A class of uniquely (strongly) clean rings

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Abstract: In this paper we call a ring \( R \) \( \delta_r \)-clean if every element is the sum of an idempotent and an element in \( \delta(R_R) \) where \( \delta(R_R) \) is the intersection of all essential maximal right ideals of \( R \). If this representation is unique (and the elements commute) for every element we call the ring uniquely (strongly) \( \delta_r \)-clean. Various basic characterizations and properties of these rings are proved, and many extensions are investigated and many examples are given. In particular, we see that the class of \( \delta_r \)-clean rings lies between the class of uniquely clean rings and the class of exchange rings, and the class of uniquely strongly \( \delta_r \)-clean rings is a subclass of the class of uniquely strongly clean rings. We prove that \( R \) is \( \delta_r \)-clean if and only if \( R/\delta_r(R_R) \) is Boolean and \( R/Soc(R_R) \) is clean where \( Soc(R_R) \) is the right socle of \( R \).

Key words: Clean ring, strongly clean ring, uniquely clean ring, strongly \( J \)-clean ring

1. Introduction

Clean rings have been studied by many ring and module theorists since 1977, and it is still a very popular subject. They were defined by Nicholson as a subclass of exchange rings. An associative ring with unity is called clean if every element is the sum of an idempotent and a unit [14]. If this representation is unique for every element, Nicholson and Zhou [17] call the ring uniquely clean. They proved that a ring \( R \) is uniquely clean if and only if for all \( a \in R \) there exists a unique idempotent \( e \in R \) such that \( a - e \in J(R) \) where \( J(R) \) is the Jacobson radical of \( R \) (we call the ring with this property uniquely \( J \)-clean). Chen et al. [7] call a ring uniquely strongly clean if every element can be written uniquely as the sum of an idempotent and a unit that commute. They proved that \( R \) is uniquely strongly clean if and only if for every \( a \in R \), there exists a unique idempotent \( e \in R \) such that \( a - e \in J(R) \) and \( ae = ea \) (we call the ring with this property uniquely strongly \( J \)-clean). Recently, Chen [6] defined strongly \( J \)-clean rings. A ring \( R \) is called strongly \( J \)-clean if for all \( a \in R \) there exists an idempotent \( e \in R \) such that \( a - e \in J(R) \) and \( ea = ae \) [6]. Note that strongly \( J \)-clean rings are strongly clean but the converse need not be true [6, Proposition 2.1 and Example 2.2].

These results motivate us to define the class of uniquely \( \delta(R_R) \)-clean and uniquely strongly \( \delta(R_R) \)-clean rings where \( \delta(R_R) \) is the ideal defined by Zhou [21]. These classes of rings give some new classes of uniquely clean and uniquely strongly clean rings and also give some ideas on the cleanness of \( R/Soc(R_R) \) where \( Soc(R_R) \) is the right socle of \( R \). Firstly basic properties of \( \delta(R_R) \)-clean rings are given in Section 2. Interestingly we see that the class of \( \delta(R_R) \)-clean rings lies between the class of uniquely clean rings and exchange rings. We also

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prove that if \( R \) is \( \delta(R_R) \)-clean, then \( R/Soc(R_R) \) is clean and partially unit regular, i.e. every regular element is unit regular. In Section 3, uniquely \( \delta(R_R) \)-clean rings are studied. We see that any uniquely \( \delta(R_R) \)-clean ring is uniquely clean. Contrary to the result in [17] saying that \( R \) is uniquely clean if and only if \( R[[x]] \) is uniquely clean, just the necessity is true for uniquely \( \delta(R_R) \)-clean rings. Section 4 is devoted to uniquely strongly \( \delta(R_R) \)-clean rings (USDC for short). Any uniquely \( \delta(R_R) \)-clean ring is USDC, and any USDC ring is uniquely strongly clean. We prove that if \( R \) is a commutative ring, then \( R \) is USDC if and only if the ring of \( 2 \times 2 \) upper triangular matrices, \( T_2(R) \), is USDC. In the last section \( \delta(R_R) \)-cleanness of the formal triangular matrix ring is investigated.

Recall some definitions. Following [21], a submodule \( N \) of a module \( M \) is called \( \delta \)-small in \( M \) (denoted by \( N \ll_\delta M \)) if \( N + K \neq M \) for any submodule \( K \) of \( M \) with \( M/K \) singular. Denote \( \delta(M) \) to be the sum of all \( \delta \)-small submodules of \( M \) (see [21, Lemma 1.5]). We use \( \delta_r \) (or \( \delta_r(R) \)) for \( \delta(R_R) \) for a ring \( R \). Clearly \( J(R) \subseteq \delta_r(R) \ll_\delta R_R \). If \( S \) is simple and \( M \) is essential, then \( S \cap M \) must equal \( S \) (as it cannot be zero). Since every simple right ideal is contained in every essential right ideal, then \( S_r := Soc(R_R) \subseteq \delta_r(R) \) (see also [21, Lemma 1.9]). By view of [21, Corollary 1.7], \( J(R/S_r) = \delta_r/S_r \); in particular, \( R \) is semisimple if and only if \( \delta(R_R) = R \).

A ring \( R \) is an exchange ring if, for every \( a \in R \), there exists an idempotent \( e \in aR \) such that \( 1 - e \in (1 - a)R \) (see [14]). For example, (von Neumann) regular rings and clean rings are exchange. If \( I \) is a left ideal of a ring \( R \), idempotents lift modulo \( I \) if, given \( a \in R \) with \( a^2 - a \in I \), there exists \( e^2 = e \in R \) such that \( a - e \in I \) [14]. Note that \( R \) is an exchange ring if and only if idempotents lift modulo every left ideal of \( R \) [14, Corollary 1.3]. A ring \( R \) is called \( \delta \)-semiregular if \( R/\delta_r \) is a regular ring and idempotents lift modulo \( \delta_r \) [21, Theorem 3.5]. A ring \( R \) is called abelian if every idempotent of \( R \) is central.

Throughout this article, all rings are associative with unity and all modules are unitary. We denote \( S_r = Soc(R_R) \) and \( Z_r = Z(R_R) \) for the right socle and the right singular ideal of a ring \( R \). We write \( J \) (or \( J(R) \)) for the Jacobson radical of \( R \). \( U(R) \) is the set of all units in \( R \). The ring of integers modulo \( n \) is denoted by \( \mathbb{Z}_n \), and we write \( M_n(R) \) (resp. \( T_n(R) \)) for the rings of all (resp., all upper triangular) \( n \times n \) matrices over the ring \( R \).

2. \( \delta_r \)-clean rings

Chen [6] calls a ring \( R \) strongly \( J \)-clean if for every element \( a \in R \) there exists an idempotent \( e \in R \) such that \( a - e \in J \) and \( ce = ae \). Call a ring \( R \) \( J \)-clean if for any element \( a \in R \), there exists an idempotent \( e \in R \) such that \( a - e \in J \).

Any \( J \)-clean ring is clean. Let \( a \in R \) and \( a = e + w \) where \( e^2 = e \in R \), \( w \in J \). Then \( a = (1 - e) + (2e - 1 + w) \). Since \((2e - 1)^2 = 1\) we see that \( a - (1 - e) \in U(R) \) (see [6, Proposition 2.1]). It is easy to give an example of a ring that is clean but not \( J \)-clean (e.g., \( \mathbb{Z}_3 \)). Now we introduce the notion of \( \delta_r \)-clean rings.

**Definition 2.1** A ring \( R \) is called \( \delta_r \)-clean if for every element \( a \in R \) there exists an idempotent \( e \in R \) such that \( a - e \in \delta_r \).

The class of \( \delta_r \)-clean rings contains Boolean rings, semisimple rings, and \( J \)-clean rings. Clearly, \( R \) is \( \delta_r \)-clean if and only if \( R/\delta_r \) is Boolean and idempotents lift modulo \( \delta_r \). Note that there exists a ring \( R \) with \( R/\delta_r \) is Boolean but such that idempotents do not lift modulo \( \delta_r \). There is a ring \( R \) with \( R/J(R) \) Boolean
but such that idempotents do not lift modulo $J(R)$ (see [13, Example 15]). In this ring, idempotents do not lift modulo $\delta_r$, for, if they did, then $R$ would be $\delta_r$-clean and therefore exchange, by Theorem 2.2 below. Then idempotents would lift modulo $J(R)$, a contradiction.

On the other hand, if $R$ is $\delta_r$-clean, then $R/J$ need not be a Boolean ring. For example, $\mathbb{Z}_3$ is semisimple but not Boolean.

**Theorem 2.2** If $R$ is a $\delta_r$-clean ring, then

1) $R/S_r$ is a semiregular ring, i.e. $R$ is $\delta_r$-semiregular;

2) $R$ is an exchange ring;

3) $R/S_r$ is a clean ring;

4) $Z_r \subseteq J$.

**Proof**

1) Since $R/\delta_r$ is a Boolean ring and idempotents lift modulo $\delta_r$, $R$ is $\delta$-semiregular. By [19, Theorem 1.4], $R$ is $\delta_r$-semiregular if and only if $R/S_r$ is semiregular.

2) If $R/S_r$ is semiregular, then $R$ is exchange by [19, Corollary 1.5].

3) If $R$ is $\delta_r$-clean, then $R/S_r$ is $J(R/S_r)$-clean since $J(R/S_r) = \delta_r/S_r$. Any $J$-clean ring is clean. We thus conclude that $R/S_r$ is a clean ring.

4) Since $R$ is $\delta_r$-semiregular, $Z_r \subseteq \delta_r$ by [16, Theorem 1.2]. Then $Z_r$ is $\delta_r$-small in $R$. This gives that $Z_r$ is small in $R$. Hence, $Z_r \subseteq J$. \hfill $\square$

**Example 2.3** If $R$ is a semisimple ring that is not a Boolean ring (e.g., $\mathbb{Z}_3$), then $R$ is $\delta_r$-clean but not $J$-clean since $J = 0$ and $\delta_r = R$.

**Example 2.4** There exist clean rings that are not $\delta_r$-clean.

**Proof**

1) Let $V_D$ be a nonzero vector space over a division ring $D$ and let $R = \text{End}_D(V)$. Then $R$ is regular (see [1, Exercise 15.13]) and clean [15, Lemma 1] (see also [3, Lemma 3.1]) and $S_r = S_l = \{f \in R | \text{rank} f < \infty\}$ (see [1, Exercise 18.4]). Since $J(R/S_r) = \delta_r/S_r$ and $R$ is regular, we have that $\delta_r = S_r$.

Now assume that $V_D$ is a countably infinite dimensional vector space and let $\{v_1, v_2, \ldots\}$ be a basis of $V$. Define the shift operator $f$ on $V$ by $f(v_n) = v_{n+1}$ for $n = 1, 2, 3, \ldots$. Then $f^2 - f \not\in S_r$. This shows that $R/S_r = R/\delta_r$ is not Boolean. Hence, $R$ is not $\delta_r$-clean.

2) Let $p$ be a prime integer and consider the local ring $\mathbb{Z}_{(p)} = \{m/n | m, n \in \mathbb{Z}, (m, n) = 1, p \nmid n\}$. Since $\mathbb{Z}_{(p)}$ is not semisimple, $J = \delta_r = p\mathbb{Z}_{(p)}$. Then $\mathbb{Z}_{(p)}$ is clean but not $\delta_r$-clean, because $\mathbb{Z}_{(p)}/\delta_r$ is not Boolean. \hfill $\square$

Note that any clean ring is exchange [14, Proposition 1.8]. Bergman’s example is an example of an exchange ring that is not clean. We prove below that this ring is not $\delta_r$-clean, and so we pose the following question.

**Question:** Is any $\delta_r$-clean ring clean?

**Example 2.5 (Bergman)** Let $F$ be a field with $\text{char}(F) \neq 2$, and $A = F[[x]]$. Let $Q$ be the field of fractions of $A$. Define

$$R = \{r \in \text{End}_F(A) | \exists q \in Q \text{ and } \exists n > 0 \text{ with } r(a) = qa \text{ for all } a \in x^nA\}.$$
Then $R$ is a regular (so exchange) ring [10], but not clean [4]. There is also an epimorphism $\theta : R \to Q$ given by $r \mapsto q$, where $r$ agrees with $q$ on $x^n A$ for some $n > 0$ with $\ker \theta = S_r = \delta_r$ (see [12, Example 1]). Now assume that $R$ is $\delta_r$-clean. Then, for any $r \in R$, there exists an idempotent $e \in R$ such that $e - e \in \delta_r$. This gives that $\theta(r - e) = \theta(r) - \theta(e) = 0$ and $\theta(r) = \theta(e)$ is an idempotent in $Q$. Since $Q$ is a field, $\theta(r) = 0$ or 1, which contradicts the fact that $\theta$ is an epimorphism. Therefore, $R$ is not $\delta_r$-clean.

Thus we conclude that

\[
\{ \text{ Boolean } \} \subsetneq \{ J\text{-clean}\} \subsetneq \{ \delta_r\text{-clean}\} \subsetneq \{ \text{ exchange } \}.
\]

Now we give a few conditions for a $\delta_r$-clean ring to be clean or $J$-clean. First note that Baccella [2] proved the important fact that idempotents lift modulo $S_r$ for any ring $R$.

**Proposition 2.6** Any $\delta_r$-clean ring $R$ is $J$-clean if

1) $R/J$ is Boolean, or 2) $S_r \subseteq J$.

**Proof** 1) Assume that $R$ is $\delta_r$-clean and $R/J$ is Boolean. Let $a \in R$. Then $a^2 - a \in J$. By Theorem 2.2, idempotents lift modulo $J$. Hence, there exists an idempotent $e \in R$ such that $a - e \in J$.

2) Assume that $R$ is $\delta_r$-clean. If $S_r \subseteq J$, then $J/S_r = J(R/S_r) = \delta_r/S_r$, and we have that $J = \delta_r$. Hence, $R$ is $J$-clean. \qed

**Proposition 2.7** If $R$ is $\delta_r$-clean and $R/J$ is abelian, then $R$ is clean.

**Proof** Assume that $R$ is $\delta_r$-clean. According to Theorem 2.2, $R$ is exchange and so $R/J$ is exchange and idempotents lift modulo $J$ by [14, Corollary 1.3]. Thus, $R/J$ is abelian exchange and it is clean by [14, Proposition 1.8]. By [9, Proposition 6], $R$ is clean. \qed

Recall that a ring $R$ is called right quasi-duo if every maximal right ideal is a 2-sided ideal. If $R$ is an exchange ring, then $R/J$ is right quasi-duo iff $R/J$ is reduced iff $R/J$ is abelian [20, Proposition 4.1]. Hence, the following corollary is immediate.

**Corollary 2.8** If $R$ is $\delta_r$-clean and right (or left) quasi-duo, then $R$ is clean.

**Proposition 2.9** Let $R$ be a ring with only trivial idempotents (e.g., a local ring). Then $R$ is $\delta_r$-clean if and only if $R$ is either a division ring or $R/J(R) \cong \mathbb{Z}_2$.

**Proof** Assume that $R$ is $\delta_r$-clean. Then $R$ is exchange by Theorem 2.2. Since $R$ is exchange and has only trivial idempotents, $R$ is local. Then either $J(R) = 0$ or $J(R) = \delta_r$. If $J(R) = 0$, then $R$ is a division ring. If $J(R) = \delta_r$, then $R$ is $J$-clean and so $R$ is strongly $J$-clean by hypothesis. Hence, $R/J(R) \cong \mathbb{Z}_2$ by [6, Lemma 4.2]. Conversely, if $R$ is a division ring, then $R$ is semisimple and so $R$ is $\delta_r$-clean. If $R/J(R) \cong \mathbb{Z}_2$, then $R$ is $J$-clean by [17, Theorem 15] and so $R$ is $\delta_r$-clean. \qed

A characterization of $\delta_r$-clean rings can be given as follows.

**Theorem 2.10** Let $R$ be a ring. The following statements are equivalent.

1) $R$ is $\delta_r$-clean.

2) $R/S_r$ is $J$-clean.
3) \( R/\delta_r \) is Boolean and \( R/S_r \) is clean.

**Proof** Since \( J(R/S_r) = \delta_r/S_r \), (1) \( \Leftrightarrow \) (2). By Theorem 2.2, (1) \( \Rightarrow \) (3).

(3) \( \Rightarrow \) (1) Let \( a \in R \). Then \( a^2 - a \in \delta_r \). Since \( R = R/S_r \) is clean, idempotents of \( R/J(R) \) lift to idempotents of \( R \). By [19, Lemma 1.3], idempotents of \( R/\delta_r \) lift to idempotents of \( R \). Hence, there exists \( e^2 = e \in R \) such that \( a - e \in \delta_r \). Thus, \( R \) is \( \delta_r \)-clean.

Bergman's example (see Example 2.5) also shows that if \( R/S_r \) is a clean ring, then \( R \) need not be clean [12, Example 1].

Recall that a ring \( R \) is said to have stable range 1, written \( sr(R) = 1 \), if given \( a, b \in R \) for which \( aR + bR = R \), there exists a \( y \in R \) such that \( a + by \in U(R) \). It is obvious that \( sr(R) = 1 \) if and only if \( sr(R/J) = 1 \).

**Lemma 2.11** Let \( R \) be a ring. Then \( sr(R/\delta_r) = 1 \) if and only if \( sr(R/S_r) = 1 \).

**Proof** It can be easily seen by the fact that \( J(R/S_r) = \delta_r/S_r \). \( \square \)

Recall that an element \( a \) of a ring \( R \) is called regular (resp., unit regular) if there exists \( u \in R \) (resp., \( u \in U(R) \)) such that \( a = auu \). A ring \( R \) is called partially unit regular if every regular element of \( R \) is unit regular. These rings are also called \( IC \)-ring in [11].

**Theorem 2.12** If \( R \) is a \( \delta_r \)-clean ring, then \( R/S_r \) is partially unit regular.

**Proof** Since \( R/\delta_r \) is a Boolean ring, \( sr(R/\delta_r) = 1 \). By Theorem 2.2, \( R \) is an exchange ring. Hence, by Lemma 2.11 and [5, Theorem 3], \( R/S_r \) is partially unit regular. \( \square \)

The following example shows that if \( R \) is \( \delta_r \)-clean, then \( R/S_r \) need not be a regular ring in general.

**Example 2.13** Let \( R = \mathbb{Z}_8 \). Then \( Soc(R) = 4R \) and \( J = 2R \). It is clear that \( R \) is \( J \)-clean, but since \( J \nsubseteq Soc(R) \), \( R/Soc(R) \) is not regular.

3. Uniquely \( \delta_r \)-clean rings

**Definition 3.1** A ring \( R \) is called uniquely \( \delta_r \)-clean if for every element \( a \in R \) there exists a unique idempotent \( e \in R \) such that \( a - e \in \delta_r \).

Let \( I \) be an ideal of \( R \). Then idempotents lift uniquely modulo \( I \) if whenever \( a^2 - a \in I \), there exists a unique idempotent \( e \in R \) such that \( e - a \in I \) [17]. This condition implies that if \( e - f \in I \), \( e^2 = e \), \( f^2 = f \), then \( e = f \); in particular, 0 is the only idempotent in \( I \).

Clearly, \( R \) is uniquely \( \delta_r \)-clean if and only if \( R/\delta_r \) is Boolean and idempotents lift uniquely modulo \( \delta_r \).

**Theorem 3.2** If \( R \) is uniquely \( \delta_r \)-clean, then the following hold.

1) \( \delta_r = J \).

2) \( R \) is uniquely clean.
Proof 1) Since idempotents lift uniquely modulo $\delta_r$, by the remark above, the only idempotent in $\delta_r$ is 0. Now let $a \in \delta_r$. Then there exists a semisimple right ideal $Y$ of $R$ such that $R = (1-a)R \oplus Y$ by [21, Theorem 1.6]. Since $Y \subseteq S_r \subseteq \delta_r$, we have that $Y = 0$. Hence $1-a$ is right invertible in $R$, and so $a \in J$.

2) It is clear by (1) and [17, Theorem 20].

Examples 3.3 1) No semisimple ring is uniquely $\delta_r$-clean, for, if $R$ is a semisimple ring, then $\delta_r = R$ and for any $a \in R$, $a - 0 \in R$ and $a - 1 \in R$.

2) If $R \ncong \mathbb{Z}_2$, then $R/J \cong \mathbb{Z}_2$ if and only if $R$ is local uniquely $\delta_r$-clean, for, if $R/J \cong \mathbb{Z}_2$, then $J = \delta_r$ and $R$ is uniquely clean by [17, Theorem 15] and so $R$ is uniquely $\delta_r$-clean. The converse is also true by Proposition 2.9.

Therefore, for example, the rings $R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$, $R = \left\{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \mid x \in \mathbb{Z}_4, y \in \mathbb{Z}_4 \oplus \mathbb{Z}_4 \right\}$, or $R = \mathbb{Z}_{2^n}$ where $1 \neq n \in \mathbb{N}$ are uniquely $\delta_r$-clean.

Uniquely clean rings need not be uniquely $\delta_r$-clean.

Example 3.4 1) $\mathbb{Z}_2$ is uniquely clean but not uniquely $\delta_r$-clean.

2) Let $R = \prod_{i=1}^{\infty} R_i$ where $R_i \cong \mathbb{Z}_2$ for all $i = 1, 2, \ldots$. Then $R$ is a Boolean ring with $S_r = \bigoplus_{i=1}^{\infty} R_i$. Since $R/S_r$ is Boolean, $J(R/S_r) = 0$ and so $S_r = \delta_r$. Clearly $R$ is uniquely $J$-clean, that is, uniquely clean but not uniquely $\delta_r$-clean.

It is easy to see that every uniquely clean ring is $\delta_r$-clean by the fact that $R$ is uniquely clean if and only if $R/J$ is uniquely $J$-clean [17, Theorem 20]. But if $R$ is a semisimple ring that is not Boolean, then $R$ is $\delta_r$-clean but not uniquely clean (see Example 2.3).

Thus, we conclude that

$\{ \text{uniquely $\delta_r$-clean} \} \subsetneq \{ \text{uniquely clean} \} \subsetneq \{ \delta_r\text{-clean} \} \subsetneq \{ \text{exchange} \}$.

If $S_r \subseteq J$ for a ring $R$, then $J/S_r = J(R/S_r) = \delta_r/S_r$ and so $J = \delta_r$. Hence, Proposition 3.5 below is obvious by Proposition 2.6.

Proposition 3.5 If $R$ is a uniquely clean ring with $S_r \subsetneq J$, then $R$ is uniquely $\delta_r$-clean.

By [17, Theorem 20] we know that $R$ is uniquely clean if and only if $R/J$ is Boolean, $R$ is abelian, and idempotents lift modulo $J$. However, this result cannot be restated for $\delta_r$ in general. The following theorem and examples prove our claim.

Theorem 3.6 Let $R$ be a ring and consider the following conditions.

1) $R$ is uniquely $\delta_r$-clean.

2) $R/\delta_r$ is Boolean, $R$ is abelian, and idempotents lift modulo $\delta_r$.

3) $R/\delta_r$ is Boolean, $R/S_r$ is abelian, and idempotents lift modulo $\delta_r$.
4) \(R/S_r\) is uniquely clean.

Then (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) \(\iff\) (4).

**Proof** (1) \(\Rightarrow\) (2) Since \(R\) is uniquely clean, it is abelian by [17, Lemma 4].

(2) \(\Rightarrow\) (3) Since idempotents always lift modulo \(S_r\), it is clear.

(3) \(\iff\) (4) It is by [17, Theorem 20]. Note that idempotents lift modulo \(J\) [19, Lemma 1.3].

In Theorem 3.6, (2) \(\not\iff\) (1) in general.

**Example 3.7** We consider again the ring \(R = \prod_{i=1}^{\infty} R_i\) where \(R_i \cong \mathbb{Z}_2\), \(i = 1, 2, \ldots\) (see Example 3.4). Since \(R\) is uniquely clean, \(R\) is abelian and \(\delta_r\)-clean. But \(R\) is not uniquely \(\delta_r\)-clean.

In Theorem 3.6, (4) \(\not\iff\) (2) in general.

**Example 3.8** Let \(R = \left[ \begin{array}{ll} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{array} \right]\). Then \(S_r = \delta_r = \left[ \begin{array}{ll} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{array} \right]\) and \(R/S_r \cong \mathbb{Z}_2\) is Boolean. Obviously \(R\) is \(\delta_r\)-clean but not abelian.

**Theorem 3.9** If \(R\) is uniquely \(\delta_r\)-clean and \(e^2 = e \in R\), then \(eRe\) is uniquely \(\delta_r\)-clean.

**Proof** Since \(R\) is abelian, \(\delta_r(eRe) = e\delta_r e\) by [18, Theorem 3.11]. By Theorem 3.2, \(\delta_r = J\), so we have that \(J(eRe) = eJe = \delta_r(eRe)\). If \(R\) is uniquely \(\delta_r\)-clean, then \(R\) is uniquely clean by Theorem 3.2. By [17, Corollary 6], \(eRe\) is uniquely clean. By [17, Theorem 20], \(eRe\) is uniquely \(\delta_r\)-clean.

Although every factor ring of a uniquely clean ring is uniquely clean [17, Theorem 22], the same property does not hold for uniquely \(\delta_r\)-clean.

**Remark 3.10** 1) If \(R\) is a uniquely \(\delta_r\)-clean ring, then factor rings of \(R\) need not be uniquely \(\delta_r\)-clean in general. For example, if \(R \not\cong \mathbb{Z}_2\) and \(R/J \cong \mathbb{Z}_2\), then \(R\) is uniquely \(\delta_r\)-clean by Example 3.3, but \(R/J\) is not uniquely \(\delta_r\)-clean.

(2) Since matrix ring \(M_n(R)\) and upper triangular matrix ring \(T_n(R)\) are not abelian for \(n \geq 2\), they are not uniquely \(\delta_r\)-clean by Theorem 3.2.

Let \(R\) be a ring and \(V\) an \((R, R)\)-bimodule that is a general ring (possibly with no unity) in which \((vw)r = v(wr), (vr)w = v(rw), and (rv)w = r(vw)\) hold for all \(v, w \in V\) and \(r \in R\). Then the ideal-extension (also called the Dorroh extension) \(I(R; V)\) of \(R\) by \(V\) is defined to be the additive abelian group \(I(R; V) = R \oplus V\) with multiplication \((r, v)(s, w) = (rs, rw + vs + vw)\).

Uniquely clean ideal-extensions are considered in [17, Proposition 7]. Now we deal with uniquely \(\delta_r\)-clean ideal-extensions.

**Proposition 3.11** An ideal-extension \(S = I(R; V)\) is uniquely \(\delta_r\)-clean if the following conditions are satisfied:

1) \(R\) is uniquely \(\delta_r\)-clean;

2) if \(e^2 = e \in R\) then \(ev = ve\) for all \(v \in V\);

3) if \(v \in V\) then \(v + w + vw = 0\) for some \(w \in V\).
Proof Assume that (1), (2), and (3) are satisfied. Since $R$ is uniquely $\delta_r$-clean, $R$ is uniquely clean by Theorem 3.2 and so $S$ is uniquely clean by [17, Proposition 7]. Then $S$ is $\delta_r$-clean. Note by the proof of [17, Proposition 7] that any idempotent in $S$ is of the form $(e, 0)$ where $e^2 = e \in R$. Now suppose that $(e, 0) + (u, v) = (e_1, 0) + (u_1, v_1)$ in $S$ where $(e, 0)$ and $(e_1, 0)$ are idempotents and $(u, v), (u_1, v_1) \in \delta_r(S)$. Then $e + u = e_1 + u_1$ in $R$ where $e$ and $e_1$ are idempotents in $R$ and $u, u_1 \in \delta_r(R)$ by the following result, and so $(e, 0) = (e_1, 0)$ by (1).

Claim. If $(u, v) \in \delta_r(S)$ then $u \in \delta_r(R)$.

Proof. Let $(u, v) \in \delta_r(S)$. Then $(u, 0) \in \delta_r(S)$ because $(0, V) \subseteq J(S) \subseteq \delta_r(S)$ by (3). Let $L$ be a right ideal of $R$ such that $uR + L = R$. It is enough to show that $L$ is a direct summand of $R$ by [21, Theorem 1.6]. Since $(u, 0)S + (L + V) = S$ and $(u, 0) \in \delta_r(S)$, we have that $L + V$ is a direct summand of $S$ and so is generated by an idempotent $(e, 0) \in S$ where $e^2 = e \in R$. Then we see that $L = eR$, and hence $L$ is a direct summand of $R$, as desired.

Example 3.12 Let $R$ be a uniquely $\delta_r$-clean ring and let $S = \{a_{ij} \in T_n(R) \mid a_{11} = \ldots = a_{nn} \}$. Then $S$ is uniquely $\delta_r$-clean and is noncommutative if $n \geq 3$.

Proof. If $V = \{a_{ij} \in T_n(R) \mid a_{11} = \ldots = a_{nn} = 0 \}$, then $S \cong I(R; V)$. The conditions in Proposition 3.11 hold as in [17, Example 8].

If $R$ is a ring and $\sigma : R \to R$ is a ring endomorphism, let $R[[x, \sigma]]$ denote the ring of skew formal power series over $R$, that is, all formal power series in $x$ with coefficients from $R$ with multiplication defined by $xr = \sigma(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, 1_R]]$ is the ring of formal power series over $R$. Since $R[[x, \sigma]] \cong I(R; < x >)$ where $< x >$ is the ideal generated by $x$, the proof of [17, Example 9] and Proposition 3.11 give the next results.

Corollary 3.13 Let $R$ be a ring and $\sigma : R \to R$ a ring endomorphism and $e = \sigma(e)$ for all $e^2 = e \in R$. If $R$ is uniquely $\delta_r$-clean, then $R[[x, \sigma]]$ is uniquely $\delta_r$-clean.

Corollary 3.14 If $R$ is a uniquely $\delta_r$-clean ring, then $R[[x]]$ is uniquely $\delta_r$-clean.

Corollary 3.14 can be proven by using Proposition 3.15 below, for, if $R$ is uniquely $\delta_r$-clean, then $R[[x]]$ is a uniquely clean ring by Theorem 3.2 and [17, Corollary 10]. By Proposition 3.15, $J(R[[x]]) = J(R) + < x > \subseteq \delta_r(R[[x]]) \subseteq \delta_r(R) + < x >$. Then since $J(R) = \delta_r(R)$ by Theorem 3.2(1), $J(R[[x]]) = \delta_r(R[[x]])$. Hence, $R[[x]]$ is a uniquely $\delta_r$-clean ring.

Proposition 3.15 Let $R$ be a ring. Then $\delta_r(R[[x]]) \subseteq \delta_r(R) + < x >$.

Proof. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots \in \delta_r(R[[x]])$. Since $< x > \subseteq J(R[[x]])$, $a_0 \in \delta_r(R[[x]])$. Let $L$ be a right ideal of $R$ such that $a_0 R + L = R$. It is enough to show that $L$ is a direct summand of $R$ by [21, Theorem 1.6]. Since $a_0 R[[x]] + L[[x]] = R[[x]]$ and $a_0 \in \delta_r(R[[x]])$, we have that $L[[x]]$ is a direct summand of $R[[x]]$ and so is generated by an idempotent $e(x) = e_0 + e_1 x + e_2 x^2 + \ldots \in R[[x]]$. Then $e_0$ is an idempotent in $R$ and it can be seen that $L = e_0 R$. Thus, $a_0 \in \delta_r(R)$, as desired.

Note that $J(\mathbb{Z}_2[[x]]) = \delta_r(\mathbb{Z}_2[[x]]) \subseteq \delta_r(\mathbb{Z}_2) + < x > = \mathbb{Z}_2[[x]]$. 47
Corollary 3.16  If $R[[x]]$ is $\delta_r$-clean, then $R$ is $\delta_r$-clean.

Proof  Let $a \in R$. Then there exist $e(x)^2 = e(x) \in R[[x]]$ and $w(x) \in \delta_r(R[[x]])$ such that $a = e(x) + w(x)$ and so $w(0) \in \delta_r(R)$ by Proposition 3.15. Thus, $a = e(0) + w(0)$ where $e(0)^2 = e(0) \in R$, as asserted. \hfill $\blacksquare$

If $R[[x]]$ is uniquely $\delta_r$-clean, then $R$ need not be uniquely $\delta_r$-clean. For example, $\mathbb{Z}_2$ is not uniquely $\delta_r$-clean but since $\mathbb{Z}_2[[x]]/J(\mathbb{Z}_2[[x]]) \cong \mathbb{Z}_2$, $\mathbb{Z}_2[[x]]$ is uniquely $\delta_r$-clean by Example 3.3(2).

4. Uniquely strongly $\delta_r$-clean rings

Uniquely strongly clean rings were studied in [7]. A ring $R$ is called uniquely strongly clean if for every element $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in U(R)$ and $ea = ae$. In Theorem 17 of [7] it is proven that a uniquely strongly clean ring is exactly the same as a uniquely strongly $J$-clean, i.e. for any $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in J$ and $ea = ae$.

Definition 4.1  A ring $R$ is called uniquely strongly $\delta_r$-clean if for every element $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in \delta_r$ and $ea = ae$.

Proposition 4.2  A ring $R$ is uniquely strongly $\delta_r$-clean if and only if $R$ is an abelian USDC ring.

Proof  Since uniquely $\delta_r$-clean rings are abelian by Theorem 3.6, the proof is obvious. \hfill $\blacksquare$

Proposition 4.3  Let $R$ be a USDC ring. Then the following hold:

1) If $e^2 = e \in \delta_r$ then $e = 0$.

2) $R/J$ is Boolean.

3) $\delta_r = J$.

4) $R$ is uniquely strongly clean.

Proof  1) Let $e^2 = e \in \delta_r$. Then $e + 0 = 0 + e$ and $0.e = e.0$ yield $e = 0$.

2) $R$ is exchange by Theorem 2.2. If we show that every nonzero idempotent of $R$ is not the sum of 2 units, then by [13, Theorem 13], $R/J$ will be Boolean. Let $e$ be a nonzero idempotent in $R$. Write $e = u + v$, where $u, v \in U(R)$. Since $R$ is USDC, $R/\delta_r$ is Boolean and so $2 \in \delta_r$. Therefore, $u$ and $v$ are congruent to 1, modulo $\delta_r$, which means that their sum is in $\delta_r$. This contradicts with (1).

3) Let $a \in \delta_r$. Since $R/J$ is Boolean, $a^2 - a \in J$. By Theorem 2.2, $R$ is exchange and so idempotents lift modulo $J$. Thus, there exist $e^2 = e \in R$ such that $a - e \in J$. Since $J \subseteq \delta_r$, $e = 0$ by (1). Hence, $a \in J$, as asserted.

4) It is clear by (3) and [7, Theorem 17].

However, a uniquely strongly clean ring need not be USDC. The ring $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ is uniquely strongly clean by [7, Theorem 10] but not USDC by Example 3.8.

Thus, we conclude that

\{ uniquely $\delta_r$-clean \} $\subsetneq$ \{ USDC \} $\subsetneq$ \{ uniquely strongly clean \} $\subsetneq$ \{ $\delta_r$-clean \}. 

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The first and the last containments above are proper because, for example, the ring \(\mathbb{Z}_p\), where \(2 \neq p\) is a prime is \(\delta_-\)-clean but not uniquely strongly clean because \(J(\mathbb{Z}_p) = 0\) and \(\mathbb{Z}_p\) is not Boolean. If \(R\) is a commutative uniquely \(\delta_-\)-clean ring, then \(T_n(R)\) is USDC by Theorem 4.5 for any \(n \in \mathbb{N}\), but \(T_n(R)\) is never uniquely \(\delta_-\)-clean by Remark 3.10(2).

Any factor ring of any USDC ring need not be USDC. For example, since \(\mathbb{Z}_4\) is uniquely \(\delta_-\)-clean by Example 3.3, it is USDC by Proposition 4.2. However, \(\mathbb{Z}_4/J(\mathbb{Z}_4) \cong \mathbb{Z}_2\) is not USDC by Proposition 4.2 and Example 3.3.

**Proposition 4.4** Let \(e\) be an idempotent of a ring \(R\) such that \(eR = eRe\) (i.e. right semicentral) or \(ReR = R\) (i.e. full idempotent). If \(R\) is USDC, then \(eRe\) is USDC.

**Proof** Assume that \(R\) is USDC. For any idempotent \(e\) of \(R\), \(eRe\) is uniquely strongly clean by Proposition 4.3(4) and [7, Example 5]. Since uniquely strongly clean rings are uniquely strongly \(J\)-clean, for any \(a \in eRe\), there exists an idempotent \(f \in eRe\) and \(v \in \delta_e(eRe)\) such that \(a = f + v\) and \(fv = vf\). It remains to show the uniqueness. Let \(a = f + v = g + w\) where \(f\) and \(g\) are idempotents in \(eRe\) and \(v, w \in \delta_e(eRe)\) such that \(fv = vf\) and \(gw = wg\). If \(e\) is an idempotent as in the hypothesis, then \(\delta_e(eRe) \subseteq e\delta_e \subseteq \delta_e(R)\) by [18, Theorems 3.9 and 3.11]. Hence, by assumption, \(f = g\).

Since \(M_n(R)\) is never uniquely strongly clean by [7, Lemma 6], \(M_n(R)\) is never USDC.

**Theorem 4.5** Let \(R\) be a commutative ring. Then the following are equivalent.

1. \(R\) is USDC.
2. \(R\) is uniquely \(\delta_-\)-clean.
3. \(T_n(R)\) is USDC for all \(n \geq 1\).
4. \(T_2(R)\) is USDC.

**Proof** (1) \(\Leftrightarrow\) (2) This follows by Proposition 4.2.

(3) \(\Rightarrow\) (4) It is clear.

(4) \(\Rightarrow\) (1) Suppose that \(T_2(R)\) is USDC and let \(e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in T_2(R)\). Since \(e\) is right semicentral and \(eT_2(R)e \cong R\), \(R\) is USDC by Proposition 4.4.

(1) \(\Rightarrow\) (3) If \(R\) is USDC, then \(T_n(R)\) is uniquely strongly clean by Proposition 4.3(4) and [7, Theorem 10]. According to Proposition 4.3(3) and Lemma 5.1, \(\delta_e(T_n(R)) = J(T_n(R))\) and so \(T_n(R)\) is USDC by [7, Theorem 17]. Therefore, the proof is completed.

5. On the formal triangular matrix rings

Let \(S\) and \(T\) be any ring, \(M\) an \((S,T)\)-bimodule, and \(R\) the formal triangular matrix ring \(\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}\). It is well known that \(J(R) = \begin{bmatrix} J(S) & M \\ 0 & J(T) \end{bmatrix}\) (e.g., [8, Corollary 2.2]), but for \(\delta_e(R)\) the similar property does
not hold in general. For example, if $S = M = T = F$ is a field, then $\delta_r(R) = \text{Soc}_r(R) = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ since $R/\text{Soc}_r(R)$ has zero Jacobson radical, but $\begin{bmatrix} \delta_r(S) & M \\ 0 & \delta_r(T) \end{bmatrix} = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix} = R$. Now we prove the following.

**Lemma 5.1** Let $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ where $S$, $T$ are any ring and $M$ is an $(S,T)$-bimodule. Then $\delta_r(R) \subseteq \begin{bmatrix} \delta_r(S) & M \\ 0 & \delta_r(T) \end{bmatrix}$.

**Proof** Let $r = \begin{bmatrix} s & m \\ 0 & t \end{bmatrix} \in \delta_r(R)$ where $s \in S$, $t \in T$ and $m \in M$. We claim that $s \in \delta_r(S)$. Let $I$ be a right ideal of $S$ such that $sS + I = S$. It is enough to show that $I$ is a direct summand of $S$ by [21, Theorem 1.6]. Since $rR + \begin{bmatrix} I & M \\ 0 & T \end{bmatrix} = R$ and $r \in \delta_r(R)$, we have that $\begin{bmatrix} I & M \\ 0 & T \end{bmatrix}$ is a direct summand of $R$ and so is generated by an idempotent $e \in R$. Let $e = \begin{bmatrix} g & n \\ 0 & f \end{bmatrix}$ where $g \in S$, $f \in T$ and $n \in M$. Then $g$ is an idempotent in $S$ and we see that $I = gS$, and hence $I$ is a direct summand of $S$, as desired. By a similar argument we see that $t \in \delta_r(T)$. Hence, the proof is completed.

According to [8, Proposition 6.3], $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ is clean if and only if $S$ and $T$ are clean. This result also holds for $J$-clean ring.

**Proposition 5.2** Let $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. Then $R$ is $J$-clean if and only if $S$ and $T$ are $J$-clean.

**Proof** Since $S$ and $T$ are factor rings of $R$, the necessity is obvious. Now assume that $S$ and $T$ are $J$-clean.

Let $r = \begin{bmatrix} s & m \\ 0 & t \end{bmatrix} \in R$ where $s \in S$, $t \in T$ and $m \in M$. Then $s = e + w$ where $e^2 = e \in S$ and $w \in J(S)$, and $t = f + v$ where $f^2 = f \in T$ and $v \in J(T)$. This gives that $\begin{bmatrix} s & m \\ 0 & t \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} + \begin{bmatrix} w & m \\ 0 & v \end{bmatrix}$ where $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$ is an idempotent in $R$ and $\begin{bmatrix} w & m \\ 0 & v \end{bmatrix} \in J(R)$. Hence, $R$ is $J$-clean.

If $S$ and $T$ are local rings with nonzero maximal left ideal, then $J(S) = \delta_r(S)$ and $J(T) = \delta_r(T)$. By Lemma 5.1, one can thus deduce that $J(R) = \delta_r(R)$. Hence, the following corollary is immediate from Proposition 5.2.

**Corollary 5.3** Let $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ where $S$ and $T$ are local rings with nonzero maximal left ideals. Then $R$ is $\delta_r$-clean if and only if $S$ and $T$ are $\delta_r$-clean.

If $R = \begin{bmatrix} Z_3 & Z_3 \\ 0 & Z_3 \end{bmatrix}$, then $Z_3$ is a $\delta_r$-clean ring, but $R$ is not $\delta_r$-clean since no quotient of it is Boolean.
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