Appl Categor Struct (2008) 16:249–254
DOI 10.1007/s10485-006-9025-8

On Non-M-Cosingular Completely $\bigoplus$-Supplemented Modules

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Received: 25 April 2006 / Accepted: 5 July 2006 / Published online: 2 August 2006

Abstract In this paper, it is shown that any non-M-cosingular $\oplus$-supplemented module M is (D_3) if and only if M has the summand intersection property. Let $N \in \sigma[M]$ be any module such that $\overline{Z}_M(N)$ has a coclosure in N. Then we prove that N is (completely) $\oplus$-supplemented if and only if $N = \overline{Z}^2_M(N) \oplus K$ for some submodule K of N such that $\overline{Z}^2_M(N)$ and K both are (completely) $\oplus$-supplemented.

Key words pseudo projective module · non-M-cosingular module · completely $\bigoplus$-supplemented module

1. Introduction

Throughout this paper all rings are associative with identity and all modules are unital right modules. Let R be a ring. We denote the category of all right R-modules by Mod-R. Let M be a module and $S \subseteq M$. Then $S \ll M$ denotes that S is a small submodule of M. Let K be a submodule of any module M. K is called *coclosed* in M if whenever $K/A \ll M/A$ for any submodule A of M with $A \subseteq K$, $K = A$. If A and B are submodules of any module M with $A \subseteq B$, we say that A is a *coclosure* or an *s-closure* of B in M if $B/A \ll M/A$ and A is a coclosed submodule of M (See, [13] and [9]).

Let M be a module. Following [4], M has the *summand sum property* if the sum of any two direct summands of M is a direct summand of M and following [5], M has the *summand intersection property* if the intersection of any two direct summands of M is a direct summand of M.

Let M be any module. Let X be a submodule of M. A *supplement* of X in M is a submodule K of M minimal with respect to the property $M = X + K$, equivalently,

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$M = X + K$ and $X \cap K \ll K$. A submodule K of M is called a *supplement submodule* in M provided there exists a submodule X of M such that K is a supplement of X in M. If every submodule of M has a direct summand supplement in M, then M is called $\oplus$-*supplemented*. $\oplus$-supplemented modules are studied in [6, 7] and [8]. We don't know if any direct summand of any $\oplus$-supplemented module is $\oplus$-supplemented. According to [6], we call any module M *completely* $\oplus$-*supplemented* if every direct summand of M is $\oplus$-supplemented. Any module M is called *amply supplemented* (according to [11], supplemented) if for any two submodules A and B of M with $M = A + B$, there exists a supplement P of A in M which is contained in B. By [9, Proposition 1.5], if M is amply supplemented, then every submodule of M has a coclosure in M.

Let M be any module. M is called a *lifting* module (or *satisfies* (D_1)) if for any submodule X of M, there exists a direct summand K of M such that $K \leq X$ and $X/K \ll M/K$, equivalently, for every submodule X of M there exist submodules K and K' of M such that $M = K \oplus K'$, $K \leq X$ and $X \cap K' \ll K'$. By [14, 41.12], M is lifting iff M is amply supplemented and every coclosed submodule of M is a direct summand of M. Clearly, every lifting module is $\oplus$-supplemented and so every lifting module is completely $\oplus$-supplemented by [11, Lemma 4.7]. The module M is called *quasi-discrete* if M is lifting and satisfies the following condition:

(D_3) For every direct summands K and L of M with $M = K + L$, $K \cap L$ is a direct summand of M.

The module M is called *discrete* if M is lifting and satisfies the following condition:

(D_2) For any submodule K of M with M/K isomorphic to a direct summand of M, K is a direct summand of M.

Let M be an R-module. An R-module N is said to be *subgenerated by M* if N is isomorphic to a submodule of an M-generated module. We denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are all R-modules subgenerated by M (see [14]). The injective hull of any module $N \in \sigma[M]$ is denoted by $\widehat{N}$. The module $N \in \sigma[M]$ is said to be *M-small* if $N \ll \widehat{N}$. It is easy to see that N is M-small if and only if there exists a module $L \in \sigma[M]$ such that $N \ll L$.

Talebi and Vanaja define the following set for $N \in \sigma[M]$ in [13]:

$$\overline{Z}_M(N) = \bigcap\{\mathrm{Ker}g : g \in Hom(N, L), L \in \mathcal{S}\}$$

where $\mathcal{S}$ denotes the class of all M-small modules. They call N *non-M-cosingular* if $\overline{Z}_M(N) = N$ and call N *M-cosingular* if $\overline{Z}_M(N) = 0$. Clearly, every M-small module in $\sigma[M]$ is M-cosingular. Any $\oplus$-supplemented module with (D_3) is a completely $\oplus$-supplemented module by [6, Proposition 2.3]. In Section 2, we continue to study completely $\oplus$-supplemented modules. In particular, we prove that any $\oplus$-supplemented module with the summand sum property is completely $\oplus$-supplemented. Let M be a non-M-cosingular $\oplus$-supplemented module. We prove that M is (D_3) if and only if M has the summand intersection property.

Let $N \in \sigma[M]$. In [13], Talebi and Vanaja set $\overline{Z}^0_M(N) = N$, $\overline{Z}^1_M(N) = \overline{Z}_M(N)$ and define inductively $\overline{Z}^\alpha_M(N)$ for any ordinal α. If α is not a limit ordinal they set $\overline{Z}^\alpha_M(N) = \overline{Z}_M(\overline{Z}^{\alpha-1}_M(N))$, while if α is a limit ordinal they set $\overline{Z}^\alpha_M(N) = \bigcap_{\beta<\alpha} \overline{Z}^\beta_M(N)$. This gives the descending sequence $N = \overline{Z}^0_M(N) \supseteq \overline{Z}_M(N) \supseteq \overline{Z}^2_M(N) \supseteq \cdots$ of submodules of N. In Section 3, we decompose any $\oplus$-supplemented module $N \in \sigma[M]$

in terms of the submodule $\overline{Z}^2_M(N)$. We prove that if $N \in \sigma[M]$ is any module such that $\overline{Z}_M(N)$ has a coclosure in N, then N is (completely) $\oplus$-supplemented if and only if $N = \overline{Z}^2_M(N) \oplus K$ for some submodule K of N such that K and $\overline{Z}^2_M(N)$ are (completely) $\oplus$-supplemented.

2. Non-M-Cosingular $\oplus$-Supplemented Modules

We will start with some characterizations of $\oplus$-supplemented modules. Recall that M_1 and M_2 are *relatively projective* if M_1 is M_2-projective and M_2 is M_1-projective. Any module M is called a *pseudo projective* module if for every $A \leq M$ any epimorphism $f : M \longrightarrow M/A$ can be lifted to M. It is obvious that a pseudo projective module is (D_2) and any quasi-projective module is pseudo projective. But the converses are not true(see, for example, [3] and [2, 4.45(8)]). In this direction, Ganesan and Vanaja prove in [3] that any lifting pseudo projective module is quasi-projective [3, Theorem 3.3]. Note that in [2] and [3], they use the name epi-projective instead of pseudo projective.

LEMMA 2.1. *Assume that M is $\oplus$-supplemented such that whenever $M = M_1 \oplus M_2$ then M_1 and M_2 are relatively projective. Then M is lifting.*

Proof. Follows from 26.2 (c)⇒ (d) in [2]. □

The following corollary can be compared with 26.7 (h) in [2].

COROLLARY 2.2. *A module M is quasi-discrete if and only if M is $\oplus$-supplemented such that whenever $M = M_1 \oplus M_2$ then M_1 and M_2 are relatively projective.*

Proof. Necessity: by definition, every quasi-discrete module is $\oplus$-supplemented. Assume $M = M_1 \oplus M_2$ for the submodules M_1 and M_2 of M. By [11, Lemma 4.23], M_1 and M_2 are relatively projective. Sufficiency: by Lemma 2.1, M is lifting. It is easy to see that M is a (D_3)-module. Thus M is quasi-discrete. □

LEMMA 2.3. *If $M_1 \oplus M_2$ is pseudo projective, then M_1 and M_2 are relatively projective.*

Proof. Assume that $M_1 \oplus M_2$ is pseudo projective. Let $f : M_1 \to M_2/X$ be a homomorphism and $\pi : M_2 \to M_2/X$ be the natural epimorphism where $X \leq M_2$. Define $g : M_1 \oplus M_2 \to M_2/X$ with $g(m_1 + m_2) = f(m_1) + m_2 + X$. Clearly g is epic. By assumption, there exists a homomorphism $h : M_1 \oplus M_2 \to M_2$ such that $\pi h = g$. Therefore $h\,|_{M_1}$ is the desired homomorphism. Hence M_1 is M_2-projective. Similarly, M_2 is M_1-projective. □

Note that the above lemma is contained in 4.24 of [2].

LEMMA 2.4. *Let M be a pseudo projective module. Then the following are equivalent:*

(1) *M is discrete.*
(2) *M is quasi-discrete.*
(3) *M is lifting.*
(4) *M is completely $\oplus$-supplemented.*
(5) *M is $\oplus$-supplemented. In this case M is quasi-projective.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are clear. (5) $\Rightarrow$ (1). Then M is $\oplus$-supplemented. By Lemmas 2.1 and 2.3, M is lifting. On the other hand, since M is a (D_2)-module, it is discrete. The last statement follows from [3, Theorem 3.3]. □

THEOREM 2.5.

(1) *Assume that N is $\oplus$-supplemented and E is a submodule of N. If for every direct summand T of N, $(E+T)/E$ is a direct summand of N/E, then N/E is $\oplus$-supplemented. In particular, if N has the summand sum property, then N is completely $\oplus$-supplemented.*

(2) *Assume that N is (D_3) . Let D and E be direct summands of N such that $N/(D \cap E)$ is non-N-cosingular. If N/E is $\oplus$-supplemented, then $(D+E)/E$ is a direct summand of N/E*

(3) *Assume that N is non-N-cosingular (D_3). If N is (completely) $\oplus$-supplemented then it has the summand sum property.*

Proof.

(1) Consider $A/E \leq N/E$. By hypothesis, A has a direct summand supplement B in N. Then $(B+E)/E$ is a supplement of A/E in N/E which is a direct summand of N/E by hypothesis. Any direct summand of N is isomorphic to N/E for some direct summand E of N. Therefore if N has the summand sum property, then N is completely $\oplus$-supplemented.

(2) By hypothesis, $N/E = K/E + (D+E)/E$ where K/E is a direct summand of N/E and $(D+E)/E \cap K/E \ll N/E$. That is $(E+(D\cap K))/E$ is a small submodule of N/E and hence is an N-small module. Then $(D\cap K)/(D\cap E) \cong (E+(D\cap K))/E$ is also an N-small module and since $N/(D\cap E)$ is non-N-cosingular, $D\cap K = D\cap E$ and hence $N/E = (D+E)/E \oplus K/E$.

(3) Since N has (D_3), N is completely $\oplus$-supplemented if and only if it is $\oplus$-supplemented (see, [6, Proposition 2.3]). Suppose D and E are direct summands of N. Then N/E (being isomorphic to a direct summand of N) is $\oplus$-supplemented and $N/(D\cap E)$ is non-N-cosingular. By (2), $(D+E)/E$ is a direct summand of N/E and hence $D+E$ is a direct summand of N. □

PROPOSITION 2.6. *Let M have the summand sum property with (D_3). Then M has the summand intersection property.*

Proof. By [1, Lemma 19(2)]. □

COROLLARY 2.7. *Let M be a non-M-cosingular $\oplus$-supplemented module. Then M is (D_3) if and only if M has the summand intersection property.*

Proof. Clear by Theorem 2.5, Proposition 2.6 and the definition of (D_3)-module. □

In Corollary 2.7, the non-M-cosingularity condition is essential as we see in the following example.

EXAMPLE 2.8. (see also, [1, Example 1]) Let F be a field and let R denote the ring $R = \left\{ \begin{bmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \end{bmatrix} \mid a, b, x, y \in F \right\}$. Then the right R-module R_R is (D_3) since every quasi-projective module is (D_3). The Jacobson radical J of R consists of all matrices in R with a zero diagonal, and $R/J \cong FxF$. Therefore J is nonzero and hence R is not a right V-ring. By [13, Corollary 2.6], R_R is not non-R-cosingular. We also know that R is a QF-ring (see, for example, [10, 16.19(4)]). Hence R_R is $\oplus$-supplemented. Consider the right ideals $K = \left\{ \begin{bmatrix} 0&0&0&0 \\ 0&b&0&0 \\ 0&0&b&x \\ 0&0&0&0 \end{bmatrix} \mid b, x \in F \right\}$ and $N = \left\{ \begin{bmatrix} 0&b&0&0 \\ 0&b&0&0 \\ 0&0&b&x \\ 0&0&0&0 \end{bmatrix} \mid b, x \in F \right\}$ of R. Clearly, K and N are direct summands of R_R. But since $N \cap K = \left\{ \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&x \\ 0&0&0&0 \end{bmatrix} \mid x \in F \right\}$ is a nilpotent right ideal of R, it is not a direct summand of R_R. Thus R_R does not have the summand intersection property.

3. $\oplus$-Supplemented Modules and $\bar{Z}^2_M$

Let $N \in \sigma[M]$. Note that for every direct summand A of N, $\bar{Z}^2_M(A) = \bar{Z}^2_M(N) \cap A$ by [13, Proposition 2.1(4)]. Then for each decomposition $N = N_1 \oplus N_2$ of N we have that $\bar{Z}^2_M(N) = [\bar{Z}^2_M(N) \cap N_1] \oplus [\bar{Z}^2_M(N) \cap N_2]$. Therefore, if N is $\oplus$-supplemented, then $N/\bar{Z}^2_M(N)$ is $\oplus$-supplemented and if moreover $\bar{Z}^2_M(N)$ is a direct summand of N, then $\bar{Z}^2_M(N)$ is also $\oplus$-supplemented (see, for example, [7, Proposition 2.5]). Now we can give the following Theorem which is dual to Theorem 2.7 in [12].

THEOREM 3.1. *Let $N \in \sigma[M]$ be any module such that $\bar{Z}_M(N)$ has a coclosure in N. Then N is $\oplus$-supplemented if and only if $N = \bar{Z}^2_M(N) \oplus K$ for some submodule K of N such that $\bar{Z}^2_M(N)$ and K both are $\oplus$-supplemented.*

Proof. Sufficiency: by [6, Theorem 1.4]. Necessity: since N is $\oplus$-supplemented, there exist submodules K and K' of N such that $N = K \oplus K' = K + \bar{Z}^2_M(N)$ and $K \cap \bar{Z}^2_M(N) = \bar{Z}^2_M(K) \ll K$. Then $\bar{Z}^2_M(K)$ is M-small and so, M-cosingular. On the other hand, by [13, Corollary 3.4], $\bar{Z}^2_M(N)$ is a non-M-cosingular submodule of N and it is the unique coclosure of $\bar{Z}_M(N)$ in N. So, by [13, Proposition 2.4], $\bar{Z}^2_M(K)$ is non-M-cosingular. Hence $\bar{Z}^2_M(K) = 0$. Therefore $N = K + \bar{Z}^2_M(N) = K \oplus \bar{Z}^2_M(N)$. Now from the above remark, K and $\bar{Z}^2_M(N)$ are $\oplus$-supplemented submodules of N. □

COROLLARY 3.2. *Let $N \in \sigma[M]$ be amply supplemented. Then N is $\oplus$-supplemented if and only if $N = \bar{Z}^2_M(N) \oplus K$ for some submodule K of N such that $\bar{Z}^2_M(N)$ and K both are $\oplus$-supplemented.*

The following Theorem is dual to Theorem 2.15 in [16].

THEOREM 3.3. *Let $N \in \sigma[M]$ be any module such that $\overline{Z}_M(N)$ has a coclosure in N. Then N is completely $\oplus$-supplemented if and only if $N = \overline{Z}^2_M(N) \oplus K$ for some submodule K of N such that $\overline{Z}^2_M(N)$ and K both are completely $\oplus$-supplemented.*

Proof. Assume N is completely $\oplus$-supplemented. Then N is $\oplus$-supplemented and by Theorem 3.1, $N = \overline{Z}^2_M(N) \oplus K$ for some submodule K of N. Since all direct summands of $\overline{Z}^2_M(N)$ and K are also direct summands of N, $\overline{Z}^2_M(N)$ and K are completely $\oplus$-supplemented. Conversely, let $N = \overline{Z}^2_M(N) \oplus K$ for some submodule K of N with K and $\overline{Z}^2_M(N)$ completely $\oplus$-supplemented. By [6, Theorem 1.4], N is $\oplus$-supplemented. Suppose $N = D \oplus D'$. Then $N = \overline{Z}^2_M(N) \oplus K$ and $\overline{Z}^2_M(N) = \overline{Z}^2_M(D) \oplus \overline{Z}^2_M(D')$ implies $D = \overline{Z}^2_M(D) \oplus T$ and $D' = \overline{Z}^2_M(D') \oplus T'$. $T \oplus T' \cong K$ and K is completely $\oplus$-supplemented implies both T and T' are $\oplus$-supplemented. $\overline{Z}^2_M(N)$ is completely $\oplus$-supplemented implies both $\overline{Z}^2_M(D)$ and $\overline{Z}^2_M(D')$ are $\oplus$-supplemented. Hence both D and D' are $\oplus$-supplemented. □

COROLLARY 3.4. *Let $N \in \sigma[M]$ be amply supplemented. Then N is completely $\oplus$-supplemented if and only if $N = \overline{Z}^2_M(N) \oplus K$ for some submodule K of N such that $\overline{Z}^2_M(N)$ and K both are completely $\oplus$-supplemented.*

Note that the above corollary is also given in [15].

Acknowledgments The author would like to express her gratefulness to Professor R. Wisbauer for his valuable suggestions which improved the presentation of the paper. Also the author would like to thank Hacettepe University for the 05 G 602 001 Project.

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