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Cohereditary Modules in $\sigma[M]$

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This paper is dedicated to Professor Robert Wisbauer on his 65th birthday

Abstract. A module $N \in \sigma[M]$ is called cohereditary in $\sigma[M]$ if every factor module of N is injective in $\sigma[M]$. This paper explores the properties and the structure of some classes of cohereditary modules. Among others, we prove that any cohereditary lifting semi-artinian module in $\sigma[M]$ is a direct sum of Artinian uniserial modules. We show that over a commutative ring a lifting module N with small radical is cohereditary in $\sigma[M]$ if and only if N is semisimple M-injective. It is also shown that if E is an indecomposable injective module over a commutative Noetherian ring R with associated prime ideal p, then E is cohereditary lifting if and only if there is only one maximal ideal m over p and the ring R_m is a discrete valuation ring.

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1. Introduction

Throughout this article, R denotes an associative ring with identity and all modules will be unitary right R-modules. Mod-R denotes the category of all right R-modules. Let M be an R-module and $A \leq M$. The notation $A \ll M$ means that A is a small submodule of M. Let $End_R(M)$ denotes the endomorphism ring of M. We will denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are isomorphic to a submodule of an M-generated module. A module $N \in \sigma[M]$ is said to be M-*small* if N is small in its injective hull in $\sigma[M]$. It is easy to see that N is M-small if and only if there exists a module $L \in \sigma[M]$ such that $N \ll L$. A module $N \in \sigma[M]$ is called *cohereditary* in $\sigma[M]$ if every factor module of N is M-injective. The module N is called *cohereditary* if N is cohereditary in Mod-R ([5],[25]). Recall that a module M is called *lifting* if for every submodule A of M there exists a direct summand B of M such that $B \leq A$ and $A/B \ll M/B$.

In Section 2 some examples of cohereditary modules are given. In particular, we show that a direct sum of cohereditary modules in $\sigma[M]$ need not be in general cohereditary in $\sigma[M]$. We also prove that if M is a generator in $\sigma[M]$, then M is cohereditary in $\sigma[M]$ if and only if M is semisimple (Proposition 2.4).

Section 3 is devoted to the study of cohereditary lifting modules in $\sigma[M]$. We prove that any cohereditary lifting semi-artinian module in $\sigma[M]$ is a direct sum of Artinian uniserial modules (Corollary 3.8). It is also shown that over a commutative ring a lifting module $N \in \sigma[M]$ with small radical is cohereditary in $\sigma[M]$ if and only if N is semisimple M-injective (Corollary 3.12). We give an example of a finitely generated cohereditary lifting right R-module M over a noncommutative ring R such that M is not semisimple.

Section 4 deals with the cohereditary lifting modules over commutative Noetherian rings. Our main result shows that if E is an indecomposable injective module over a commutative Noetherian ring R with associated prime ideal p, then E is cohereditary lifting if and only if there is only one maximal ideal m over p and the ring R_m is a discrete valuation ring (Corollary 4.13).

2. Cohereditary modules in $\sigma[M]$

Recall that a module $N \in \sigma[M]$ is called *cohereditary* in $\sigma[M]$ if every factor module of N is M-injective. The module N is called *cohereditary* if N is cohereditary in Mod-R. Obviously factor modules of cohereditary modules in $\sigma[M]$ are again cohereditary in $\sigma[M]$. Of course, this property of N depend on the surrounding category $\sigma[M]$. For example, if M is a semisimple R-module, then for every module $N \in \sigma[M]$, N is cohereditary in $\sigma[M]$ by [24, 20.3] but N need not be cohereditary in Mod-R. The ring R is called right *cohereditary* if the module R_R is cohereditary. In [19], Osofsky proved that the ring R is semisimple if and only if every cyclic right R-module is injective. Therefore right cohereditary rings are precisely the semisimple rings. On the other hand, it is clear from the next example that a cohereditary module need not be semisimple and a semisimple module need not be cohereditary.

Examples 2.1. (1) Let R be a Dedekind domain which is not a field. The quotient field of R will be denoted by K. Let $\mathbb{P}$ denote the set of all non-zero prime ideals of R. If $P \in \mathbb{P}$, let $R(P^\infty)$ denote the set of all $x \in \frac{K}{R}$ such that $P^n x = 0$ for some integer $n \geq 0$. By [24, 39.16], a ring R is hereditary if and only if every injective R-module is cohereditary. In particular, over a Dedekind domain R, it is clear that a module M is cohereditary if and only if M is injective if and only if M is radical if and only if M is a direct sum of copies of K and $R(P^\infty)$, for various $P \in \mathbb{P}$ (see [8, Lemma 2.1]). So a $\mathbb{Z}$-module M is cohereditary if and only if M is a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$, for various primes p.

(2) It is clear that a semisimple module is cohereditary if and only if it is injective.

Proposition 2.1. *Let M be an R-module and N a cohereditary module in $\sigma[M]$. Then every submodule of an N-projective module in $\sigma[M]$ is N-projective.*

Proof. Let T be an N-projective module in $\sigma[M]$ and $A \leq T$. Let $f : A \to K$ be a homomorphism and $\pi : N \to K$ be an epimorphism, where K is any module. Since K is injective in $\sigma[M]$, there exists a homomorphism $g : T \to K$ such that $gi = f$, where $i : A \to T$ is the inclusion map. Since T is N-projective, there exists a homomorphism $h : T \to N$ such that $\pi h = g$. hi is the desired homomorphism and $\pi hi = f$. Hence A is N-projective. □

Theorem 2.2. *Let M be a projective module in $\sigma[M]$. Then the following are equivalent for a module N in $\sigma[M]$:*

1. *N is cohereditary in $\sigma[M]$.*
2. *Every submodule of an N-projective module in $\sigma[M]$ is N-projective and N is injective in $\sigma[M]$.*

Proof. (1) $\Rightarrow$ (2). By Proposition 2.1.

(2) $\Rightarrow$ (1). Since M is N-projective, every submodule of M is N-projective. Now the result follows by [24, 39.2 (2)]. □

The following lemma is [25, Proposition 6.2(1)]. We give its proof for completeness.

Lemma 2.3. *Let M be an R-module and $\{N_i\}_{i=1}^{n}$ be a family of modules in $\sigma[M]$. Then $N = \oplus_{i=1}^{n} N_i$ is cohereditary in $\sigma[M]$ if and only if all N_i are cohereditary in $\sigma[M]$.*

Proof. It is sufficient to show that the direct sum of two cohereditary modules N_1, N_2 in $\sigma[M]$ is again cohereditary. Let $N = N_1 \oplus N_2$ and $K \leq N$. We will show that N/K is M-injective. Since N_1 is cohereditary in $\sigma[M]$, $(N_1 + K)/K$ is M-injective. By [24, 16.3], $(N_1 + K)/K$ is a direct summand of N/K. There exists a submodule T/K of N/K such that $N/K = (N_1 + K)/K \oplus T/K$. Then $T/K \cong N_2/[(N_1 + K) \cap N_2]$. Since N_2 is cohereditary in $\sigma[M]$, T/K is M-injective. Then N/K is M-injective. □

A module $P \in \sigma[M]$ is called *hereditary* in $\sigma[M]$ if every submodule of P is projective in $\sigma[M]$. A ring R is right *hereditary* if the module R_R is hereditary in Mod-R. It is well known that a ring R is right hereditary if and only if every injective module is cohereditary [24, 39.16]. The following Remark shows that a direct sum of cohereditary modules in $\sigma[M]$ need not be in general cohereditary in $\sigma[M]$.

Remark 2.1. Let M be a projective hereditary module in $\sigma[M]$ which is not a locally Noetherian R-module. By [24, 39.8], every injective module in $\sigma[M]$ is cohereditary in $\sigma[M]$. By [24, 27.3], there exists a family of M-injective modules $(N_i)_{i \in I}$ such that $N = \oplus_{i \in I} N_i$ is not M-injective. This proves that a direct sum of cohereditary modules in $\sigma[M]$ need not be in general cohereditary in $\sigma[M]$. In particular, if R is a hereditary ring which is not Noetherian, then there exists a family of cohereditary R-modules $(N_i)_{i \in I}$ such that $N = \oplus_{i \in I} N_i$ is not cohereditary. As an example of hereditary non-Noetherian ring we can take the ring $\begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{Q} \end{bmatrix}$.

In fact, by [2, Example 28.12], R is a hereditary semiprimary ring that is neither left nor right Artinian. Therefore R is neither left nor right Noetherian by [2, Theorem 15.20].

Proposition 2.4. *Let M be an R-module such that M is a generator in $\sigma[M]$. Then M is cohereditary in $\sigma[M]$ if and only if M is semisimple.*

Proof. Assume M is semisimple. Then M is cohereditary in $\sigma[M]$ by [24, 20.3]. Conversely, assume M is cohereditary in $\sigma[M]$. Let X be a cyclic module in $\sigma[M]$. Since M is a generator in $\sigma[M]$, there exist a finite index set I and an epimorphism $f : M^{(I)} \to X$. By Lemma 2.3, $M^{(I)}$ is cohereditary in $\sigma[M]$. Therefore X is injective in $\sigma[M]$. Since every cyclic module in $\sigma[M]$ is injective in $\sigma[M]$, M is semisimple by [6, Corollary 7.14]. □

Corollary 2.5. *Let M be a right R-module such that M is a generator in $Mod-R$. Then M is cohereditary in Mod-R if and only if M is semisimple if and only if R is semisimple.*

Example 2.2. Consider the $\mathbb{Z}$- module $\mathbb{Q}$. It is well known that $\mathbb{Q}$ is an injective module. Since $\mathbb{Z}$ is a Dedekind ring, $\mathbb{Q}$ is a cohereditary $\mathbb{Z}$-module by [24, 40.5]. In particular, $\mathbb{Q}$ is cohereditary in $\sigma[\mathbb{Q}]$. On the other hand, it is clear that $\mathbb{Q}$ is not semisimple. Thus by [24, 20.3], $\sigma[\mathbb{Q}]$ contains a module which is not cohereditary in $\sigma[\mathbb{Q}]$. Moreover, this example shows that the condition "M is a generator in $\sigma[M]$" is not superfluous in Proposition 2.4.

A module M has the *summand sum property* (*SSP*) (resp. *summand intersection property* (*SIP*)) if the sum (resp. intersection) of two direct summands of M is a direct summand of M.

Theorem 2.6. (See [1, Theorem 8]). *M has the SSP iff for every decomposition $M = A \oplus B$ and every homomorphism $f : A \longrightarrow B$, Imf is a direct summand of B.*

Recall that a module M is said to be a (D_3)-*module* if for every two direct summands U, V of M with $U + V = M$, the submodule $U \cap V$ is also a direct summand of M. Note that every quasi-projective module is (D_3).

Lemma 2.7. (See [1, Lemma 19]). *If M has the SSP and M is a (D_3)-module, then M has the SIP.*

Proposition 2.8.

(1) *Let N be a cohereditary module in $\sigma[M]$. Then N has the SSP.*
(2) *Let N be a quasi-projective cohereditary module in $\sigma[M]$. Then N has the SIP.*

Proof. (1) Let $N = A \oplus B$ and let $f : A \longrightarrow B$ be any homomorphism. Clearly, $N/Kerf = (A/Kerf) \oplus ((B + Kerf)/Kerf)$. Therefore $A/Kerf \cong Imf$ is injective in $\sigma[M]$. Thus Imf is a direct summand of B. Hence by Theorem 2.6, M has the SSP.

(2) Clear by Lemma 2.7 and (1). □

3. Cohereditary lifting modules in $\sigma[M]$

Let M and N be right R-modules. In [22], Talebi and Vanaja define

$$\bar{Z}_M(N) = \cap\{Kerg \mid g : N \to L, L \in \mathcal{S}\}$$

where $\mathcal{S}$ denotes the class of all M-small modules. They call N an *M-cosingular (non-M-cosingular) module* if $\bar{Z}_M(N) = 0$ ($\bar{Z}_M(N) = N$). It is easy to see that a module $N \in \sigma[M]$ is non-M-cosingular if and only if every nonzero factor module of N is not M-small. Note that if $M = R$, then we say that "M is non-cosingular" instead of "M is non-R-cosingular".

Lemma 3.1. *Let $N \in \sigma[M]$. If N is cohereditary in $\sigma[M]$, then N is non-M-cosingular.*

Proof. Straightforward. □

A module M is called *discrete* if M is a lifting module such that every submodule N of M with M/N isomorphic to a direct summand of M is itself a direct summand of M. Note that every quasi-projective lifting module is discrete. A nonzero module M is called *hollow* if every proper submodule of M is small in M. The module M is called *local* if it is hollow and cyclic.

If every injective module in $\sigma[M]$ is lifting, then M is called a *Harada module.* Note that if M is a non-M-cosingular Harada module in $\sigma[M]$, then M is cohereditary in $\sigma[M]$ by [20, Theorem 2.2].

The following example shows that a non-M-cosingular module need not be cohereditary in $\sigma[M]$.

Example 3.1. Let K be a field and let $R = \prod_{n\geq 1} K_n$ with $K_n = K$ for all $n \geq 1$. Then the ring R is a von Neumann regular ring which is not semisimple (see [13, p. 264]). Hence the R-module R is not cohereditary in $Mod - R$ by Corollary 2.5. On the other hand, by [22, Corollary 2.6], the R-module R is non-cosingular.

Proposition 3.2. *Let M be an R-module. The following are equivalent for a module N in $\sigma[M]$:*

1. *N is cohereditary in $\sigma[M]$.*
2. *N is non-M-cosingular and every non-M-cosingular finitely N-generated module in $\sigma[M]$ is injective in $\sigma[M]$.*

Proof. (1) ⇒ (2). N is non-M-cosingular by Lemma 3.1. Let T be a non-M-cosingular finitely N-generated module in $\sigma[M]$. Then there exist a finite index set I and an epimorphism $f : N^{(I)} \to T$. By Lemma 2.3, T is injective in $\sigma[M]$.

(2) ⇒ (1). Let $K \leq N$. By [22, Proposition 2.4], N/K is non-M-cosingular. Since N/K is a finitely N-generated non-M-cosingular module, N/K is injective in $\sigma[M]$. Therefore N is cohereditary in $\sigma[M]$. □

Lemma 3.3. *Let $N \in \sigma[M]$. If N is non-M-cosingular discrete, then $End_R(N)$ is von Neumann regular.*

Proof. By [17, Theorem 5.4], $\mathrm{End}_R(N)/J(\mathrm{End}_R(N))$ is von Neumann regular and $J(\mathrm{End}_R(N)) = \{f : N \to N \mid \mathrm{Im}f \ll N\}$. Let $f \in J(\mathrm{End}_R(N))$. Then $\mathrm{Im}f$ is M-small. On the other hand, $N/\mathrm{Ker}f \cong \mathrm{Im}f$ implies that $\mathrm{Im}f = 0$ since N is non-M-cosingular. Therefore $J(\mathrm{End}_R(N)) = 0$ and hence $\mathrm{End}_R(N)$ is von Neumann regular. □

Corollary 3.4. *Let N be a cohereditary discrete module in $\sigma[M]$. Then $End_R(N)$ is von Neumann regular.*

Proof. By Lemma 3.1 and Lemma 3.3. □

Proposition 3.5.

(1) *Let N be a self-injective lifting R-module. Then N has a decomposition $N = \oplus_{i\in I} N_i$ such that each N_i is hollow with local endomorphism ring.*

(2) *Let H be a hollow non-M-cosingular module in $\sigma[M]$. Then H is discrete if and only if $End_R(H)$ is a division ring.*

Proof. (1) By [5, 22.20], [24, 19.9] and [17, Corollary 4.9].

(2) Let $0 \neq f : H \to H$ be any homomorphism. Assume $f(H) \neq H$. Then $f(H) \ll H$. Therefore $f = 0$, a contradiction. Therefore f is epic. Now $H/\mathrm{Ker}f \cong H$ implies that $\mathrm{Ker}f$ is a direct summand of H since H is discrete. Therefore $\mathrm{Ker}f = 0$. Thus $\mathrm{End}_R(H)$ is a division ring. The converse is a consequence of [17, Lemma 5.1]. □

We say that a module M has the *strong summand sum property*, denoted by *SSSP*, if the sum of any family of direct summands of M is a direct summand of M.

Proposition 3.6. *Let N be a cohereditary module in $\sigma[M]$. If N is lifting, then N has SSSP.*

Proof. By Proposition 2.8, N has the SSP. Therefore N has the SSSP by [7, Proposition 4.9]. □

Theorem 3.7. *Let N be a cohereditary module in $\sigma[M]$. If N is lifting, then N is a direct sum of uniserial modules.*

Proof. By Proposition 3.5, $N = \oplus_{i\in I} N_i$, where each N_i is hollow. Now we show that each N_i is uniserial. Let $i \in I$. Assume that T is any submodule of N_i. Then N_i/T is indecomposable injective in $\sigma[M]$. Therefore every factor module of N_i is uniform by [24, 19.9]. Thus for every factor module L of N_i, $Soc(L)$ is simple or zero. By [24, 55.1], each N_i is uniserial. □

Corollary 3.8. *Let N be a cohereditary module in $\sigma[M]$. If N is lifting semi-artinian, then N is a direct sum of Artinian uniserial modules.*

Proof. Let U be a uniserial semi-artinian module. By [24, 55.1], every factor module of U has a simple essential socle. Therefore U is Artinian. The result follows from Theorem 3.7. □

Example 3.2. Let R be as in Example 2.1. Let M be an R-module. By [17, Proposition A.7 and Proposition A.8], M is injective lifting if and only if $M \cong \oplus_{P\in\mathbb{P}}[R(P^\infty)]^{n_P}$ with for each $P \in \mathbb{P}$, n_P is a natural number which vary with P. In particular, a $\mathbb{Z}$-module N is injective lifting if and only if $N \cong \oplus_p[\mathbb{Z}(p^\infty)]^{m_p}$ where all p are primes and for each prime p, m_p is a natural number which vary with p. It is clear that every p-primary component of N is Artinian and N is cohereditary semi-artinian. In fact, let X be any proper submodule of N. Since X is torsion, it is a direct sum of primary groups X_p (see, e.g., [12, Theorem 1]). It is clear that $X_p \leq N_p$ where N_p is the p-component of N. Hence $\frac{N}{X} \cong \oplus_p(\frac{N_p}{X_p})$. Let q be a prime such that $\frac{N_q}{X_q} \neq 0$. Since N_q is Artinian as a finite direct sum of $\mathbb{Z}(q^\infty)$, we have $Soc(\frac{N_q}{X_q}) \neq 0$. Therefore N is semi-artinian.

Lemma 3.9. *Any local module over a commutative ring is discrete.*

Proof. By [17, Lemma 5.1] and [5, 4.27]. ☐

Lemma 3.10. *Suppose that the ring R is commutative and let M be an R-module. A local module L in $\sigma[M]$ is non-M-cosingular if and only if L is a simple M-injective module.*

Proof. Suppose that R is a commutative ring and let L be a local R-module. Let I be an ideal of R such that $L \cong \frac{R}{I}$. By Lemma 3.9, L is a discrete module. Since L is non-M-cosingular, $End_R(L)$ is a division ring by Proposition 3.5. But it is not hard to see that the ring $\frac{R}{I}$ is isomorphic to $End_R(\frac{R}{I})$. Therefore the ring $\frac{R}{I}$ is a division ring. Thus L is a simple R-module. By [16, Proposition 5.1.4], L is M-injective. The converse is clear. ☐

Theorem 3.11. *Suppose that the ring R is commutative and let M be an R-module. The following are equivalent for a module $N = \oplus_{i\in I}N_i$ in $\sigma[M]$ which is a direct sum of local submodules N_i:*

(1) *N is non-M-cosingular.*
(2) *For every $i \in I$, N_i is a simple M-injective module.*

Proof. (1) ⇒ (2). By Lemma 3.10, each N_i is a simple M-injective module.

(2) ⇒ (1). Since N is semisimple, N is quasi-injective lifting. By Lemma 3.10, every N_i is non-M-cosingular. Then N is non-M-cosingular by [22, Proposition 2.4]. ☐

Corollary 3.12. *Suppose that the ring R is commutative and let M be an R-module. The following are equivalent for a lifting R-module N in $\sigma[M]$ with $Rad(N) \ll N$:*

(1) *N is non-M-cosingular M-injective.*
(2) *N is cohereditary in $\sigma[M]$.*
(3) *N is semisimple M-injective.*

Proof. If N satisfies any of these three conditions, then N is a direct sum of hollow submodules by Proposition 3.5. But $\mathrm{Rad}(N) \ll N$. Then M is a direct sum of local submodules. The equivalence of the three statements follows from Theorem 3.11. ☐

Corollary 3.13. *Suppose that the ring R is commutative and let M be an R-module. The following are equivalent for a finitely generated lifting module N in $\sigma[M]$:*

(1) *N is non-M-cosingular.*
(2) *N is cohereditary in $\sigma[M]$.*
(3) *N is semisimple M-injective.*

Proof. It is a consequence of Theorem 3.11 and the fact that every finitely generated lifting module is a direct sum of local submodules. ☐

The following example shows that, when the ring R is not commutative, there may exist cohereditary and lifting cyclic modules which are not semisimple. Thus the commutativity of the ring R is necessary in Corollary 3.12.

Example 3.3. Let F be any field. Let R denote the ring of all upper triangular 2×2 matrices with entries in F. We know that R is a left and right hereditary Artinian ring (see [6, Example 13.6]). Consider the right R-module $M = [\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}]$. It is easy to see that M is cyclic Artinian. Clearly M is a direct summand of $E(R) = [\begin{smallmatrix} F & F \\ F & F \end{smallmatrix}]$. Therefore M is an injective right R-module. Since R is right hereditary, M is cohereditary by [24, 39.16]. Since $R = [\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}] \oplus [\begin{smallmatrix} 0 & 0 \\ 0 & F \end{smallmatrix}]$, M is projective. Since R is a right perfect ring, M is a lifting right R-module by [17, Theorem 4.41]. It is not hard to see that $\mathrm{Soc}(M) = [\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}]$. Therefore M is not semisimple.

Let M and N be two modules. If for every module F, any epimorphism $f : M \to F$ and any homomorphism $h : N \to F$ either there exists a homomorphism $\varphi : N \to M$ with $f\varphi = h$ or there exist a nonzero direct summand M_1 of M and $\varphi : M_1 \to N$ with $h\varphi = f\,|_{M_1}$, then N is called *almost M-projective* (see [3]).

Lemma 3.14. *Let U be a simple module and N an indecomposable module. If U is almost N-projective, then U is almost N/X-projective for every submodule X of N.*

Proof. Let U be almost N-projective and $X \leq N$. Let $f : U \to (N/X)/(Y/X)$ be any nonzero homomorphism and $\pi : N/X \to (N/X)/(Y/X)$ be the natural epimorphism, where $Y/X \leq N/X$. Consider the isomorphism $\alpha : (N/X)/(Y/X) \to N/Y$ and the natural epimorphism $\nu : N \to N/X$. Now, we have the homomorphism $\alpha f : U \to N/Y$ and the epimorphism $\alpha\pi\nu : N \to N/Y$. Assume that there exists a homomorphism $g : U \to N$ such that $\alpha\pi\nu g = \alpha f$. Therefore νg lifts f. Now, assume that there exists a nonzero direct summand N_1 of N and a homomorphism $h : N_1 \to U$ such that $\alpha f h = \alpha\pi\nu\,|_{N_1}$. Since N is indecomposable, $N_1 = N$ and hence $h : N \to U$ and $\alpha f h = \alpha\pi\nu$. Since U is simple, h is epic and f is monic. Therefore $\mathrm{Ker} h = Y$. Define the homomorphism $\bar{h} : N/X \to U$ with $\bar{h}(n + X) = h(n)$ where $n \in N$. It is easy to check that $f\bar{h} = \pi$. Thus U is almost N/X-projective. ☐

Theorem 3.15. *Let N be a cohereditary lifting module in $\sigma[M]$ with the decomposition $N = \oplus_{i\in I} N_i$, where each N_i is hollow. Assume that every simple subfactor S of N_i is almost N_i-projective ($i \in I$). Then N is a direct sum of Artinian hollow modules with local endomorphism rings.*

Proof. We want to show that every N_i is Artinian. As we saw in the proof of Theorem 3.7, every factor module of each N_i ($i \in I$) is a uniform module with zero or simple socle. Let $i \in I$. Let N_i/T be a nonzero factor of N_i. We will prove that N_i/T has nonzero socle. Suppose that N_i/T is not simple. Then there exists a nonzero element $x + T \in N_i/T$ with $\bar{x}R = (x + T)R \neq N_i/T$. Let P/T be a maximal submodule of $\bar{x}R$. Now, $U = (xR + T)/P$ is simple. Therefore by hypothesis, Lemma 3.14 and [3, Theorem 1], $U \oplus N_i/T$ is lifting. Consider the inclusion map $f : U \to N_i/P$ and the epimorphism $\alpha : N_i/T \to N_i/P$. By [14, Lemma 1], there exists a homomorphism $g : U \to N_i/T$ with $\alpha g = f$. Therefore $0 \neq g(U)$ is the only simple submodule in N_i/T. Hence every N_i is Artinian. □

4. Cohereditary lifting modules over commutative Noetherian rings

Throughout this section, R will be a commutative and Noetherian ring. The full ring of quotients of R will be denoted by $Q(R)$. If p is a prime ideal of R, we will denote by R_p the localization of R at p. If M is an R-module, we will mean by M *is cohereditary* that M is cohereditary in $Mod - R$.

Proposition 4.1. *Let M be a cohereditary lifting R-module. Then $M = \oplus_{i\in I} H_i$ is a direct sum of uniserial modules H_i in which each $H_i (i \in I)$ is either radical or simple and isomorphic to $E(\frac{R}{p_i})$, for some prime ideal p_i of R.*

Proof. By [21, Corollary p. 53], Theorem 3.7 and Lemma 3.10. □

Lemma 4.2. (See [27, Folgerung 2.7]) *Let I be an ideal of R. Then the following are equivalent:*

(1) *$\frac{R}{I}$ is non-cosingular.*
(2) *$\frac{R}{I}$ is semisimple and I is a direct summand of R.*

Remarks 4.1. (1) If R is a local ring which is not a field or R is a ring with $Soc(R) = 0$ then, by Lemma 4.2, for every proper ideal I of R, the R-module $\frac{R}{I}$ is not non-cosingular and hence is not cohereditary. Therefore every cohereditary lifting R-module is a direct sum of radical uniserial modules such that each of them is isomorphic to $E(\frac{R}{p_i})$ for some prime ideal p_i of R.

(2) If R is a local ring which is not a field, then every cohereditary lifting R-module is a finite direct sum of radical uniserial modules (see [11, Proposition 2.2]).

(3) If R is a local Artinian ring with maximal ideal m, then $E(\frac{R}{m})$ is cohereditary hollow if and only if R is a field. In this case, we have $E(\frac{R}{m}) \cong R$ (see [9, Remark 2] and Corollary 2.5).

Now our purpose is to investigate the structure of hollow cohereditary R-modules.

Lemma 4.3. (See [15, Theorem 3.72]) *Let* m *be a maximal ideal in a commutative ring* R (*not necessarily Noetherian*). *Then* $\frac{R}{m}$ *is an injective* R*-module if and only if* R_m *is a field.*

Proposition 4.4. *Suppose that* R *is a commutative ring* (*not necessarily Noetherian*). *Let* p *be a prime ideal of* R. *Then the following are equivalent:*

(1) $E(\frac{R}{p})$ *is cohereditary local.*

(2) p *is maximal and* R_p *is a field.*

In this case, we have $E(\frac{R}{p}) \cong \frac{R}{p}$.

Proof. (1) ⇒ (2). By Lemma 3.10, $E(\frac{R}{p})$ is simple. Then $E(\frac{R}{p}) \cong \frac{R}{p}$ and p is a maximal ideal of R. Therefore R_p is a field by Lemma 4.3.

(2) ⇒ (1). By Lemma 4.3, $E(\frac{R}{p}) \cong \frac{R}{p}$ is simple injective. This completes the proof. □

Corollary 4.5. *Suppose that* R *is a commutative local ring* (*not necessarily Noetherian*) *with maximal ideal* m *and let* p *be a prime ideal of* R. *Then the following are equivalent:*

(1) $E(\frac{R}{p})$ *is cohereditary local.*

(2) R *is a field.*

Proof. (1) ⇒ (2). By Proposition 4.4, $p = m$ and R_m is a field. But $R_m \cong R$. Then R is a field.

(2) ⇒ (1). Clear. □

Corollary 4.6. *Suppose that* R *is a commutative ring* (*not necessarily Noetherian*). *Then the following statements are equivalent:*

(1) R *is von Neumann regular.*

(2) *For every maximal ideal* m *of* R, $E(\frac{R}{m})$ *is cohereditary local.*

Proof. (1) ⇒ (2). By [15, Theorem 3.71] and Proposition 4.4.

(2) ⇒ (1). From Proposition 4.4 we conclude that for every maximal ideal m of R, R_m is a field. Therefore R is von Neumann regular by [15, Theorem 3.71]. □

As in [23], we say that an R-module M is *almost finitely generated* (a.f.g.) if M is not finitely generated but every proper R-submodule of M is finitely generated.

A ring R is *almost DVR* if R is a local Noetherian domain of Krull dimension 1 and the integral closure R' of R in $Q(R)$ is a discrete valuation ring which is finitely generated as R-module [23, page 194].

Proposition 4.7. *Let* E *be a cohereditary hollow radical* R*-module. Then* E *is almost finitely generated.*

Proof. Let F be any proper submodule of E. Then there exists an element x of E such that $x \notin F$. But E is uniserial (see Proposition 4.1). Thus F is contained in Rx. Since Rx is a Noetherian R-module, F is a finitely generated R-module. □

Definitions 4.8. An element of R that is not a zero divisor in R will be called a *regular element* of R. We shall say that an ideal of R is *regular* if it contains a regular element of R.

We will say that a ring R is a 1-*dimensional Cohen-Macaulay ring* if it is a commutative Noetherian ring of Krull dimension 1 such that every maximal ideal of R contains a regular element. A Noetherian domain of Krull dimension 1 is a 1-dimensional Cohen-Macaulay ring.

Let R be a local, 1-dimensional Cohen-Macaulay ring. Let I be a regular ideal of R. Then I is said to be a *canonical ideal* for R if $\frac{Q(R)}{I}$ is an injective R-module.

Theorem 4.9. *Suppose that R is local with maximal ideal m. Then the following are equivalent:*

(1) $E = E(\frac{R}{m})$ *is cohereditary hollow radical.*
(2) R *is a discrete valuation ring.*

If these conditions hold, then $E(\frac{R}{m}) \cong \frac{Q(R)}{R}$.

Proof. (1) ⇒ (2). Following [9, Proposition 4] and Proposition 4.7, R is an almost DVR. By [9, Proposition 3], there exists a nonzero ideal I of R such that $E \cong \frac{Q(R)}{I}$. So $\frac{Q(R)}{R}$ is cohereditary hollow. Let X be any proper nonzero R-submodule of $Q(R)$ and let x be a nonzero element in X. Then we have the exact sequence $\frac{Q(R)}{Rx} \to \frac{Q(R)}{X} \to 0$. Since $\frac{Q(R)}{Rx} \cong \frac{Q(R)}{R}$, $\frac{Q(R)}{X}$ is indecomposable injective Artinian (see [23, Proposition 1.4]). Therefore $\frac{Q(R)}{X} \cong E$. Thus for every nonzero ideal I of R, we have $\frac{Q(R)}{I} \cong E$, and hence I is a canonical ideal. By [18, Theorem 15.8], for every nonzero ideal I of R, we have $I \cong R$. So R is a principal ideal ring. The result follows.

(2) ⇒ (1). If R is a discrete valuation ring, it is well known that $E(\frac{R}{m}) \cong \frac{Q(R)}{R}$ and so $E(\frac{R}{m})$ is cohereditary hollow. □

Let Ω be the set of all maximal ideals of R. As in [28, page 53], given an $m \in \Omega$ and an R-module M, we will denote the m-local component of M by $K_m(M) = \{x \in M \mid x = 0 \text{ or the only maximal ideal over } \operatorname{Ann}_R(x) \text{ is } m\}$.

We call M m-local if $K_m(M) = M$, or equivalently if m is the only maximal ideal over each $p \in \operatorname{Ass}(M)$. In this case M is an R_m-module with the following operation: $(\frac{r}{s})x := rx'$ with $x = sx' (r \in R,\ s \in R\backslash m)$. The submodules of M over R and over R_m are identical.

For $K(M) = \{x \in M \mid \frac{R}{\operatorname{Ann}_{R(x)}} \text{ is semiperfect}\}$ we always have the decomposition $K(M) = \oplus_{m\in\Omega} K_m(M)$ by [28, Satz 2.3].

Lemma 4.10. ([28, Lemma 1.5(b)]) *The following are equivalent for an R-module M:*

(1) $M = K(M)$.

(2) $\frac{R}{p}$ *is a local ring for all* $p \in Ass(M)$.

Lemma 4.11. *Let p be a prime ideal of R such that $\frac{R}{p}$ is a local ring and let m be the only maximal ideal over p. Then:*

(1) $E(\frac{R}{p})$ *has the structure of an R_m-module.*

(2) *The submodules of $E(\frac{R}{p})$ over R and over R_m are identical.*

Moreover, as R_m-module, $E(\frac{R}{p})$ is isomorphic to an injective envelope of $\frac{R_m}{S^{-1}p}$ where $S = R\backslash m$.

Proof. By Lemma 4.10, $E(\frac{R}{p})$ is m-local. The rest of the proof runs as the proof of [10, Proposition 5.9]. □

Remarks 4.2. By Theorem 4.9, Lemma 4.11 and [21, Proposition 5.5], we can easily get the following results:

(1) Let p be a prime ideal of R such that $\frac{R}{p}$ is a local ring and let m be the only maximal ideal over p. Then $E(\frac{R}{p})$ has the structure of an R_m-module such that the submodules of $E(\frac{R}{p})$ over R and over R_m are identical, and the following are equivalent:

(i) $E(\frac{R}{p})$ is cohereditary hollow as an R-module.

(ii) $E(\frac{R}{p})$ is cohereditary hollow as an R_m-module.

(2) Let m be a maximal ideal of R. The following are equivalent:

(i) $E(\frac{R}{m})$ is cohereditary hollow radical.

(ii) The ring R_m is a discrete valuation ring.

A principal ideal ring is called *special* if it has only one prime ideal $p \neq R$ and p is nilpotent [26, page 245].

Theorem 4.12. *Suppose that R is local with maximal ideal m and let p be a prime ideal of R different from m. The following are equivalent:*

(1) $E(\frac{R}{p})$ *is cohereditary hollow radical.*

(2) *R is a discrete valuation ring.*

In this case we have $p = 0$ and $E(R) \cong Q(R)$.

Proof. (1) ⇒ (2). Since $E(\frac{R}{p})$ is hollow, $E(\frac{R}{p}) \cong Q(\frac{R}{I})$ with $I = \mathrm{Ann}_R(E(\frac{R}{p}))$, by [9, Proposition 5]. By Proposition 4.1, $Q(\frac{R}{I})$ is uniserial. Thus $\frac{R}{I}$ is a uniserial R-module. Therefore $\frac{R}{I}$ is a uniserial ring. Since $\frac{R}{I}$ is Noetherian, $\frac{R}{I}$ is a principal ideal ring. By [26, Chapter IV, Section 15, Theorem 33], $\frac{R}{I}$ is a discrete valuation ring or a special principal ideal ring. But $E(\frac{R}{p})$ is radical. Then $\frac{R}{I}$ is a discrete valuation ring (see [2, Corollary 15.21]). Since $\frac{p}{I}$ is the only minimal prime ideal of $\frac{R}{I}$, by [9, Proposition 5], $p = I$ and $E(\frac{R}{p}) \cong Q(\frac{R}{p})$. By [23, Proposition 1.4],

$Q(\frac{R}{p})/\frac{R}{p}$ is Artinian, as an $(\frac{R}{p})$-module. So $Q(\frac{R}{p})/\frac{R}{p}$ is Artinian, as an R-module. But $Q(\frac{R}{p})$ is a cohereditary R-module. Then $Q(\frac{R}{p})/\frac{R}{p}$ is an Artinian injective R-module. By [21, Corollary p. 53], $Q(\frac{R}{p})/\frac{R}{p} \cong E(\frac{R}{m})$, as R-modules. It follows that $E(\frac{R}{m})$ is a cohereditary hollow radical R-module. By Theorem 4.9, R is a discrete valuation ring. It is clear that $p = 0$.

(2) ⇒ (1). This is obvious. □

Corollary 4.13. *Let p be a prime ideal of R. The following are equivalent:*

(1) $E(\frac{R}{p})$ *is cohereditary hollow radical.*

(2) *There is only one maximal ideal m over p and the ring R_m is a discrete valuation ring.*

Proof. If p is maximal, then (1) ⇔ (2) by Remarks 4.2(2). Suppose that p is not maximal.

(1) ⇒ (2). By [28, Satz 2.3 and Satz 2.5], $E(\frac{R}{p}) = K(E(\frac{R}{p}))$. By Lemma 4.10, $\frac{R}{p}$ is a local ring. Therefore there is only one maximal ideal m over p. The result follows from Lemma 4.11, Remarks 4.2(1) and Theorem 4.12.

(2) ⇒ (1). By Lemma 4.11, Remarks 4.2(1) and Theorem 4.12. □

Proposition 4.14. *Let R be a commutative Noetherian domain which is not a field. Then the following statements are equivalent:*

(1) *R is a Dedekind domain.*

(2) *For every maximal ideal m of R, $E(\frac{R}{m})$ is cohereditary hollow radical.*

Proof. By Remarks 4.2(2) and [4, Théorème 1 p. 217]. □

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