Acta Math. Hungar.
83 (1–2) (1999), 161–169.

ON ⊕-SUPPLEMENTED MODULES

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Abstract. Let R be a ring and M a right R-module. M is called *⊕-supplemented* if every submodule of M has a supplement that is a direct summand of M, and M is called *completely ⊕-supplemented* if every direct summand of M is ⊕-supplemented. In this paper various properties of these modules are developed. It is shown that (1) Any finite direct sum of ⊕-supplemented modules is ⊕-supplemented. (2) If M is ⊕-supplemented and (D3) then M is completely ⊕-supplemented.

1. ⊕-supplemented modules

In this note all rings are associative with identity and all modules are unital right modules. Let R be a ring and let M be an R-module. $N \leqq M$ will mean N is a submodule of M. $E(M)$, $\operatorname{Rad}(M)$, $Z(M)$ will indicate injective hull, Jacobson radical and singular submodule of M, respectively. We set $Z^*(M) = \{ m \in M \,:\, mR \text{ is small in } E(mR) \}$, which is a submodule of M. Let N and K be submodules of M. N is called a *supplement* of K in M if it is minimal with respect to $M = N + K$, equivalently $M = N + K$ and $N \cap K$ is small in N. Following [9], M is called *supplemented* if every submodule of M has a supplement in M and is called *amply supplemented* if for any two submodules A and B with $M = A + B$, B contains a supplement of A. (In [7], supplemented modules are called *weakly supplemented* and amply supplemented modules are called *supplemented.*) It is clear that every amply supplemented module is supplemented.

A module M is said to satisfy $AB5^*$ if $\cap_I(N + L_i) = N + \cap_I L_i$ for every submodule N and inverse system of submodules L_i $(i \in I)$. For example, Artinian modules and more generally linearly compact modules satisfy $AB5^*$ (see, for example [9, 29.8]). Recall that a module M is *linearly compact* if for every index set I, elements m_i in M and submodules N_i $(i \in I)$ such that the cosets $m_i + N_i$ $(i \in I)$ satisfy the finite intersection property, $\cap_I(m_i + N_i)$ is non-empty. By Zorn's Lemma any module satisfying $AB5^*$ is supplemented. In addition if M is a module with a finite collection of supplemented submodules M_i $(1 \leqq i \leqq n)$ such that $M = M_1 + \cdots + M_n$ then M is supplemented [9, 41.2]. In particular, any finite direct sum of supplemented modules is supplemented.

Let M be any $\mathbf{Z}$-module. For any prime p, $M(p)$ will denote the p-primary component of M, i.e.

$$M(p) = \{m \in M \ : \ p^n m = 0 \text{ for some positive integer } n\}.$$

Zöschinger [13] proved that a $\mathbf{Z}$-module M is supplemented if and only if M is a torsion module and for every prime p the submodule $M(p)$ is a direct sum of an Artinian module and a module with bounded order. If the $\mathbf{Z}$-module M is a direct sum of an infinite number of copies of the Prüfer p-group $\mathbf{Z}(p^\infty)$ then M is a direct sum of supplemented modules but is not supplemented.

Following [7], we call a module M *$\oplus$-supplemented* if every submodule of M has a supplement that is a direct summand of M. A non-zero module M is called *hollow* if every proper submodule is small in M and is called *local* if the sum of all proper submodules of M is also a proper submodule of M. Note that local modules are hollow and hollow modules are $\oplus$-supplemented.

Clearly $\oplus$-supplemented modules are supplemented, but the converse is false in general [7, Lemma A.4(2)]. Zöschinger has proved that if R is a Dedekind domain then an R-module M is supplemented if and only if M is $\oplus$-supplemented (see [7, Proposition A.7] and [Proposition A.8]). If R is a PI-ring (i.e. a ring with polynomial identity), in particular a commutative ring, then we have the following simple fact.

LEMMA 1.1. *Let R be a prime PI-ring. Then a torsion-free injective R-module M is supplemented if and only if M is $\oplus$-supplemented.*

PROOF. Suppose that M is supplemented. Let N be a submodule of M and let L be a supplement of N in M. Let c be any non-zero central element of R. Then [5, Proposition 6.12] gives $M = Mc = Nc + Lc \leqq N + Lc$. Thus $M = N + Lc$. By the choice of L we have $L = Lc$. Thus $L = Lc$ for every non-zero central element c in R. By [6, Theorem 6.4], L is a divisible R-module and hence L is injective [5, Proposition 6.12]. Thus L is a direct summand of M. It follows that M is $\oplus$-supplemented.

Recall that a projective module M is *semiperfect* if every homomorphic image of M has a projective cover. Then we also have the following lemma.

LEMMA 1.2. *Let M be a projective module. Then the following statements are equivalent.*

1. *M is semiperfect.*
2. *M is supplemented.*
3. *M is $\oplus$-supplemented.*

PROOF. (1) $\Leftrightarrow$ (2) is proved in [7, Corollary 4.43]. (1) $\Leftrightarrow$ (3) is proved in [2, Proposition 1.4].

Let M be a module. We consider the following conditions.

(D1) For every submodule N of M, M has a decomposition with $M = M_1 \oplus M_2$, $M_1 \leqq N$ and $M_2 \cap N$ is small in M_2.

(D3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

By [7, Lemma 4.6 and Proposition 4.38], every quasi-projective module has (D3).

Let M be a module. Then by [7, Proposition 4.8], M has (D1) if and only if M is amply supplemented and every supplement submodule of M is a direct summand. Therefore every (D1)-module is $\oplus$-supplemented.

LEMMA 1.3. *Let N and L be submodules of a module M such that $N + L$ has a supplement H in M and $N \cap (H + L)$ has a supplement G in N. Then $H + G$ is a supplement of L in M.*

PROOF. Let H be a supplement of $N + L$ in M and let G be a supplement of $N \cap (H + L)$ in N. Then $M = (N + L) + H$ such that $(N + L) \cap H$ is small in H and $N = \left[N \cap (H + L)\right] + G$ such that $(H + L) \cap G$ is small in G. Since $(H + G) \cap L \leqq \left[(G + L) \cap H\right] + \left[(H + L) \cap G\right]$, $H + G$ is a supplement of L in M.

THEOREM 1.4. *For any ring R, any finite direct sum of $\oplus$-supplemented R-modules is $\oplus$-supplemented.*

PROOF. Let n be any positive integer and let M_i be a $\oplus$-supplemented R-module for each $1 \leqq i \leqq n$. Let $M = M_1 \oplus \cdots \oplus M_n$. To prove that M is $\oplus$-supplemented it is sufficient by induction on n to prove this is the case when $n = 2$. Thus suppose $n = 2$.

Let L be any submodule of M. Then $M = M_1 + M_2 + L$ so that $M_1 + M_2 + L$ has a supplement 0 in M. Let H be a supplement of $M_2 \cap (M_1 + L)$ in M_2 such that H is a direct summand of M_2. By Lemma 1.3, H is a supplement of $M_1 + L$ in M. Let K be a supplement of $M_1 \cap (L + H)$ in M_1 such that K is a direct summand of M_1. Again applying Lemma 1.3, we have that $H + K$ is a supplement of L in M. Since H is a direct summand of M_2 and K is a direct summand of M_1 it follows that $H + K = H \oplus K$ is a direct summand of M. Thus $M = M_1 \oplus M_2$ is $\oplus$-supplemented.

COROLLARY 1.5. *Any finite direct sum of modules with* (D1) *is $\oplus$-supplemented.*

COROLLARY 1.6. *Any finite direct sum of hollow (or local) modules is $\oplus$-supplemented.*

2. Completely ⊕-supplemented modules

While the properties (D1), amply supplemented and supplemented are inherited by summands, it is unknown (and unlikely) that the same is true for the property $\oplus$-supplemented. In this vein we call a module M *completely $\oplus$-supplemented* if every direct summand of M is $\oplus$-supplemented.

Let R be a Dedekind domain. We have already remarked that an R-module M is supplemented if and only if M is $\oplus$-supplemented. Therefore an R-module M is $\oplus$-supplemented if and only if M is completely $\oplus$-supplemented.

Let R be any ring. It is clear that every (D1)-module is completely $\oplus$-supplemented. But in general the converse is false as the following example shows.

EXAMPLE 2.1. Let R be a discrete valuation ring with field of fractions K, let P be the unique maximal ideal of R and let $P = Ra$ for some element $a \in P$. Let M denote the R-module $(K/R) \oplus (R/P)$. Then M is completely $\oplus$-supplemented, but is not (D1). Moreover M satisfies (D3).

PROOF. Let $M_1 = K/R$, $M_2 = R/P$. By [7, Proposition A.7], M is $\oplus$-supplemented and is not (D1). Let N be any direct summand of M. If $N \cap M_2 \neq 0$ then $N = (N \cap M_1) \oplus M_2$ and $N \cap M_1 = K/R$ or 0. Thus $N = M$ or $N = 0 \oplus M_2$. In either case N is $\oplus$-supplemented. Now suppose that $N \cap M_2 = 0$. Let $\pi : M \to M_1$ denote the canonical projection. Then $N \cong \pi(N) = (Ra^{-n})/R$, for some positive integer n, or $N \cong \pi(N) = K/R$. Since $(Ra^{-n})/R \cong R/Ra^n$ it follows that N is $\oplus$-supplemented in either case.

To show that M satisfies (D3) we investigate N a bit more. Suppose that $N \cap M_2 = 0$ and $\pi(N) = K/R$. Let $(x + R, b + P) \in N$ for some $x \in K$, $b \in R - P$. There exists $y \in K$ such that $x = ay$. Then $(y + R, c + P) \in N$ for some $c \in R$ and hence $(0, b + P) = (x + R, b + P) - a(y + R, c + P) \in N$. It follows that $(0, b + P) \in N \cap M_2$, a contradiction. Thus $N = (K/R) \oplus 0$.

Now suppose that $N \cap M_2 = 0$, $\pi(N) = (Ra^{-n})/R$. If $n = 0$ then $N = 0$. Suppose that $n > 0$. If $n > 1$ then $(a^{-2} + R, d + P) \in N$ for some $d \in R$. Hence $(a^{-1} + R, 0) \in N$. But $M = N \oplus X$ for some direct summand X of M and clearly $M_1 = \pi(N) + \pi(X)$ so that $\pi(X) = K/R$ and $X = (K/R) \oplus 0$ by the above argument. But $(a^{-1} + R, 0) \in N \cap X = 0$, a contradiction. Thus $n = 1$. There exists $f \in R$ such that $(a^{-1} + R, f + P) \in N$. Since $M = N \oplus X$ and $X = (K/R) \oplus 0$ we have $f \in R - P$. If $\omega \in N$ then $\omega = (ra^{-1} + R, g + P)$ for some $r \in R$, $g \in R$. Now $\big(0, (g - rf) + P\big) = \omega - r(a^{-1} + R, f + P) \in N \cap M_2$, so that $\omega = r(a^{-1} + R, f + P)$. It follows that $N = R(a^{-1} + R, f + P)$ for some unit f in R. Thus the direct summands N of M are 0, $M_1 \oplus 0$, $0 \oplus M_2$, M, N_x $(x \in R/P)$ where $N_x = R(a^{-1} + R, x)$ for each $x \in R/P$. Note that the submodules N_x are distinct and since $aN_x = 0$, N_x is simple for all $x \in R/P$. It is now easy to show that M satisfies (D3).

Given a positive integer n, the modules M_i $(1 \leqq i \leqq n)$ are called *relatively projective* if M_i is M_j-projective for all $1 \leqq i \neq j \leqq n$.

THEOREM 2.2. *Let M_i $(1 \leqq i \leqq n)$ be any finite collection of relatively projective modules. Then the module $M = M_1 \oplus \cdots \oplus M_n$ is $\oplus$-supplemented if and only if M_i is $\oplus$-supplemented for each $1 \leqq i \leqq n$.*

PROOF. The sufficiency is proved in Theorem 1.4. Conversely, we only prove M_1 to be $\oplus$-supplemented. Let $A \leqq M_1$. Then there exists $B \leqq M$

such that $M = A + B$, B is a direct summand of M and $A \cap B$ is small in B. Since $M = A + B = M_1 + B$, by [7, Lemma 4.47], there exists $B_1 \leqq B$ such that $M = M_1 \oplus B_1$. Then $B = B_1 \oplus (M_1 \cap B)$. Note that $M_1 = A + (M_1 \cap B)$ and $M_1 \cap B$ is a direct summand of M_1. Therefore $A \cap B = A \cap (M_1 \cap B)$ is small in $M_1 \cap B$ by [7, Lemma 4.2]. Hence M_1 is ⊕-supplemented.

PROPOSITION 2.3. *Let M be a ⊕-supplemented module with* (D3). *Then M is completely ⊕-supplemented.*

PROOF. Let N be a direct summand of M and A a submodule of N. We show A has a supplement in N that is direct summand of N. Since M is ⊕-supplemented, there exists a direct summand B of M such that $M = A + B$ and $A \cap B$ is small in B. Hence $N = A + (N \cap B)$. Furthermore $N \cap B$ is a direct summand of M because M has (D3). Then $A \cap (N \cap B) = A \cap B$ is small in $N \cap B$.

Let M be a module. A submodule N of M is *closed* in M if N has no proper essential extensions in M. In [8], P. F. Smith calls a module M a *UC-module* if every submodule of M has a unique closure in M. M is called *extending module* if every closed submodule of M is a direct summand of M.

LEMMA 2.4. *Let M be a UC extending module. Then M has* (D3).

PROOF. Let M_1 and M_2 be direct summands of M with $M = M_1 + M_2$. From [11, Proposition 1.1], $M_1 \cap M_2$ is a closed submodule of M. Since M is extending, $M_1 \cap M_2$ is a direct summand of M. Hence M has (D3).

THEOREM 2.5. *Let M be a UC extending module. Then M is ⊕-supplemented if and only if M is completely ⊕-supplemented.*

PROOF. Sufficiency is clear. Conversely, assume that M is ⊕-supplemented. By Lemma 2.4, M has (D3). Hence M is completely ⊕-supplemented from Proposition 2.3.

Let M be a module. M is called *monoform* if each non-zero partial endomorphism of M is monomorphism. M is called *polyform* if each partial endomorphism has closed kernel. M is called *locally finite dimensional* if every finitely generated submodule has finite Goldie dimension, following [11]. Note that polyform extending modules have (D3) [3, Lemma 1.11] and every monoform module is polyform.

COROLLARY 2.6. *Let M be a polyform (monoform) extending module. Then M is ⊕-supplemented if and only if M is completely ⊕-supplemented.*

PROOF. By [11, Proposition 2.2], M is a (UC)-module. Then by Theorem 2.5, we have the result.

THEOREM 2.7. *Suppose that M is a locally finite dimensional polyform module. If M is quasi-injective, then for any index set I, $M^{(I)}$ is ⊕-supplemented if and only if $M^{(I)}$ is completely ⊕-supplemented.*

PROOF. Suppose that $M^{(I)}$ is $\oplus$-supplemented. Since M is polyform, $M^{(I)}$ is polyform from [12, Proposition 3.3] and $M^{(I)}$ is quasi-injective from [11, Corollary 3.4]. Hence $M^{(I)}$ is extending. Thus by Corollary 2.6, $M^{(I)}$ is completely $\oplus$-supplemented.

LEMMA 2.8. *Let M be a supplemented module and let N be a submodule of M such that $N \cap \operatorname{Rad}(M) = 0$. Then N is semisimple.*

PROOF. By [9, 41.2(3)], $M/\operatorname{Rad}(M)$ is semisimple. Hence N is semisimple.

PROPOSITION 2.9. *Let M be a supplemented module. Then $M = M_1 \oplus M_2$, where M_1 is a semisimple module and M_2 is a module with* $\operatorname{Rad}(M_2)$ *essential in M_2.*

PROOF. Let M_1 be a submodule of M such that $\operatorname{Rad}(M) \oplus M_1$ is essential in M. Since M is supplemented, there exists a submodule M_2 of M such that $M = M_1 + M_2$ and $M_1 \cap M_2$ is small in M_2. Hence $M_1 \cap M_2$ is a submodule of both $\operatorname{Rad}(M)$ and M_1. It follows that $M = M_1 \oplus M_2$, and then $\operatorname{Rad}(M) = \operatorname{Rad}(M_2)$ is essential in M_2, and by Lemma 2.8, M_1 is semisimple.

PROPOSITION 2.10. *Let M be a $\oplus$-supplemented module. Then $M = M_1 \oplus M_2$, where M_1 is a module with* $\operatorname{Rad}(M_1)$ *small in M_1 and M_2 is a module with* $\operatorname{Rad}(M_2) = M_2$.

PROOF. Suppose that M is a $\oplus$-supplemented module. There exists a direct summand M_1 of M such that $M = M_1 + \operatorname{Rad}(M)$ and $M_1 \cap \operatorname{Rad}(M)$ is small in M_1 and $M = M_1 \oplus M_2$ for some submodule M_2 of M. Then $M = \operatorname{Rad}(M_2) \oplus M_1$. Hence $M_2 = \operatorname{Rad}(M_2)$, and $\operatorname{Rad}(M_1) = M_1 \cap \operatorname{Rad}(M)$ is small in M_1.

PROPOSITION 2.11. *Let M be a $\oplus$-supplemented module. Then $M = M_1 \oplus M_2$, where M_1 is a module with $Z^*(M_1)$ small in M_1 and M_2 is a module with $Z^*(M_2) = M_2$.*

PROOF. Since M is $\oplus$-supplemented, there exists a direct summand M_1 of M such that $M = Z^*(M) + M_1$, $Z^*(M_1) = M_1 \cap Z^*(M)$ is small in M_1 and $M = M_1 \oplus M_2$ for some submodule M_2 of M. Since $Z^*(M) = Z^*(M_1) \oplus Z^*(M_2)$, then $Z^*(M_2) = M_2$.

THEOREM 2.12. *For a module M with* (D3) *the following statements are equivalent.*

(i) *M is completely $\oplus$-supplemented.*

(ii) *M is $\oplus$-supplemented.*

(iii) *$M = M_1 \oplus M_2$, where M_1 is a semisimple module and M_2 is a $\oplus$-supplemented module with* $\operatorname{Rad}(M_2)$ *essential in M_2.*

(iv) *$M = M_1 \oplus M_2$, where M_1 is a $\oplus$-supplemented module with* $\operatorname{Rad}(M_1)$ *small in M_1 and M_2 is a $\oplus$-supplemented module with* $\operatorname{Rad}(M_2) = M_2$.

(v) *$M = M_1 \oplus M_2$, where M_1 is a $\oplus$-supplemented module with $Z^*(M_1)$ small in M_1 and M_2 is a $\oplus$-supplemented module with $Z^*(M_2) = M_2$.*

PROOF. (i) $\Rightarrow$ (ii). Clear from definition.

(ii) $\Rightarrow$ (i). It follows from Proposition 2.3.

(i) $\Rightarrow$ (iii). By Proposition 2.9, $M = M_1 \oplus M_2$, where M_1 is semisimple and $\operatorname{Rad}(M_2)$ is essential in M_2. By (i), M_2 is $\oplus$-supplemented.

(i) $\Rightarrow$ (iv). By Proposition 2.10, $M = M_1 \oplus M_2$, where $\operatorname{Rad}(M_1)$ is small in M_1 and $\operatorname{Rad}(M_2) = M_2$. Thus M_1 and M_2 are $\oplus$-supplemented by (i).

(i) $\Rightarrow$ (v). By Proposition 2.11, we have $M = M_1 \oplus M_2$, where $Z^*(M_1)$ is small in M_1 and $Z^*(M_2) = M_2$, and hence M_1 and M_2 are $\oplus$-supplemented by (i).

(iii) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (ii) follow by Theorem 1.4.

THEOREM 2.13. *The following statements are equivalent for a projective module M.*

(i) *M is a direct sum of $\oplus$-supplemented modules and* $\operatorname{Rad}(M)$ *has finite Goldie dimension.*

(ii) *$M = M_1 \oplus M_2$ for some semisimple module M_1 and module M_2 such that M_2 has finite Goldie dimension and M_2 is a (finite) direct sum of local modules.*

In this case, M is a semiperfect module.

PROOF. (ii) $\Rightarrow$ (i). Clear.

(i) $\Rightarrow$ (ii). Assume $M = \oplus_{i \in I} M_i$, M_i is $\oplus$-supplemented and $\operatorname{Rad}(M)$ has finite Goldie dimension. Since $\operatorname{Rad}(M) = \oplus_{i \in I} \operatorname{Rad}(M_i)$, then there is a finite subset J of I such that $\operatorname{Rad}(M_i) = 0$ for all $i \in I - J$. Therefore M_i is semisimple for all $i \in I - J$. Hence there is a semisimple submodule M_1 of M such that $M = M_1 \oplus (\oplus_{j \in J} M_j)$. By Proposition 2.9, without loss of generality, we may assume $\operatorname{Rad}(M_j)$ is essential in M_j $(j \in J)$. Then M_j $(j \in J)$ has finite Goldie dimension by [4, Proposition 3.20]. Next we prove each M_j is local or a finite direct sum of local modules, for $j \in J$. Set $H = M_j$ for any $j \in J$. First, note that $\operatorname{Rad}(H) \neq H$ because H is projective [1, Proposition 17.14]. Assume H has Goldie dimension 1, and take some $x \in H - \operatorname{Rad}(H)$. Since H is $\oplus$-supplemented, there exists a submodule K of H such that $H = xR + K$, $xR \cap K$ is small in K and $H = K \oplus K_1$ for some submodule K_1 of M. Then $K = 0$ or $K_1 = 0$. If $K_1 = 0$, then xR becomes a submodule of $\operatorname{Rad}(H)$. This is a contradiction. Hence $K = 0$, thus $H = xR$. It follows that H is local. Let $n > 1$ be a positive integer and assume each M_j having Goldie dimension k $(1 \leqq k < n)$ is local or a finite direct sum of local submodules. Let $j \in J$ and $H = M_j$ and assume H has Goldie dimension n. Suppose H is not local. Let $x \in H - \operatorname{Rad}(H)$ such that $H \neq xR$. Then since H is $\oplus$-supplemented there exists submodules K, K_1 of H such that $H = xR + K = K \oplus K_1$ and $xR \cap K$ is small in K. It is clear that $K_1 \neq 0$. Also $K \neq 0$. Since projective modules satisfy (D3) and then by Proposition 2.3, any direct summand of M is $\oplus$-supplemented. Thus K and

K_1 are $\oplus$-supplemented. By induction, K and K_1 are local or finite direct sums of local submodules. This completes the proof of (i) $\Rightarrow$(ii). To prove the last part, by (ii), M is a finite direct sum of $\oplus$-supplemented modules, by Theorem 1.4, M is $\oplus$-supplemented. Hence M is semiperfect by Lemma 1.2.

LEMMA 2.14. *Let M be an indecomposable module. Then M is hollow if and only if M is completely $\oplus$-supplemented.*

PROOF. Clear from definitions.

PROPOSITION 2.15. *Let $M = U \oplus V$ such that U and V have local endomorphism rings. Then M is completely $\oplus$-supplemented if and only if U and V are hollow modules.*

PROOF. The necessity is clear from Lemma 2.14. Conversely, let K be a direct summand of M. If $K = M$ then by Corollary 1.6, K is $\oplus$-supplemented. Assume $K \neq M$. Then either $K \cong U$ or $K \cong V$ by Krull–Schmidt–Azumaya Theorem [1, Corollary 12.7]. In either case K is $\oplus$-supplemented. Thus M is completely $\oplus$-supplemented.

EXAMPLE 2.16. Let a be any integer and denote M the $\mathbf{Z}$-module $(\mathbf{Z}/a^i\mathbf{Z}) \oplus (\mathbf{Z}/a^j\mathbf{Z})$ $(i,j \in \mathbf{N})$. Then M is completely $\oplus$-supplemented.

PROOF. By [9, 31.14], $\mathbf{Z}/a^i\mathbf{Z}$ and $\mathbf{Z}/a^j\mathbf{Z}$ have local endomorphism rings. Thus M is completely $\oplus$-supplemented by Proposition 2.15.

THEOREM 2.17. *Let M be a non-zero module with finite Goldie dimension. Then the following statements are equivalent.*

(i) *Every direct summand of M is a finite direct sum of hollow modules.*

(ii) *M is a completely $\oplus$-supplemented module.*

PROOF. (i) $\Rightarrow$ (ii). Clear by Corollary 1.6.

(ii) $\Rightarrow$ (i). Let N be a direct summand of M. Since N has finite Goldie dimension, N has a decomposition $N = L_1 \oplus \cdots \oplus L_n$, where each L_i is indecomposable for $1 \leqq i \leqq n$ for some finite integer $1 \leqq n$. Hence each L_i $(1 \leqq i \leqq n)$ is hollow from Lemma 2.14.

COROLLARY 2.18. *Let M be a linearly compact module. Then the following statements are equivalent.*

(i) *Every direct summand of M is a finite direct sum of hollow modules.*

(ii) *M is a completely $\oplus$-supplemented module.*

PROOF. By [10, Proposition 3.4] and Theorem 2.17.

Acknowledgement. This paper was written during a visit of the third author to the Hacettepe University under a grant supporting graduate students from TUBITAK. The authors wish to thank this organisation for its support.

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(Received September 19, 1997)

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