A Generalization of Semiregular and Almost Principally Injective Rings

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Dedicated to Professor Abdullah Harmancı, on his 65th birthday.

Abstract. In this article, we call a ring $R$ right almost $I$–semiregular if, for any $a \in R$, there exists a left $R$–module decomposition $l_{R}R(a) = P \oplus Q$ such that $P \subseteq Ra$ and $Q \cap Ra \subseteq I$, where $I$ is an ideal of $R$, $l$ and $r$ are the left and right annihilators, respectively. This definition generalizes the right almost principally injective rings defined by Page and Zhou [10], $I$–semiregular rings defined by Nicholson and Yousif [7], and right generalized semiregular rings defined by Xiao and Tong [11]. We prove that $R$ is $I$–semiregular if and only if, for any $a \in R$, there exists a decomposition $l_{R}R(a) = P \oplus Q$, where $P = Re \subseteq Ra$ for some $e^2 = e \in R$ and $Q \cap Ra \subseteq I$. Among the results for right almost $I$–semiregular rings, we are able to show that if $I$ is the left socle $\text{Soc}(R_{R})$ or the right singular ideal $Z(R_{R})$, then $R$ being right almost $I$–semiregular implies that $R$ is right almost $J$–semiregular, where $J$ is the Jacobson radical of $R$. We show that $\delta(eRe) = e\delta(R_{R})e$ for any idempotent $e$ of $R$ satisfying $ReR = R$ and, for such an idempotent, $R$ being right almost $\delta(R_{R})$–semiregular implies that $eRe$ is right almost $\delta(eRe)$–semiregular.

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1 Introduction

Throughout this paper, $R$ denotes an associative ring with identity and all modules are unitary right $R$–modules.

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Let $M$ be an $R$–module and $F$ a submodule of $M_R$. Following Alkan and Özcan [1], $M$ is called $F$–semiregular if, for any $m \in M$, there exists a decomposition $M = P \oplus Q$ such that $P$ is projective, $P \subseteq mR$ and $Q \cap mR \subseteq F$. If $F$ is a fully–invariant submodule of $M_R$, then $M$ is $F$–semiregular if and only if, for any $m \in M$, there exists a decomposition $mR = P \oplus S$ such that $P$ is a projective (direct) summand of $M$ and $S \subseteq F$. A ring $R$ is called $I$–semiregular for an ideal $I$ of $R$ if $R_R$ is an $I$–semiregular module. Such rings are studied in [7] and [9]. Note that being $I$–semiregular for an ideal $I$ of a ring $R$ is left-right symmetric by [9, Lemma 27 and Theorem 28].

A module $M$ is said to be principally injective (or $P$–injective for short) if $l_{MR}(a) = Ma$ for all $a \in R$, where $l$ and $r$ are the left and right annihilators, respectively. As a generalization of $P$–injective modules, Page and Zhou [10] call a module $M$ almost principally injective (or $AP$–injective for short) if, for any $a \in R$, there exists an $S$–submodule $X_a$ of $M$ such that $l_{MR}(a) = Ma \oplus X_a$ as $S$–modules, where $S = \text{End}_R(M)$. A ring $R$ is called right $AP$–injective if $R_R$ is $AP$–injective.

In [13], $M$ is called almost principally quasi–injective (or $APQ$–injective for short) if, for any $m \in M$, there exists an $S$–submodule $X_m$ of $M$ such that $l_{MR}(m) = Sm \oplus X_m$, where $S = \text{End}_R(M)$. Then $R_R$ is $APQ$–injective if and only if $R_R$ is $AP$–injective.

In this article, we call a right $R$–module $M$ almost $F$–semiregular if, for any $m \in M$, there exists an $S$–module decomposition $l_{MR}(m) = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \subseteq F$, where $S = \text{End}_R(M)$ and $F$ is a submodule of $sM$. A ring $R$ is called right almost $I$–semiregular for an ideal $I$ of $R$ if $R_R$ is almost $I$–semiregular. If $sM$ is $F$–semiregular, then $M_R$ is almost $F$–semiregular. An $APQ$–injective module $M_R$ is almost $F$–semiregular for any $S$–submodule $F$ of $M$. Moreover,

$$M_R \text{ is } APQ\text{–injective} \Leftrightarrow M_R \text{ is almost } 0\text{–semiregular.}$$

Right almost $J$–semiregular rings are examined in [11] and named as right generalized semiregular rings.

In Section 2, firstly we give a new characterization of $F$–semiregular modules by modifying the definition of almost $F$–semiregular modules. Next, we give conditions under which a right almost $I$–semiregular ring is $I$–semiregular. Some of the results in [11] are extended. We also prove that if $R$ is a right almost $I$–semiregular ring, then $eRe$ is a right almost $eIe$–semiregular ring for a right semicentral idempotent $e$ of $R$ (i.e., $eR = eRe$) or an idempotent $e$ of $R$ satisfying $ReR = R$. If the matrix ring $M_n(R)$ is right almost $M_n(I)$–semiregular for an ideal $I$ of $R$, then $R$ is right almost $I$–semiregular.

In [1, Corollary 4.6], it is shown that if $M_R$ is projective and $\text{Soc}(M)$–semiregular, then $M$ is semiregular (i.e., for any $m \in M$, there exists a decomposition $M = A \oplus B$ such that $A$ is projective, $A \subseteq mR$ and $B \cap mR \ll M$).
In the last section, we prove that if $M_R$ is almost $Soc(SM)$–semiregular, then $M_R$ is almost semiregular, i.e., for any $m \in M$, there exists an $S$–module decomposition $l_M r_R(m) = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \ll SM$. We also consider right almost $I$–semiregular rings for some ideals such as the socle, the singular ideal and the ideal $\delta$. If $R$ is right almost $Z_r$–semiregular, then $R_R$ satisfies $(C2)$ and is almost semiregular.

The following implications hold for a ring $R$.

$S_l$–semiregular $\Rightarrow$ right almost $S_l$–semiregular $\Rightarrow$ right almost $\delta_r$–semiregular and right almost $\delta_l$–semiregular.

$Z_r$–semiregular $\Rightarrow$ right almost $Z_r$–semiregular $\Rightarrow$ right almost $\delta_r$–semiregular and right almost $\delta_l$–semiregular.

Counterexamples to each of the inverse implications are given.

It is well known that $J(eRe) = eJe$ for any idempotent $e \in R$. But $\delta_r(eRe) \neq e\delta_r(R)e$ even for a right semicentral idempotent $e$ (see Example 3.13). However if $e \in R$ is an idempotent with $ReR = R$, then $\delta_r(eRe) = e\delta_r(R)e$. Consequently, if $R$ is right almost $\delta(rR)$–semiregular and $ReR = R$, then $eRe$ is right almost $\delta_l(eRe)$–semiregular.

The symbols $Rad(M)$, $Soc(M)$ and $Z(M)$ will stand for the Jacobson radical, the socle and the singular submodule of a module $M$, respectively. In the ring case we use the abbreviations: $S_r = Soc(R_R)$, $S_l = Soc(R_R)$, $Z_r = Z(R_R)$ and $Z_l = Z(R_R)$. We write $J = J(R)$ for the Jacobson radical of $R$. For a small (resp. an essential) submodule $K$ of $M$, we write $K \ll M$ (resp. $K \leq_r M$). For any non-empty subset $X$ of $R$, $l_M(X)$ (resp. $r_M(X)$) is used for the left (resp. right) annihilator of $X$ in $M$. For any subset $N$ of $M$, $l_R(N)$ (resp. $r_R(N)$) will denote the left (resp. right) annihilator of $N$ in $R$.

Following [12], a submodule $N$ of a module $M$ is called $\delta$–small in $M$, denoted by $N \ll_\delta M$, if $N + K \neq M$ for any submodule $K$ of $M$ with $M/K$ singular. Let

$$\delta(M) = \cap \{N \subseteq M : M/N \text{ is singular simple}\}.$$ 

Then $\delta(M)$ is the sum of all $\delta$–small submodules of $M$ and is a fully invariant submodule of $M$ [12, Lemma 1.5]. Clearly $Rad(M) \leq \delta(M)$. If $M$ is a projective module, then $Soc(M) \subseteq \delta(M)$ [12, Lemma 1.9]. We use $\delta_r$ for $\delta(rR)$ and $\delta_l$ for $\delta(rR)$. Note that $\delta_r$ need not be equal to $\delta_l$. For example, if $R$ is the ring of $2 \times 2$ upper triangular matrices over a field $F$, then $\delta_r = S_r$ and $\delta_l = S_l$.

2 Almost $F$–semiregular Modules

**Definition 2.1.** Let $M$ be a right $R$–module, $S = End_R(M)$ and $F$ a submodule of $SM$. The module $M_R$ is called almost $F$–semiregular if, for any $m \in M$,
there exists an $S$–module decomposition $l_{M}r_{R}(m) = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \subseteq F$. A ring $R$ is called right almost $I$–semiregular for an ideal $I$ of $R$ if $R_{I}$ is almost $I$–semiregular.

If $M_{R}$ is $APQ$–injective, then $M_{R}$ is almost $F$–semiregular for any submodule $F$ of $S M$. Moreover, $M_{R}$ is almost $0$–semiregular if and only if $M_{R}$ is $APQ$–injective.

**Proposition 2.2.** Let $M$ be a right $R$–module, $S = \text{End}_{R}(M)$ and $F$ any submodule of $S M$. If $S M$ is $F$–semiregular, then $M_{R}$ is almost $F$–semiregular.

**Proof.** Let $m \in M$. Then there exists a decomposition $S M = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \subseteq F$. Since $l_{M}r_{R}(m) = l_{M}r_{R}(m) \cap M$, by the modular law, we have $l_{M}r_{R}(m) = P \oplus (l_{M}r_{R}(m) \cap Q)$ and $(l_{M}r_{R}(m) \cap Q) \cap Sm = Q \cap Sm \subseteq F$. Hence, $M_{R}$ is almost $F$–semiregular. \hfill $\square$

In particular, if $S M$ is semiregular, then $M_{R}$ is almost $\text{Rad}(S M)$–semiregular. If $R$ is an $I$–semiregular ring for an ideal $I$, then it is right and left almost $I$–semiregular, because the notion of $I$–semiregular rings is left–right symmetric.

When we take the summand $P$ of $l_{M}r_{R}(m)$ as a summand of $M$ in Definition 2.1, we have the following result.

**Theorem 2.3** Let $M$ be a right $R$–module and $S = \text{End}_{R}(M)$. If $S M$ is projective and $S F$ is a fully–invariant submodule of $S M$, then the following are equivalent:

1. $S M$ is $F$–semiregular.
2. For any $m \in M$, there exists an $S$–module decomposition $l_{M}r_{R}(m) = P \oplus Q$, where $P \subseteq Sm$, $P$ is a summand of $M$ and $Q \cap Sm \subseteq F$.

**Proof.** (1) $\Rightarrow$ (2) Follows from the proof of Proposition 2.2.

(2) $\Rightarrow$ (1) Let $m \in M$ and $l_{M}r_{R}(m) = P \oplus Q$, where $P \subseteq Sm$, $P$ is a summand of $M$ and $Q \cap Sm \subseteq F$. Then $Sm = P \oplus (Q \cap Sm)$, where $P$ is a projective summand of $M$ and $Q \cap Sm \subseteq F$. Hence, $S M$ is $F$–semiregular. \hfill $\square$

By Theorem 2.3, we obtain the following characterization of $I$–semiregular rings for an ideal $I$.

**Corollary 2.4** Let $I$ be an ideal of a ring $R$. The following are equivalent:

1. $R$ is $I$–semiregular.
2. For any $a \in R$, there exists a decomposition $l_{R}r_{R}(a) = P \oplus Q$, where $P = Re \subseteq Ra$ for some $e^{2} = e \in R$ and $Q \cap Ra \subseteq I$.
3. For any $a \in R$, there exists a decomposition $r_{R}l_{R}(a) = P \oplus Q$, where $P = eR \subseteq aR$ for some $e^{2} = e \in R$ and $Q \cap aR \subseteq I$. 
Now we consider the module-theoretic version of right generalized semiregular rings defined by Xiao and Tong [11].

**Definition 2.5** Let $M$ be a right $R$–module and $S = \text{End}_R(M)$. $M$ is called *almost semiregular* if, for any $m \in M$, there exists an $S$–module decomposition $l_Mr_R(m) = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \ll M$. A ring $R$ is called a right almost semiregular if $R_R$ is almost semiregular. Clearly, $R$ is right almost $J$–semiregular if and only if $R$ is right almost semiregular. Semiregular or right $AP$–injective rings are right almost semiregular by [11, Proposition 1.2]. Example 1.3 in [11] shows that right almost semiregular rings need not be right $AP$–injective or semiregular.

Let $M$ be a right $R$–module and $S = \text{End}_R(M)$. If $SM$ is semiregular, then $MR$ is almost semiregular by a proof similar to that of Proposition 2.2. Moreover, if $MR$ is almost semiregular, then it is almost $\text{Rad}(SM)$–semiregular. The converse is true if $\text{Rad}(SM) \ll SM$.

The following result generalizes [11, Lemma 1.4].

**Proposition 2.6** Let $I$ be an ideal of a ring $R$. If $R$ is right almost $I$–semiregular and there exists $e^2 = e \in R$ such that $r_R(a) = r_R(e)$ for any $a \in R$, then $R$ is $I$–semiregular.

**Proof.** Let $a \in R$. Then there exists a decomposition $l_Rr_R(a) = P \oplus Q$ such that $P \subseteq Ra$ and $Q \cap Ra \subseteq I$ as left ideals. Since $r_R(a) = r_R(e)$ for some $e^2 = e \in R$, $Re = P \oplus Q$ and $a = ae$. Let $e = p + q$, where $p = ra \in P$ and $q \in Q$. Then $a = ae = ara + aq$ and $ra = raa + raq$. Since $ra - raa = raq \in P \cap Q = 0$, $ra$ is an idempotent. Also, we have $a(1 - ra) = a - ara = aq \in Q \cap Ra \subseteq I$. Hence, $R$ is $I$–semiregular. □

**Corollary 2.7** If $l_Rr_R(a)$ is a summand of $R$ for any $a \in R$ and $R$ is right almost $I$–semiregular for an ideal $I$, then $R$ is $I$–semiregular.

**Proof.** Let $a \in R$. By hypothesis $l_Rr_R(a) = Re$ for some idempotent $e$. Then $r_R(a) = r_R(e)$ and the claim holds by Proposition 2.6. □

A ring $R$ is called a right $PP$–ring if every principal right ideal of $R$ is projective ([2]), or equivalently, for any $a \in R$, $r_R(a) = eR$ for some idempotent $e \in R$. Hence, we have the following result.

**Corollary 2.8** Let $R$ be a right $PP$–ring. If $R$ is a right almost $I$–semiregular ring for an ideal $I$, then $R$ is $I$–semiregular.
Nicholson and Zhou [9, Proposition 41] prove that if $R$ is $I$–semiregular for an ideal $I$, then $eRe$ is $eIe$–semiregular for any idempotent $e$ of $R$. We consider this property for almost $I$–semiregular rings.

An idempotent $e \in R$ is called right semicentral if $eRe = eRe [3]$.

**Theorem 2.9** If $R$ is a right almost $I$–semiregular ring for an ideal $I$ and $e$ is a right semicentral idempotent of $R$, then $eRe$ is a right almost $eIe$–semiregular ring.

**Proof.** Let $a \in eRe$. Then there is a decomposition $l_{Re}r_{Re}(a) = P \oplus Q$ such that $P \subseteq Ra$ and $Q \cap Ra \subseteq I$. Since $e$ is right semicentral, by the proof of [11, Proposition 1.11], $l_{eRe}r_{eRe}(a) = eP \oplus eQ$. Then $eP \subseteq eRa = eRea$ and $eQ \cap eRea \subseteq e(eQ \cap eRea)e$. Hence, $eQ \cap eRea \subseteq Q \cap Ra \subseteq I$ implies that $eQ \cap eRea \subseteq eIe$. □

**Theorem 2.10** Let $e$ be an idempotent of $R$ such that $ReR = R$. If $R$ is a right almost $I$–semiregular ring for an ideal $I$, then $eRe$ is a right almost $eIe$–semiregular ring.

**Proof.** Follows from the proof of [11, Theorem 1.15]. □

**Proposition 2.11** Let $S$ be a right almost $I$–semiregular ring for an ideal $I$ of $S$. If $\varphi : S \to R$ is a ring isomorphism, then $R$ is a right almost $\varphi(I)$–semiregular ring.

**Proof.** Let $a \in R$. Then there is a decomposition $l_{Sr}r_{Sr}(\varphi^{-1}(a)) = P \oplus Q$ such that $P \subseteq S\varphi^{-1}(a)$ and $Q \cap S\varphi^{-1}(a) \subseteq I$. If $x \in l_{Rr}r_{Rr}(a)$, then $\varphi^{-1}(x) \in l_{Sr}r_{Sr}(\varphi^{-1}(a))$. Then we obtain a decomposition $l_{Rr}r_{Rr}(a) = \varphi(P) \oplus \varphi(Q)$, where $\varphi(P) \subseteq Ra$ and $\varphi(Q) \cap Ra \subseteq \varphi(I)$. Hence, $R$ is a right almost $\varphi(I)$–semiregular ring. □

The following result generalizes [11, Corollary 1.16].

**Corollary 2.12** Let $I$ be an ideal of a ring $R$ and let $n \geq 1$. If $M_n(R)$ is right almost $M_n(I)$–semiregular, then $R$ is right almost $I$–semiregular.

**Proof.** Let $S = M_n(R)$. Then $Se_{11}S = S$ and $R \cong e_{11}Se_{11}$, where $e_{11}$ is the $n \times n$ matrix whose $(1,1)$-entry is 1, others are 0. By Theorem 2.10, $e_{11}Se_{11}$ is right almost $e_{11}M_n(I)e_{11}$–semiregular. Let $\varphi : e_{11}Se_{11} \to R$ be the isomorphism. Since $\varphi(e_{11}M_n(I)e_{11}) = I$, $R$ is right almost $I$–semiregular by Proposition 2.11. □
3 Special cases: Soc, $\delta$, Z

In this section, we consider a few fully invariant submodules. We begin with some examples.

Recall that if $R$ is a ring and $V$ is an $R$–$R$ bimodule, the trivial extension $R \ltimes V$ of $R$ by $V$ is the ring with additive group $R \oplus V$ and multiplication $(a, v)(b, w) = (ab, aw + vb)$.

Example 3.1 There exists a right AP-injective ring $R$ that is not semiregular. Hence, there exists a right almost $I$–semiregular ring $R$ that is not $I$–semiregular for ideals $I = J$ or $Z(R)$ or $Soc(R)$.

Proof. Let $R = \mathbb{Z} \ltimes (\mathbb{Q}/\mathbb{Z})$ be the trivial extension. So $R$ is a commutative AP-injective ring that is not semiregular by [7, Examples (8), p. 2435]. $R$ is almost $I$–semiregular for any ideal $I$, because $R$ is AP-injective. But $R$ is neither $Z(R)$–semiregular nor $Soc(R)$–semiregular by [7, Theorem 2.4] and [1, Corollary 4.6].

Example 3.2 There exists a right almost $Soc(R)$–semiregular ring $R$ that is not $Soc(R)$–semiregular.

Proof. Let $R = \mathbb{Z}/8$. Since $R$ is a self–injective ring, it is almost $I$–semiregular for any ideal $I$ of $R$. But since $2R = J \nsubseteq Soc(R) = 4R$, $R$ is not $Soc(R)$–semiregular (see [1, Example 4.21]).

Example 3.1 also shows that the class of right almost semiregular rings is not closed under homomorphic images, because $R/J \cong \mathbb{Z}$ is not right almost semiregular by [11, Example 4.8].

In [1], it is proved that if $M_R$ is a projective $Soc(M_R)$–semiregular module, then $M_R$ is semiregular.

Proposition 3.3 Let $M$ be a right $R$-module and $S = \text{End}_R(M)$. If $M_R$ is almost $Soc(SM)$–semiregular, then $M_R$ is almost semiregular.

Proof. Let $m \in M$. Then there exists a decomposition $l_{M_R}(m) = A \oplus B$ such that $A \subseteq Sm$ and $B \cap Sm \subseteq Soc(SM)$. By the modular law, $Sm = A \oplus (B \cap Sm)$. Then $B \cap Sm$ is a finite direct sum of simple $S$–submodules. If every simple submodule of $B \cap Sm$ is in $\text{Rad}(SM)$, then $B \cap Sm \leq M$ and hence $M_R$ is almost semiregular. Assume that there exists a simple submodule $S_1$ of $B \cap Sm$ such that $S_1 \nsubseteq \text{Rad}(SM)$. Then $S_1$ is a summand of $M$ and hence a summand of $B$. Let $L_1$ be such that $B = S_1 \oplus L_1$. Then $l_{M_R}(m) = A \oplus S_1 \oplus L_1$.

Similarly, $L_1 \cap Sm$ is a finite direct sum of simple submodules. If every simple submodule of $L_1 \cap Sm$ is in $\text{Rad}(SM)$, then $M_R$ is almost semiregular. Assume
that there exists a simple submodule $S_2$ of $L_1 \cap Sm$ such that $S_2 \not\subseteq \text{Rad}(SM)$. Then $S_2$ is a summand of $M$ and so there exists a submodule $L_2$ such that $L_1 = S_2 \oplus L_2$. It follows that $l_{MR}(m) = A \oplus S_1 \oplus S_2 \oplus L_2$. This process produces a strictly descending chain $B \cap Sm \supset L_1 \cap Sm \supset L_2 \cap Sm \ldots$. Since $B \cap Sm$ is semisimple and finitely generated, it is Artinian. Hence, this process must stop so that $L_n \cap Sm \subseteq \text{Rad}(SM)$ for some positive integer $n$. Hence, $l_{MR}(m) = (A \oplus S_1 \oplus \ldots \oplus S_n) \oplus L_n$, where $A \oplus S_1 \oplus \ldots \oplus S_n \leq Sm$ and $L_n \cap Sm \ll M$. Thus, $MR$ is almost semiregular. \qed

Corollary 3.4 If $R$ is right almost $S_l$–semiregular, then $R$ is right almost semiregular.

The next example shows that the converse of Corollary 3.4 is not true in general.

Example 3.5 There exists a right almost semiregular ring that is not right almost $S_l$ ($S_r$)–semiregular.

Proof. (Camillo Example) (see [8, p. 39 and p. 114]) Let $R = \mathbb{Z}_2[x_1, x_2, \ldots]$, where the $x_i$ are commuting indeterminants satisfying the relations $x_i^3 = 0$ for all $i$, $x_i x_j = 0$ for all $i \neq j$ and $x_i^2 = x_i^3$ for all $i$ and $j$. Let $m = x_1^2 = x_2^2 = \ldots$. Then $R$ is a commutative local uniform (i.e., every nonzero right ideal is essential) ring. Then $R$ is semiregular with $J = \text{Span}_{\mathbb{Z}_2} \{m, x_1, x_2, \ldots\}$ and $S_l = S_r = J^2 = \mathbb{Z}_2m$. We claim that $R$ is not (right) almost $S_l$–semiregular. Let $a = x_1 + x_2$. If $R$ is almost $S_l$–semiregular, then there exists a decomposition $l_{RR}(a) = P \oplus Q$ such that $P \subseteq Ra$ and $Q \cap Ra \subseteq S_l$. Since $l_{RR}(a)$ is uniform, either $P = 0$ or $Q = 0$. If $P = 0$, then we have that $l_{RR}(a) \cap Ra = Ra \subseteq S_l$, a contradiction. If $Q = 0$, then $l_{RR}(a) = Ra$. But since $r_R(a) = \text{Span}_{\mathbb{Z}_2} \{m, x_3, x_4, \ldots\}$, $x_1 \in l_{RR}(a)$ and $x_1 \notin Ra$. This gives a contradiction. Hence, $R$ is not almost $S_l$–semiregular. \qed

If $R$ is right almost $S_l$–semiregular, then $R$ need not be semiregular, because right $AP$–injective rings need not be semiregular (see Example 3.1).

We know from [9, Corollary 30] that $R$ is $S_l$–semiregular if and only if $R/S_l$ is (von Neumann) regular. If $R$ is right almost $S_l$–semiregular, then $(Ra + S_l)/S_l$ is a summand of $(l_{RR}(a) + S_l)/S_l$ for any $a \in R$ by [4, Lemma 18.4].

Note also that if $R$ is $S_l$–semiregular, then $R$ is semiregular, $J \subseteq S_l$ and $Z_r \subseteq S_l$ by [7, Theorem 1.2], [1, Theorem 2.3] and by the proof of [1, Theorem 4.5]. On the other hand, $J$ or $Z_r$ need not be contained in $S_l$ if $R$ is right almost $S_l$–semiregular (see Example 3.2).

According to [11], we know that if $R$ is right almost semiregular, then $Z_r \subseteq J$. Hence, if $R$ is right almost $S_l$–semiregular, then $Z_r \subseteq J$. 

\[ \text{8} \]
Because of the fact that $S_1 \subseteq \delta_1$, $R$ being right almost $\delta_1$-semiregular implies that $R$ is right almost $\delta_1$-semiregular. Also if $R$ is $\delta_1$-semiregular, then $Z_r \subseteq \delta_1$ by [7, Theorem 1.2]. We have the following result for right almost $\delta_1$-semiregular rings.

**Proposition 3.6** If $R$ is right almost $\delta_1$-semiregular and $R/S_1$ is a projective right $R$-module, then $Z_r \subseteq \delta_1$.

**Proof.** Let $a \in Z_r$. If $a \notin \delta_1$, then there exists an essential maximal left ideal $N$ of $R$ such that $a \notin N$. Then $R = Ra + N$. Write $1 = ya + n$, where $y \in R$ and $n \in N$. Since $Z_r$ is an ideal and $R \neq Z_r$, we have $n \neq 0$. Since $r_R(ya) \cap r_R(n) = 0$ and $ya \in Z_r$, we obtain that $r_R(n) = 0$. By hypothesis, $R = I_R(n) = P \oplus Q$, where $P = Re \subseteq Rn$, $Q \cap Rn \subseteq \delta_1$ and $e^2 = e \in R$.

Let $\overline{R} = R/S_1$. If $\overline{R} = 0$, then $R$ is semisimple and $Z_r = 0 \subseteq \delta_1 = R$. Assume that $\overline{R} \neq 0$. If $\overline{\pi} = \overline{1}$, then $\overline{Rn} = \overline{N} = \overline{R}$. Since $S_1 \subseteq N$, $N = R$, which is a contradiction. So $\overline{\pi} \neq \overline{1}$. Since $r_R(ya) \leq e$, $\overline{R}/r_R(ya) \cong R/(r_R(ya) + S_1)$ is a singular right $R$-module. This implies that $r_R(ya) \leq e$, $\overline{R}$ is a projective right $R$-module. Since $r_R(ya) \subseteq r_R(ya)$, we have that $r_R(ya) \leq e$, $\overline{R}$.

Now $(\overline{1} - \overline{\pi})\overline{R} \cap r_{\overline{R}}(ya) \neq 0$. Let $0 \neq (\overline{1} - \overline{\pi})\overline{\pi} \in (\overline{1} - \overline{\pi})\overline{R} \cap r_{\overline{R}}(ya)$. Let $n = se + t$, where $s \in R$ and $t \in Q$. Then $t = n - se \in Q \cap Rn \subseteq \delta_1$ and $\overline{t} \in \delta_1/S_1 = J(R/S_1)$ by [12, Corollary 1.7]. So $\overline{1} - \overline{\pi}$ is unit in $\overline{R}$. Also, we have $\overline{\pi}(\overline{1} - \overline{\pi})\overline{\pi} = (\overline{1} - \overline{\pi})\overline{\pi} = (\overline{1} - \overline{\pi})\overline{\pi}$ and $\overline{\pi}(\overline{1} - \overline{\pi})\overline{\pi} = (\overline{\pi} + \overline{\pi})(\overline{1} - \overline{\pi})\overline{\pi} = \overline{1}(\overline{1} - \overline{\pi})\overline{\pi}$. Then $(\overline{1} - \overline{\pi})\overline{\pi} \neq 0$. Hence, $(\overline{1} - \overline{\pi})\overline{\pi} = \overline{0}$, a contradiction. \qed

**Proposition 3.7** If $R$ is right almost $\delta_1$-semiregular, $R/S_1$ is a projective right $R$-module and $S_1 \subseteq Z_1$, then $Z_r \subseteq J$.

**Proof.** By a proof similar to that of Proposition 3.6. \qed

**Example 3.8** There exists a right almost $\delta_1$ (or $\delta_r$)-semiregular ring that is not right almost semiregular.

**Proof.** [12, Example 4.3] Let $F$ be a field and $I = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, and

$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) \mid n \in \mathbb{N}, x_i \in M_2(F), x \in I\}$.

Then $R$ is $\delta_r$ ($\delta_l$)-semiregular but not semiregular by [12]. Since every nonzero one-sided ideal contains a nonzero idempotent, $Z_r = Z_l = J = 0$. If $R$ was right almost semiregular, then $R$ would be regular by [11, Lemma 3.1], which is a contradiction. Hence, $R$ is not right almost semiregular. \qed
It is well known that \( J(eRe) = eJe \) for any idempotent \( e \) of \( R \). We consider this property for \( \delta \) which will be used in the forthcoming corollary. Recall by [12, Theorem 1.6] that

\[
\delta_e = \{ x \in R : \forall y \in R, \exists a \text{ semisimple right ideal } Y \text{ of } R \ni R_e = (1-xy)R \oplus Y \}
\]

\[
= \bigcap \{ \text{ideals } P \text{ of } R : R/P \text{ has a faithful singular simple module} \}
\]

**Theorem 3.9** Let \( e \) be an idempotent of \( R \) such that \( ReR = R \). Then \( \delta_1(eRe) = e\delta_i e \).

**Proof.** We know that if \( e \) is an idempotent such that \( ReR = R \), then the category of left \( R \)-modules, \( R \text{-Mod} \), and the category of left \( eRe \)-modules, \( eRe\text{-Mod} \), are Morita equivalent (see [6]) under the functors given by

\[
F : R\text{-Mod} \rightarrow eRe\text{-Mod}, \quad \mathcal{G} : eRe\text{-Mod} \rightarrow R\text{-Mod}
\]

\[
M \mapsto eM \quad T \mapsto Re \otimes_{eRe} T.
\]

By [12], \( \delta_1 \neq R \) if and only if \( R \) is semisimple. Therefore if \( \delta_1 = R \), then \( R \) is semisimple and so is \( eRe \). This gives that \( \delta_1(eRe) = eRe = e\delta_i e \).

Now assume that \( \delta_1 \neq R \). Let \( P \) be an ideal of \( R \) such that \( R/P \) has a faithful singular simple module \( N \). Denote \( \overline{R} = R/P \). Since \( \overline{eRe} = \overline{R} \), the categories \( \overline{R} \text{-Mod} \) and \( \overline{eRe}\text{-Mod} \) are Morita equivalent. So \( \overline{N} \) is a faithful \( \overline{eRe} \)-module by [6, 18.47 and 18.30], a singular \( \overline{eRe} \)-module by [5, p. 34] and a simple \( \overline{eRe} \)-module. Since \( \overline{eRe} \cong eRe/ePe \), we have that \( \delta_1(eRe) \subseteq ePe \subseteq P \). This holds for any ideal \( P \) such that \( R/P \) has a faithful singular simple module. Thus, \( \delta_1(eRe) \subseteq e\delta_i e \).

For the reverse inclusion, let \( a \in \delta_1 \). Then \( Reae \preceq_R R \). Now we claim that \( eRe(cae) \preceq_R eRe \). Let \( K \) be a left ideal of \( eRe \) such that \( eRe = eRe(cae) + K \). Write \( e = erae + k \), where \( r \in R \) and \( k \in K \). This implies that \( 1 = e + (1-e) = erae + k + (1-e) \in Reae + RK + R(1-e) \) and so \( R = Reae + RK + R(1-e) \). Since \( Reae \preceq_R R \), there exists a semisimple projective left ideal \( Y \) of \( R \) such that \( Y \subseteq Reae \) and \( R = Y \oplus [RK + R(1-e)] \) by [12, Lemma 1.2]. Hence, we obtain that \( eRe = eYe + (eRe)K = eY + K \). Since \( Y \cap RK = 0 \), we have that \( eY \cap K = 0 \). On the other hand, since \( ReR = R \), \( eY \) is a semisimple projective left \( eRe \)-module. So \( eRe = eY \oplus K \), \( eY \subseteq eRe(cae) \) and \( eY \) is a semisimple projective \( eRe \)-module. By [12, Lemma 1.2], \( eRe(cae) \preceq_R eRe \). Thus, \( e\delta_i e \subseteq \delta_1(eRe) \).

**Corollary 3.10** Let \( e \) be an idempotent of \( R \) such that \( ReR = R \). If \( R \) is right almost \( \delta_i \)-semiregular, then \( eRe \) is right almost \( \delta_i(eRe) \)-semiregular.

**Proof.** Follows from Theorems 3.9 and 2.10. □

Now we consider the ring \( eRe \), where \( e \) is a right semicentral idempotent.
Theorem 3.11 If $e$ is a right semicentral idempotent of $R$, then $e\delta_l e \subseteq \delta_l(eRe)$ and $\delta_r(eRe) \subseteq e\delta_r e$.

Proof. Let $a \in \delta_l$. Since $\delta_l$ is an ideal, $eae \in \delta_l$. By [12, Theorem 1.6], there exists a semisimple left ideal $Y$ of $R$ such that $R = R(1 - eae) \oplus Y$. Let $1 = x(1 - eae) + y$, where $x \in R$ and $y \in Y$. Then $e = ex(1 - eae)e + eye = exe(e - eae) + eye$ and so $eRe = eRe(e - eae) + eYe$. Since $e$ is right semicentral, this sum is direct. Now we claim that $eYe$ is semisimple. Let $Y = \oplus_{i=1}^n S_i$, where $S_i$ is a simple left $R$-module, for $i = 1, 2, \ldots, n$. Since $e$ is right semicentral, $eYe = \oplus_{i=1}^n eS_i e$. Let $S_1 = Re$ for some $s \in R$. Then $eS_1 e = eRe = eRe(ese) \cong eRe/l_{eRe}(ese)$. Let $K$ be a left ideal of $eRe$ such that $l_{eRe}(ese) \subseteq K$. Then there exists $k \in K$ such that $k \not\in l_{eRe}(ese)$. Since $l_{eRe}(ese) = l_{eRe}(es) = l_R(es) \cap eRe$, $k \not\in l_R(es)$. Then $kes \neq 0$. But since $l_R(s)$ is maximal in $R$, we have that $l_R(s) + Rke = R$. Let $1 = x + yke$, where $x \in l_R(s)$ and $y \in R$. Then $e = ex + eye$. Since $xs = 0$, we have $exese = 0$. Then $ex \in l_{eRe}(ese) \subseteq K$, so $ex \in K$. It follows that $e \in K$. Hence, we show that $l_{eRe}(ese)$ is a maximal left ideal of $eRe$. So $eS_1 e$ is simple. This proves that $eYe$ is semisimple. Now $eRe = eRe(e - eae) + eYe$ with $eYe$ semisimple. Since $a$ is any element in $\delta_l$, we have that $e\delta_l e \subseteq \delta_l(eRe)$.

For the other inclusion, let $P$ be an ideal of $R$ and $V$ be a faithful singular simple right $R/P$-module. Then $Ve$ is an $eRe$-module. If $Ve = 0$, then $\delta_r(eRe) \subseteq eRe \subseteq P$.

Assume that $Ve \neq 0$. Since $V$ is a simple $R$-module, $Ve$ is a simple $eRe$-module. We claim that $Ve$ is a singular $eRe$-module. Let $ve$ be the generator of $Ve$. To show that $r_{eRe}(ve) = r_R(v) \cap eRe$ is an essential right ideal of $eRe$, let $0 \neq exe \in eRe$. Since $ex \neq 0$ and $r_R(v)$ is essential in $R$, there exists $r \in R$ such that $0 \neq ext \in r_{eRe}(v)$. Then $0 \neq ext = exe \in r_{eRe}(ve)$ (e is right semicentral). Hence, $Ve$ is a singular simple $eRe$-module. Now, $V\delta_r(eRe) = Ve\delta_r(eRe) = 0$ by the definition of $\delta$. Since $V$ is a faithful $R/P$-module, we have that $\delta_r(eRe) \subseteq P$. Therefore $\delta_r(eRe) \subseteq P$ for each ideal $P$ of $R$ such that $R/P$ has a faithful singular simple module. So $\delta_r(eRe) \subseteq \delta_r$ and hence $\delta_r(eRe) \subseteq e\delta_r e$. □

Corollary 3.12 Let $e$ be a right semicentral idempotent of $R$. If $R$ is right almost $\delta_l$-semiregular, then $eRe$ is right almost $\delta_l(eRe)$-semiregular.

Proof. Follows from Theorems 3.11 and 2.9. □

The following example shows that the equality $e\delta_l e = \delta_l(eRe)$ does not hold even for a right semicentral idempotent.

Example 3.13 There exists a right semicentral idempotent $e \in R$ such that $e\delta_l e \subset \delta_l(eRe)$. 


Proof. Let $R$ be the ring of $2 \times 2$ upper triangular matrices over a field $F$ and $e = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Then $eR = eRe$ and $e\delta_l e = 0$, where $\delta_l$ is the first row of $R$. Since $eRe$ is a semisimple projective left $eRe$–module, $\delta_l(eRe) = eRe$.

Recall that $R_R$ is said to satisfy $(C2)$ if any right ideal of $R$ isomorphic to a summand of $R_R$ is itself a summand of $R$. We have the following results about right almost $Z_r(Z_l)$–semiregular rings.

**Theorem 3.14** Let $I$ be an ideal of $R$. If $R$ is right almost $I$–semiregular and $I \subseteq Z_l$, then $R_R$ satisfies $(C2)$.

**Proof.** Let $a \in R$ such that $aR \cong eR$, where $e^2 = e \in R$. By [10, Lemma 2.12], there exists an idempotent $f \in R$ such that $a = af$ and $r_R(a) = r_R(f)$. By the proof of Proposition 2.6, there exists an idempotent $h \in R$ such that $h \in Ra$ and $a(1-h) \in I$. By [9, Lemma 27], there exists an idempotent $g \in R$ such that $g \in aR$ and $(1-g)a \in I$. Then $aR = gR \oplus S$, where $S = (1-g)aR \subseteq I$. By assumption, $S$ is a singular right $R$–module. Since $aR$ is projective, we have that $S = 0$. Thus, $aR = gR$. \qed

**Corollary 3.15** Let $R$ be a right $PP$–ring and $I$ an ideal of $R$. If $R$ is right almost $I$–semiregular and $I \subseteq Z_r$, then $R$ is regular.

**Proof.** Let $a \in R$ and $r_R(a) = eR$, where $e$ is an idempotent of $R$. Then $aR \cong (1-e)R$. By Theorem 3.14, there exists an idempotent $g \in R$ such that $aR = gR$. Hence, $R$ is regular. \qed

**Corollary 3.16** If $R$ is right almost $Z_r$–semiregular, then $R_R$ satisfies $(C2)$.

We know from [7, Lemma 2.3] that if $R_R$ satisfies $(C2)$, then $Z_r \subseteq J$. Hence, we have the following result.

**Corollary 3.17** If $R$ is right almost $Z_r$–semiregular, then $R$ is right almost semiregular.

The following two examples show that the converse of Corollary 3.17 is not true in general.

**Example 3.18** There is an Artinian ring $R$ such that $R$ is $Z_l$–semiregular but not right almost $Z_r$–semiregular.

**Proof.** Let $R = \begin{bmatrix} Z_4 & Z_2 \\ 0 & Z_2 \end{bmatrix}$. Then
By [9, Example 40], \( R \) is \( Z_l \)-semiregular but not \( Z_r \)-semiregular. Now we claim that \( R \) is not right almost \( Z_r \)-semiregular. Let \( a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) in \( R \).

Then \( Ra = \begin{bmatrix} 0 & Z_2 \\ 0 & 0 \end{bmatrix} \) and \( l_{rR}(a) = \begin{bmatrix} 0 & Z_2 \\ 0 & 0 \end{bmatrix} \). If \( R \) is right almost \( Z_r \)-semiregular, then there is a decomposition \( l_{rR}(a) = P \oplus Q \), where \( P \subseteq Ra \) and \( Q \cap Ra \subseteq Z_r \). Since \( Ra \cap Z_r = 0 \), \( Q \cap Ra = 0 \). This implies that \( Ra = P \) is a summand of \( l_{rR}(a) \) which is a contradiction. Hence, \( R \) is not right almost \( Z_r \)-semiregular.

**Example 3.19** Let \( R \) be the ring of \( 2 \times 2 \) upper triangular matrices over a field \( F \). Then \( R \) is an Artinian ring which does not satisfy \((C2)\) ([8, Example 1.20]). Hence, \( R \) is right almost semiregular but not right almost \( Z_r \)-semiregular.

Recall that \( R_R \) is said to satisfy \((C1)\) if every right ideal of \( R \) is essential in a summand of \( R \). A ring \( R \) satisfying \((C1)\) and \((C2)\) as a right \( R \)-module is called right continuous. The following result generalizes [1, Corollary 3.5] in the ring case.

**Proposition 3.20** A ring \( R \) is right almost \( Z_r \)-semiregular and \( R_R \) satisfies \((C1)\) if and only if \( R \) is right continuous.

**Proof.** It is well known that if \( R_R \) is right continuous, then it is semiregular and \( Z_r = J \). Now the proof follows from Corollary 3.16.

The ring \( R \) in Example 3.19 is right almost semiregular but not right almost \( Z_l \)-semiregular, because \( Z_l = 0 \) and \( R \) is not right \( AP \)-injective.

**Proposition 3.21** If \( R \) is a right almost \( Z_l \)-semiregular and left \( PP \)-ring, then \( R \) is right \( AP \)-injective.

**Proof.** Let \( a \in R \). By hypothesis, \( Ra = P \oplus Q \), where \( P \) is a summand of \( l_{rR}(a) \) and \( Q \subseteq Z_l \). Since \( Ra \) is a projective left ideal, \( Q \) is projective, and so \( Q = 0 \). Hence, \( Ra \) is a summand of \( l_{rR}(a) \).

**Proposition 3.22** If \( R \) is right almost \( Z_l \cap \delta_l \)-semiregular, then it is right almost semiregular.
Proof. Let \( a \in R \). Then there exists a decomposition \( l_{R}r_{R}(a) = P \oplus Q \) such that \( P \subseteq Ra \) and \( Q \cap Ra \subseteq Z_{I} \cap \delta_{I} \). We claim that \( Q \cap Ra \subseteq J \). Let \( x \in Q \cap Ra \). To see that \( x \in J \), we must show that \( 1 - yx \) is left invertible in \( R \) for any \( y \in R \). Let \( u = 1 - yx \), where \( y \in R \). Since \( x \in \delta_{I} \), there exists a semisimple left ideal \( Y \) of \( R \) such that \( R(1 - yx) \oplus Y = R \) by [12, Theorem 1.6]. Let \( \varphi: R \rightarrow Y \) be the projection. Then \( \varphi(Q \cap Ra) \subseteq \varphi(Z_{I}) \subseteq Z(Y) = 0 \), and so \( Ryx \subseteq Q \cap Ra \subseteq \ker \varphi = R(1 - yx) \). Since \( R = Ryx + R(1 - yx) \), we have that \( R = R(1 - yx) \). Hence, \( x \in J \) and \( Q \cap Ra \triangleleft R \). \( \square \)

**Proposition 3.23** If \( R \) is right almost \( I \)-semiregular for an ideal \( I \) such that \( J \cap I = 0 \), then \( J \subseteq Z_{r} \).

**Proof.** Let \( a \in J \) and assume that \( a \notin Z_{r} \). Then there exists a nonzero right ideal \( K \) of \( R \) such that \( r_{R}(a) \cap K = 0 \). Take \( s \in K \) such that \( as \neq 0 \). Let \( 0 \neq u \in asR \). By hypothesis, \( l_{R}r_{R}(u) = P \oplus Q \), where \( P \subseteq Ru \), \( Q \cap Ru \subseteq I \). Without loss of generality we can assume that \( u = as \). Then it can be seen that \( r_{R}(as) = r_{R}(s) \). Then \( l_{R}r_{R}(as) = l_{R}r_{R}(s) = P \oplus Q \). Write \( s = das + x \), where \( d \in R \) and \( x \in Q \). Then \((1 - da)s = x \) and so \( u = as = a(1 - da)^{-1}x \in J \cap (Q \cap Ru) \subseteq J \cap I = 0 \), a contradiction. Hence, \( a \in Z_{r} \). \( \square \)

**Corollary 3.24** If \( R \) is right almost \( S_{I} \)-semiregular and \( R/S_{I} \) is a projective right \( R \)-module, then \( J \subseteq Z_{r} \) and \( R \) is right almost \( Z_{r} \)-semiregular.

**Proof.** Since \( S_{I} \) is a summand of \( R \), \( J \cap S_{I} = \text{Rad}(S_{I}) = 0 \). By Proposition 3.23, \( J \subseteq Z_{r} \). By Corollary 3.4, \( R \) is right almost semiregular. Then \( Z_{r} \subseteq J \) and hence \( J = Z_{r} \) and \( R \) is right almost \( Z_{r} \)-semiregular. \( \square \)

The following example shows that the assumption “\( J \cap I = 0 \)” in Proposition 3.23 is not removable in case \( I = Z_{I} \).

**Example 3.25** Let \( R \) be the ring in Example 3.18. \( R \) is a right almost \( Z_{I} \)-semiregular ring. Since \( J = \begin{bmatrix} \mathbb{Z}_{4} & \mathbb{Z}_{2} \\ 0 & 0 \end{bmatrix} \), \( J \cap Z_{I} \neq 0 \) and \( J \nsubseteq Z_{r} \).

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References


