# SUPPLEMENTARY MATERIAL
## for
## *Environmental-Induced Work Extraction (EIWE)*

In this supplementary material (SM), we first present the calculations for the work extraction in Sec I. In Sec. II, we demonstrate that a change in the curvature takes place after the work extraction process, i.e., after the entanglement is wiped out.

## I. WORK EXTRACTION

In this section, we obtain the amount of work the environmental monitoring extracts. At the first step, in Sec. I.1, we consider a two-mode (TM) squeezed thermal (Gaussian) state. We show that the extracted work is in the form $W = \xi(r) \times (\bar{n}\hbar\omega)$ at low-temperatures ($T$) —e.g., room temperature for optical modes. Here, $\xi(r)$, given in Eq. (S14), is the degree of the entanglement. In difference, e.g., to Refs. [1, 2] who employ Reyni entropy, we use von Neumann entropy ($S_V$) in our calculations. Second, in Sec. I.2, we also show that the same form, i.e., $W = \xi(r) \times (\bar{n}\hbar\omega)$, appears also for other TM Gaussian states.

### I.1. EIWE for two-mode squeezed thermal state

Initially, that is before the measurement, both modes, $a$ and $b$, are in thermal equilibrium with an environment at temperature $T$. When the environmental monitoring carries out the Gaussian measurement ($\lambda = 1$) on the $b$-mode, entropy of the $b$-mode decreases compared to the entropy at thermal equilibrium. When the $a$-mode rethermalizes with the environment it performs a work in the amount of [3, 4]

$$W = k_B T \left( S_V^{(\text{ther})} - S_V^{(\text{meas})} \right). \tag{S1}$$

Here, $S_V^{(\text{meas})}$ is the reduced entropy of the $a$-mode after the measurement in the $b$-mode is carried out. $S_V^{(\text{ther})}$ is the entropy of the $a$-mode after the rethermalization.

Entropy of a Gaussian state can be determined by its covariance matrix that contains the noise elements. Covariance matrix of a biparitite Gaussian state can be cast in the form [5–7]

$$\sigma_{ab} = \begin{bmatrix} \sigma_a & c_{ab} \\ c_{ab}^T & \sigma_b \end{bmatrix} \tag{S2}$$

via local symplectic transformations $Sp(2,\mathbb{R}) \oplus Sp(2,\mathbb{R})$, i.e., transformations altering neither entropy nor entanglement features. Here, $\sigma_a = \text{diag}(a,a)$ and $\sigma_b = \text{diag}(b,b)$ are the reduced covariance matrices of the $a$ and $b$ modes, respectively. $c_{ab} = \text{diag}(c_1, c_2)$ refers to the correlations/entanglement between the two modes. For a symmetrically squeezed two-mode thermal state, i.e., squeezing is done while both modes are in thermal equilibrium with $T$, the coefficients read $b = a$ and $c_2 = -c_1 = -c$.

The state into which the $a$-mode collapses is independent from the outcome of the $b$-mode measurement as long as a Gaussian measurement is carried out [8–11]. After the measurement, covariance matrix of the $a$-mode becomes [8–11]

$$\sigma_a^{\pi_b} = \sigma_a - c_{ab} \left( \sigma_b + \gamma^{\pi_b} \right)^{-1} c_{ab}^T \tag{S3}$$

which does not depend on the particular outcome of the $b$-measurement. Here, $\gamma^{\pi_b} = R(\phi)\ \text{diag}(\lambda/2, \lambda^{-1}/2)\ R(\phi)^T$ refers to the covariance matrix associated with a Gaussian operation (measurement) [8–11]. $\lambda$ is the measurement strength which depends solely on the measurement setup. In the case of environmental-monitoring, measurement basis is the coherent states. For coherent states $\lambda = 1$ and $\gamma^{\pi_b} = \text{diag}(1/2, 1/2)$ is independent of intramode rotations $R(\phi)$, i.e., $\hat{a}(\phi) = \hat{a}e^{i\phi}$.

The entropy of a Gaussian state is determined solely by its purity, $\mu = \frac{1}{2^n \sqrt{\text{Det}\sigma}}$, which takes the form [7]

$$S_V = \frac{1-\mu}{2\mu} \ln\left( \frac{1+\mu}{1-\mu} \right) - \ln\left( \frac{2\mu}{1+\mu} \right) \tag{S4}$$

for a single-mode state [1].

After the $b$-measurement, purity of the $a$-mode can be obtained as

$$\mu_1 \equiv \mu^{(\text{meas})} = \frac{a + 1/2}{2(a^2 - c^2 + a/2)}. \tag{S5}$$

For a TM squeezed thermal state,

$$a = (\bar{n} + \frac{1}{2}) \cosh(2r), \tag{S6}$$

$$c = (\bar{n} + \frac{1}{2}) \sinh(2r), \tag{S7}$$

the purity becomes

$$\mu_1 = \frac{a + 1/2}{2(\bar{n} + 1/2)^2 + a}, \tag{S8}$$

where $\bar{n} = (e^{\hbar\omega_a/k_B T} - 1)^{-1}$ is the occupation of the $a$-mode, which becomes $\bar{n} \to e^{-\hbar\omega_a/k_B T}$ at low T, e.g.,

[1] Please note that here we use von Neumann entropy in difference, e.g., to Refs.Refs. [1, 2], where Renyi entropy is employed.

the room temperature for optical modes. $r$ is the two-mode squeezing strength with which entanglement increases [12].

$\bar{n}$ is extremely small at low T regime. So, the purity can be approximated as

$$\mu_1 \cong 1 - \frac{2\bar{n}}{a+1/2}. \tag{S9}$$

Then, the entropy can be approximately written as

$$S_V^{(\mathrm{meas})} \cong \frac{\bar{n}}{a+1/2}[\ln(2) - \ln(2\bar{n}) + \ln(a+\frac{1}{2})] \tag{S10}$$
$$-\frac{2\bar{n}}{a+1/2},$$

where $a = (\bar{n}+1/2)\cosh(2r)$. Here, the last term origins from the $\ln\left(\frac{2\mu}{1+\mu}\right)$ term of Eq. (S4).

In Eq. (S10), the two terms in the square brackets are $\ln(\bar{n}) = \frac{\hbar\omega_a}{k_BT} \gg 1$ and $\ln(a+1/2) \cong \ln(\cosh(2r))$. Assuming that the squeezing rate (entanglement degree) is much smaller than $\frac{\hbar\omega_a}{k_BT}$, which is about ∼100 at the room temperature, i.e., $r \ll \frac{\hbar\omega_a}{k_BT}$, the entropy takes the form

$$S_V^{(\mathrm{meas})} \cong \frac{2\bar{n}}{1+\cosh(2r)}\frac{\hbar\omega_a}{k_BT}, \tag{S11}$$

where the last term in Eq. (S10) is also neglected.

Some time after the measurement, $a$-mode rethermalizes with the enviroment and reaches the equilibrium where purity becomes

$$\mu_2 \equiv \mu^{(\mathrm{ther})} = \frac{1}{1+2\bar{n}}. \tag{S12}$$

Entropy at the equilibrium can be calculated similarly and becomes

$$S_V^{(\mathrm{ther})} \cong \bar{n}\frac{\hbar\omega_a}{k_BT}. \tag{S13}$$

Therefore, the extracted work reads

$$W = \left(1 - \frac{2}{1+\cosh(2r)}\right) \times \bar{n}\hbar\omega_a, \tag{S14}$$

where the $k_BT$ term in Eq. (S1) is cancels with $1/k_BT$ appearing in (S11) and (S13). We refer the term in the parenthesis

$$\xi(r) = \left(1 - \frac{2}{1+\cosh(2r)}\right) \tag{S15}$$

as the entanglement degree which is a monotonically increasing function of two-mode squeezing (entanglement) parameter $r$ [12].

### I.2. EIWE for other Gaussian states

Above, we derived a simple form for the extracted work, i.e., $W = \xi(r) \times (\bar{n}\hbar\omega_a)$ for TM squeezed thermal states. Now, we show that a similar form appears also for other Gaussian states characterized by the covariance matrix (S2).

After the measurement in the $b$-mode, the $a$-mode collapses to a state having the covariance matrix $\sigma_a^{\pi_b}$ whose determinant is

$$\det\sigma_a^{\pi_b} = \frac{(a^2 - c_1^2 + a/2)(a^2 - c_2^2 + a/2)}{(a+1/2)^2}. \tag{S16}$$

The purity after the measurement is $\mu^{(\mathrm{meas})} = \frac{1}{2\sqrt{\det\sigma_a^{\pi_b}}}$ [7]. As a crosscheck, for $c_1 = -c_2 = c$, $\mu^{(\mathrm{meas})}$ becomes the purity given in Eq. (S5).

Here, we aim to show that the environmentally extracted work is in the "form" $W = \xi \times (\bar{n}\hbar\omega_a)$ also for other Gaussian states. Thus, we express Eq. (S16) in terms of $Sp(4,\mathbb{R})$ invariants

$$\det\sigma = (a^2 - c_1^2)(a^2 - c_2^2), \tag{S17}$$
$$\Delta = 2(a^2 + c_1c_2), \tag{S18}$$

where $\det\sigma$ is the determinant of the two-mode Gaussian state before the measurement. We do this because a random two-mode Gaussian state, Eq. (S2), can be obtained as $Sp(4,\mathbb{R})$ transformations of TM squeezed thermal states. (See Lemma 1 in Ref. [7].) The determinant in Eq. (S16) can be expressed as

$$\det\sigma_a^{\pi_b} = \frac{(a^2 - c_1^2)(a^2 - c_2^2) + \frac{a}{2}(2a^2 - c_1^2 - c_2^2)}{(\frac{1}{2}+a)^2}, \tag{S19}$$

where the first term in the numerator is $\det\sigma$ —given in Eq. (S17)— which is an $Sp(4,\mathbb{R})$ invariant. In the second term of the numerator, $I = (2a^2 - c_1^2 - c_2^2)$ can be expressed in terms of $Sp(4,\mathbb{R})$ invariants as

$$\det\sigma^{\pi_b} = \frac{(a^2 I + \Delta^2/4 - \Delta a^2)}{(a+\frac{1}{2})^2}. \tag{S20}$$

Please note that $a$ is only a local $Sp(2,\mathbb{R})$ invariant.

Below, we use a tilde, i.e., $\tilde{a}$ and $\tilde{c}_{1,2}$, for the covariance matrix elements belonging to TM squeezed states in order to distinguish them from the coefficients $a$ and $c$ which belong to general Gaussian states in Eq. (S16)-(S20).

One can realize that $\det\sigma = (\tilde{a}^2 - \tilde{c}^2)^2$ and $\Delta = 2(\tilde{a}^2 - \tilde{c}^2)$, expressed in terms of TM squeezed thermal state coefficients $\tilde{a}$ and $\tilde{c}$, are $Sp(4,\mathbb{R})$ invariant. Thus, Eq. (S20) can be recast as

$$(\tilde{a}^2 - \tilde{c}^2)^2 = a^2 I + (\tilde{a}^2 - \tilde{c}^2)^2 - \Delta a^2. \tag{S21}$$

Cancellation in Eq. (S21) results

$$I = 2a^2 - c_1^2 - c_2 = \Delta. \tag{S22}$$

Using this in Eq. (S20), we obtain the expression

$$\mu^{(\mathrm{meas})} = \frac{a+1/2}{2(z+a/2)} \tag{S23}$$

for the purity, where $z = \tilde{a}^2 - \tilde{c}^2 = (\bar{n} + 1/2)^2$. Please note that Eq. (S23) is in the same form with Eq. (S5) from which we obtain the extracted work

$$W = \xi \times (\bar{n}\hbar\omega_c) \tag{S24}$$

for TM squeezed thermal states.

Thus, above we showed that environmentally extracted work is in the form Eq. (S24) for general Gaussian states.

## II. CHANGE IN THE CURVATURE

In this section, we demonstrate that environmental-induced monitoring results a change in the background curvature in expense of diminishing entanglement. In the environmental-induced work extraction (EIWE) process, only the character of the particles' motion (in cavity $a$) changes. Before the environmental-monitoring, there is a $0 \leq \xi(r) \leq 1$ degree of entanglement. Particles (e.g., photons) in the cavity $a$ (i) moves isotropically in all directions, i.e., $v_{\text{mean}} = 0$. See Fig. S1(i). After environmental-monitoring performs the $b$ measurement, $W = \xi \times (\bar{n}\hbar\omega_a)$ amount of work is extracted in the cavity $a$. That is $\xi$ portion of the entire thermal energy is converted into directional (useful) energy. Assuming that $\xi = 1$ for the moment, all of the constituent particles (ii) move in a particular direction after the entanglement vanishes. See Fig. S1(ii).

Below, we show that the two instances, (i) and (ii), possess different curvatures in Einstein equations for a perfect relativistic fluid [13, 14]. The relativistic pressure for a fluid (i) moving isotropically in all directions and (ii) moving along a particular direction is different. We show that this creates a $\Delta R = 32\pi G p_0$ difference in the scalar curvature where $G$ is the gravitational constant and $p_0$ is the pressure of the isotropically moving fluid. That is, the initial entanglement is converted into scalar curvature.

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On a curved manifold, the full curvature tensor is the Riemann tensor $R_{\mu\alpha\nu\beta}$, of which one non-trivial trace is the Ricci tensor $R_{\mu\nu}$. Considering the underlying gravity theory to be Einstein's General Relativity, the field equations read

$$G_{\mu,\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{S25}$$

where $R$ is the scalar curvature given as $R = g^{\mu\nu} R_{\mu\nu}$ and $T_{\mu\nu}$ is the energy-momentum tensor which for a perfect fluid [13, 14] source reads as

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu + p g_{\mu\nu}, \tag{S26}$$

where $\rho$ is the energy density, $p$ is the relativistic pressure and $u^\mu$ is the four velocity of the particle normalized as $u^\mu u_\mu = -c^2$.

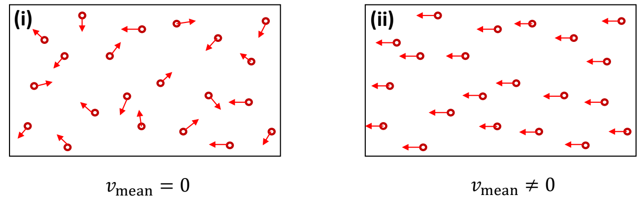


FIG. S1. Character of the motion of the constituent particles (e.g., photons) differs (i) before and (ii) after the work extraction process. (i) Thermal motion (isotropically moving in all directions) of the particles is converted into a state where all particles (ii) move along a single direction, i.e., description of work extraction. The scalar curvature $R$ differs by $\Delta R = +32\pi G p_0/c^4$ between the two instances. The initial entanglement is converted into a change in the curvature. Here, we assumed a perfect entanglement $\xi = 1$ between the cavities $a$ and $b$. For a partial entanglement degree, change in the curvature $\Delta R = \xi(r) \times \frac{32\pi G p_0}{c^4}$ is proportional to the entanglement degree.

For particles moving in the same direction, we have no pressure and the energy-momentum tensor is that of a relativistic dust

$$T_{\mu\nu} = \rho\, u_\mu u_\nu. \tag{S27}$$

If one seeks for the solutions of the Einstein equations, she/he needs an equation of state: namely one must know how $\rho$ and $p$ are related. Here, however, we are interested only in the difference between the two curvatures: (i) $R^{(i)}$ when the fluid moves isotropically in all directions ($v_{\text{mean}} = 0$) and (ii) $R^{(ii)}$ when the fluid moves in a particular direction ($v_{\text{mean}} \neq 0$).

We can obtain the Ricci curvature $R$ by multiplying the Einstein equation (S25) by the inverse metric $g^{\mu\nu}$ as

$$g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = \frac{8\pi G}{c^4} g^{\mu\nu} T_{\mu\nu}. \tag{S28}$$

We get $-R$ on the left hand side since $g^{\mu\nu} g_{\mu\nu} = 4$. Using $T_{\mu\nu}$, Eq. (S26), on the right hand side of Eq. (S28) we obtain

$$-R = \frac{8\pi G}{c^4}\left[\left(\rho + \frac{p}{c^2}\right) g^{\mu\nu} u_\mu u_\nu + p\, g^{\mu\nu} g_{\mu\nu}\right]. \tag{S29}$$

Here, $u^2 = g^{\mu\nu} u_\mu u_\nu$ is the same for a fluid (i) moving isotropically in all directions and (ii) moving in a single direction. For a *normal fluid* $u^2 = -c^2$ and for a *photon gas* it is $u^2 = 0$.

Thus, for a *normal fluid*, the scalar curvature is given by

$$R = -\frac{8\pi G}{c^4}\left[-\rho c^2 + 3p\right]. \tag{S30}$$

Therefore, the scalar curvature changes by

$$\Delta R = +3 \times \frac{8\pi G\, p_0}{c^4} \tag{S31}$$

after the work is extracted, i.e., instances (i) $p = p_0$ and (ii) $p = 0$.

Similarly, for a *photon gas*, i.e., $u^2 = 0$,

$$R = -\frac{8\pi G}{c^4} 4p \tag{S32}$$

and the Ricci curvature changes by

$$\Delta R = +4 \times \frac{8\pi G\, p_0}{c^4}. \tag{S33}$$

Above, we considered perfect entanglement, i.e., $\xi = 1$ between two instances where all thermal energy is converted into useful work $W = \bar{n}\hbar\omega_a$. If entanglement is not maximum, for instance let us take $\xi = 0.2$, $W = 0.2 \times (\bar{n}\hbar\omega_a)$ work is extracted. This means that after the work extraction, i.e., instance (ii), cavity $a$ possesses a 20% directional motion. Thus, moving to a properly chosen reference frame, 20% of the directional motion can be eliminated from the pressure. This means that there takes place a

$$\Delta R = \xi(r) \times \frac{24\pi G\, p_0}{c^4} \tag{S34}$$

change in the scalar curvature for a normal perfect fluid. For a photon gas, the change is

$$\Delta R = \xi(r) \times \frac{32\pi G\, p_0}{c^4}. \tag{S35}$$

That is, the change in the curvature is proportional to the entanglement degree $\xi$.


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