplus M_2$ such that $r(M_1) \ll M_1$ and $r(M_2) = M_2$. Therefore, $r(M)/M_2 = (r(M_1) \oplus r(M_2))/M_2 \ll M/M_2$. This means that M_2 is the (unique) s-closure of $r(M)$ in M.

Let $M = K \oplus K'$ be any decomposition of M. Then $r(M) = r(K) \oplus r(K')$. By hypothesis, there exist submodules L and L' of M such that L is an s-closure of $r(K)$ and L' is an s-closure of $r(K')$ in M. From Lemma 2.2, $L \oplus L'$ is an s-closure of $r(K) \oplus r(K')$ in M, i.e., $L \oplus L'$ is an s-closure of $r(M)$ in M. It follows that $M_2 = L \oplus L'$. Thus, $M_2 = M_2 \cap K \oplus M_2 \cap K'$. Applying Proposition 2.3, M/M_2 and M_2 are $\oplus$-supplemented. This proves the theorem. □

Corollary 2.6. *Let M be an amply supplemented R-module such that $r(M)$ has a unique s-closure in M. Then M is $\oplus$-supplemented if and only if $M = M_1 \oplus M_2$ is a direct sum of $\oplus$-supplemented modules M_1 and M_2 such that $r(M_1) \ll M_1$ and $r(M_2) = M_2$.*

Proof. By Theorem 2.5 and [4, Proposition 1.5]. □

Corollary 2.7. *Let M be an R-module such that $r(M)$ is a coclosed submodule of M. Then M is $\oplus$-supplemented if and only if $M = r(M) \oplus M'$ for some submodule M' of M and both $r(M)$ and M' are $\oplus$-supplemented.*

Proof. Suppose that M is $\oplus$-supplemented. Since $r(M)$ is a coclosed submodule of M, $r(M)$ is a supplement in M by [4, Lemma 1.1]. Hence, every direct summand of $r(M)$ is a supplement in M. Thus, every direct summand of $r(M)$ has a unique s-closure in M. By Theorem 2.5, $M = M' \oplus M''$ is a direct sum of $\oplus$-supplemented submodules M' and M'' such that $r(M') \ll M'$ and $r(M'') = M''$. Thus, $r(M) = r(M') \oplus M''$. But $r(M)$ is a supplement submodule of M. Then $r(M') \ll r(M)$ by [10, Lemma 1.1(b)]. Hence, $r(M) = M''$ and $r(M') = 0$, and the proof is complete. □

A module M is called *coatomic* if every proper submodule is contained in a maximal submodule. Let M be an R-module. By $P(M)$ we denote the sum of all radical submodules of M. If $P(M) = 0$, M is called *reduced*. P is a left preradical for the category of right modules.

Corollary 2.8. *Let R be a local commutative noetherian ring and M be an R-module. Then M is $\oplus$-supplemented if and only if $M = P(M) \oplus K$ is a direct sum of $\oplus$-supplemented modules $P(M)$ and K such that K is coatomic and reduced.*

Proof. Suppose that M is ⊕-supplemented. By [7, Corollary 2.5], $M = P(M) + X$, where X is coatomic and $P(M)$ is a sum of finitely many hollow modules. It is easily seen that $P(M)$ is a supplement of X in M (see [10, Lemma 1.3(a)]). By Corollary 2.7, $M = K \oplus P(M)$ such that $P(M)$ and K are ⊕-supplemented. Since K is supplemented, taking account of [7, Corollary 2.5], K is coatomic. It is clear that K is reduced. □

Lemma 2.9. *Let $M = M_1 \oplus M_2$. Then M_2 is ⊕-supplemented if and only if for every submodule N/M_1 of M/M_1, there exists a direct summand K of M such that $K \leq M_2$, $M = K + N$ and $N \cap K \ll M$.*

Proof. Suppose that M_2 is ⊕-supplemented. Let N/M_1 be any submodule of M/M_1. As M_2 is ⊕-supplemented, there exists a decomposition $M_2 = K \oplus K'$ such that $M_2 = (N \cap M_2) + K$ and $N \cap K \ll K$. Note that $M = (N \cap M_2) + K + M_1$ gives $M = N + K$.

Conversely, suppose that M/M_1 has the stated property. Let H be a submodule of M_2. Consider the submodule $(H \oplus M_1)/M_1 \leq M/M_1$. By hypothesis, there exists a direct summand L of M such that $L \leq M_2$, $M = (L + H) + M_1$ and $L \cap (H + M_1) \ll M$. By modularity, $M_2 = L + H$. By [6, Lemma 4.2(2)], $L \cap H \ll L$. Thus, L is a supplement of H in M_2 and it is a direct summand of M_2. Therefore, M_2 is ⊕-supplemented. □

Theorem 2.10. *Let M_2 be a direct summand of a ⊕-supplemented module M such that for every direct summand K of M with $M = K + M_2$, $K \cap M_2$ is a direct summand of M. Then M_2 is ⊕-supplemented.*

Proof. Suppose that $M = M_1 \oplus M_2$ and let $N/M_1 \leq M/M_1$. Consider the submodule $N \cap M_2$ of M. Since M is ⊕-supplemented, there exists a direct summand K of M such that $M = (N \cap M_2) + K$ and $N \cap M_2 \cap K \ll K$. Note that $M = N + M_2$. By [4, Lemma 1.2], $M = (K \cap M_2) + N$. Since $M = K + M_2$, $K \cap M_2$ is a direct summand of M by hypothesis. By Lemma 2.9, M_2 is ⊕-supplemented. □

Corollary 2.11. *Let M be a ⊕-supplemented module and K be a direct summand of M such that M/K is K-projective. Then K is ⊕-supplemented.*

Proof. Let L be a direct summand of M with $M = L + K$. Since K is a direct summand of M, $M = K \oplus K'$ for some submodule K' of M. Therefore, K' is K-projective. Then by [9, 41.14], there exists a submodule L' of L such that $M = L' \oplus K$. Now $L = L' \oplus (L \cap K)$ implies that $L \cap K$ is a direct summand of M. By Theorem 2.10, K is ⊕-supplemented. □

Corollary 2.12. *Let $M = M_1 \oplus M_2$ be a direct sum of a submodule M_1 and a ⊕-supplemented submodule M_2 such that M_2 is M_1-projective. Then M is ⊕-supplemented if and only if M_1 is ⊕-supplemented.*

Proof. If M is ⊕-supplemented, then M_1 is ⊕-supplemented by Corollary 2.11. Conversely, if M_1 is ⊕-supplemented, M is ⊕-supplemented by [2, Theorem 1.4]. □

Example 2.13. Let R be a ring which is either right noetherian or commutative, and let $M = \bigoplus_{i=1}^{n} H_i$ be a right R-module which is the direct sum of hollow artinian modules H_i. By [1, Proposition 2.63], every direct summand of M is a finite direct sum of hollow modules. Therefore, any direct summand of M is $\oplus$-supplemented by [2, Corollary 1.6].

References

[1] A. Facchini, *Module Theory: Endomorphism Rings and Direct Sum Decompositions in Some Classes of Modules*, Birkhäuser, Basel-Boston-Berlin, 1998.

[2] A. Harmancı, D. Keskin, P.F. Smith, On $\oplus$-supplemented modules, *Acta Math. Hungar.* **83** (1-2) (1999) 161–169.

[3] A. Idelhadj, R. Tribak, On some properties of $\oplus$-supplemented modules, *Int. J. Math. Math. Sci.* **69** (2003) 4373–4387.

[4] D. Keskin, On lifting modules, *Comm. Algebra* **28** (7) (2000) 3427–3440.

[5] D. Keskin, W. Xue, Generalizations of lifting modules, *Acta Math. Hungar.* **91** (3) (2000) 245–253.

[6] S.H. Mohamed, B.J. Müller, *Continuous and Discrete Modules*, London Math. Soc. Lecture Note Series 147, Cambridge Univ. Press, Cambridge, 1990.

[7] P. Rudlof, On the structure of couniform and complemented modules, *J. Pure Appl. Algebra* **74** (1991) 281–305.

[8] B. Stenström, *Rings of Quotients: An Introduction to Methods of Ring Theory*, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[9] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Philadelphia, 1991.

[10] H. Zöschinger, Komplementierte Moduln über Dedekindringen, *J. Algebra* **29** (1974) 42–56.