some additions to Input-Output formalism

$$\dot{\hat{a}}(t) = -\frac{i}{\hbar}[\hat{a}, \hat{H}_{sys}] - i\sum_{\vec{k}} g_{\vec{k}}\, e^{-i\omega_k (t-t_0)}\, b_k(t_0) - \sum_k g_k^2 \int_{t_0}^{t} dt'\, e^{-i\omega_k (t-t')}\, \hat{a}(t')$$

Adiabatic Approx. on highly detuned frequencies

- if ω_k is much different than ω_c, lets investigate these freq components seperately.

$$\dot{a}(t) = \ldots\ldots + \sum_{\substack{k \\ \omega_k \approx \omega_c}}' \ldots\ldots + \underbrace{\sum_{\substack{k \\ \omega_k \neq \omega_c}}' g_k^2 \int_{t_0}^{t} dt'\, e^{-i\Delta_k (t-t')}\, \hat{a}(t')}_{I_1}$$

- since $\Delta_k \gg \omega_{sys}$,

  $e^{-i\Delta_k (t-t')}$ oscillates very rapidly compared to $\hat{a}(t')$

$$I_1 \cong \sum_k' g_k^2\, a(t) \underbrace{\int_{t_0}^{t} dt'\, e^{-i\Delta_k (t-t')}}_{-\frac{1-e^{+i\Delta_k t}}{+i\Delta_k} \cdot e^{-i\Delta_k t}} \qquad \text{We put } t_0 = 0$$

$$\Rightarrow I_1 \cong -i \sum_k' \frac{g_k^2}{\Delta_k}\, \hat{a}(t)$$

We shall do a similar thing to the second term also:

$$J = -i \sum_{\vec{k}} g_{\vec{k}} e^{-i\omega_k (t-t_0)} b_k(t_0)$$

Oscillations with $\Delta_k \gg \omega_{sys}$ are only transients and do not couple to dynamics of $\hat{a}(t)$. Hence, we safely neglect them.

$$\Rightarrow J = -i \sum_{\substack{\vec{k} \\ \omega_k \cong \omega_c}}' g_{\vec{k}} e^{-i\Delta_k t} \cdot b_k(0)$$

Hence, one shall define $\hat{a}_{in}(t)$ in this regards!

Therefore, similar to Gardiner and Zoller Sec. 11.1.2a, we define a cut off for the integration Δ.

This term will determine the strength of Dirac-delta func, limiting it from reaching infinity. Since, similar terms will appear in different places, they will cancel.

Lets look at $\langle \delta a_{in}(t) \delta a^{+}_{in}(t') \rangle = -i \cdot i \cdot \sum'_{\vec{k}_1} \sum'_{\vec{k}_2} e^{-i\omega_{k_1} t} e^{i\omega_{k_2} t'} \underbrace{\langle b_{\vec{k}_1} b^{+}_{\vec{k}_2} \rangle}_{\delta_{\vec{k}_1, \vec{k}_2}}$

$$= \sum'_{\vec{k}} e^{-i\omega_k (t-t')}$$

$$= \int_{-\Delta/2}^{\Delta/2} d\omega \cdot D(\omega)\, e^{-i\omega(t-t')} = \Delta \cdot D(\omega) \cdot \delta(t-t')$$

$$\begin{cases} \delta(t-t') = 1 & \text{if } t = t' \\ 0 & \text{otherwise} \end{cases}$$

- It behaves as a Dirac-delta fnx, but it is not infinite!

A similar factor will apear for I:

$$I = \sum_{\vec{k}} g_k^2 \int_{t_0}^{t} dt'\, e^{-i\omega_k (t-t')}\, \hat{a}(t')$$

$$= \int_{t_0}^{t} dt'\, a(t') \int_{-\Delta/2}^{\Delta/2} d\omega\, D(\omega) \cdot g_c^2 \cdot e^{-i\omega(t-t')}$$

$$= \int_{t_0}^{t} dt'\, a(t') \cdot D(\omega)\, g_c^2 \cdot \Delta \cdot \delta(t-t')$$ (→ dimensionless)

- Lets examine this delta fnx more:

1.2 Integral Representation

To get the normalization correct for the integral representation of a delta function note that:

$$\int_{-\infty}^{\infty} dx e^{ikx} f(x) = \tilde{f}(k)$$
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-ikx} \tilde{f}(k). \tag{7}$$

Therefore taking $f(x) = \delta(x)$, $\tilde{f}(k) = 1$ and, we have that

$$\delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{ikx}. \tag{8}$$

We also get, for free, another form of the delta function:

$$\begin{aligned}
\delta(k) &= \lim_{\tau\to 0^+} \frac{1}{2\pi}\left[\int_{-\infty}^{0} dx e^{i(k-i\tau)x} + \int_{0}^{\infty} dx e^{i(k+i\tau)x}\right] \\
&= \lim_{\tau\to 0^+} \frac{1}{2\pi}\left[\frac{1}{i(k-i\tau)} - \frac{1}{i(k+i\tau)}\right] \\
&= \lim_{\tau\to 0^+} \frac{1}{2\pi}\left[\frac{1}{\tau+ik} + \frac{1}{\tau-ik}\right] \\
&= \lim_{\tau\to 0^+} \frac{1}{\pi}\frac{\tau}{\tau^2+k^2}.
\end{aligned} \tag{9}$$

$(ik+\tau)x$ $(ik-\tau)x$

$$\lim_{\tau\to 1/\Delta} = \frac{1}{\pi}\frac{1/\Delta}{1/\Delta^2+k^2}$$

if $k=0 \to \Delta$

$$[\delta a_{out}(t), \delta a^\dagger_{out}(t')] = \delta a_{out}(t)\delta a^\dagger_{out}(t') - \delta a^\dagger_{out}(t')\delta a_{out}(t)$$

$$= (\sqrt{\bar{g}_c}\, a(t) - a_{in}(t))(\sqrt{\bar{g}_c}\, a^\dagger(t') - a^\dagger_{in}(t')) - (\sqrt{\bar{g}_c}\, a^\dagger(t') - a^\dagger_{in}(t))(\sqrt{\bar{g}_c}\, a(t) - a_{in}(t))$$

$$= \bar{g}_c^2\, a(t) a^\dagger(t') + a_{in}(t) a^\dagger_{in}(t') - \bar{g}_c\,(a(t) a^\dagger_{in}(t') +$$

$x = t - t'$

$$\int_{-\Delta}^{0} e^{ikx + \tau x} + \int^{\Delta}$$

$$\frac{1 - e^{-(ik+\tau)\Delta}}{\tau + ik} + \frac{1 - e^{(ik-\tau)\Delta}}{\tau - ik} = \frac{(\tau - ik) + (\tau + ik)}{\tau^2 + k^2}$$

$$- e^{-\tau\Delta} [(\tau - ik) e^{-ik\Delta} + (\tau + ik) e^{ik\Delta}] / (\tau^2 + k^2)$$

$$[\tau 2 \cos k\Delta + ik\, i 2 \sin k\Delta]$$

$$\int_0^t dt' \frac{1}{\pi} \frac{1/\Delta}{1/\Delta^2 + (t-t')^2} = \; ?$$

$$= \frac{1}{\pi} \arctan(\Delta t)$$

- $\int \frac{1}{x^2+1} = \ln(x^2 + x + 1)$

As $x \to \infty$

$$\frac{\pi}{2} - \arctan(x) \simeq \frac{1}{x}$$

$$\int \frac{e^{ax}}{x} dx$$