Solving Simultaneous Nonlinear Equations

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Quotes of the Day

Nothing in the world can take the place of Persistence. Talent will not; nothing is more common than unsuccessful men with talent. Genius will not; unrewarded genius is almost a proverb. Education will not; the world is full of educated derelicts. Persistence and determination alone are omnipotent. The slogan 'Press On' has solved and always will solve the problems of the human race. – *Calvin Coolidge*

Victorious warriors win first and then go to war, while defeated warriors go to war first and then seek to win. – *Sun-Tzu*

Nonlinear Functions of Several Variables

- System with two nonlinear functions and two variables

  $z=f(x_1,x_2)$ and $z=g(x_1,x_2)$
- Problem is more difficult to solve for more variables $f(x_1,x_2,x_3,\ldots,x_n)$
- To find a zero of the system, find the intersection of the curves:

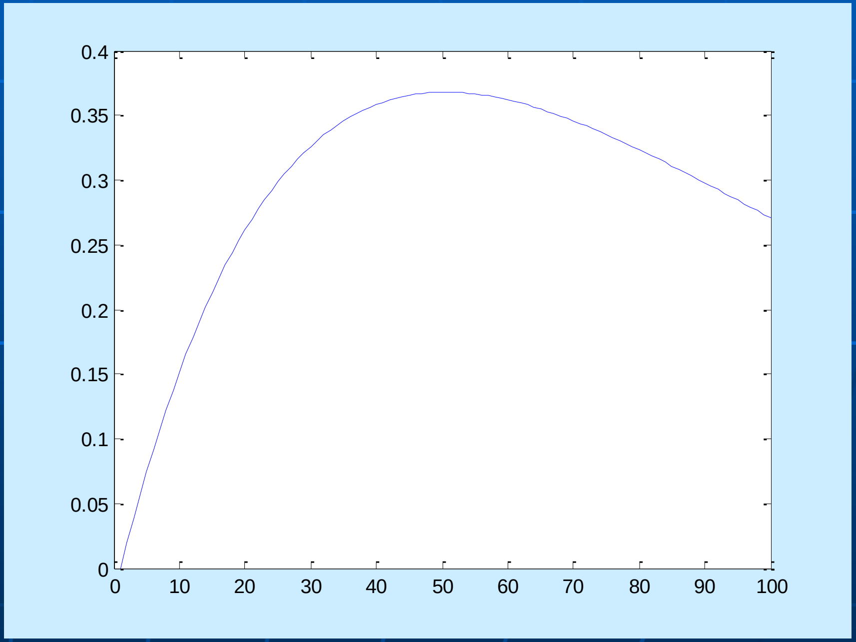
  $f(x_1,x_2)=0$ and $g(x_1,x_2)=0$
- Use all the information about the problem to find the region where the curves may intersect → SO…

Nonlinear Functions of Several Variables

- Plot a 2-D or 3-D figure of the functions in MATLAB® !

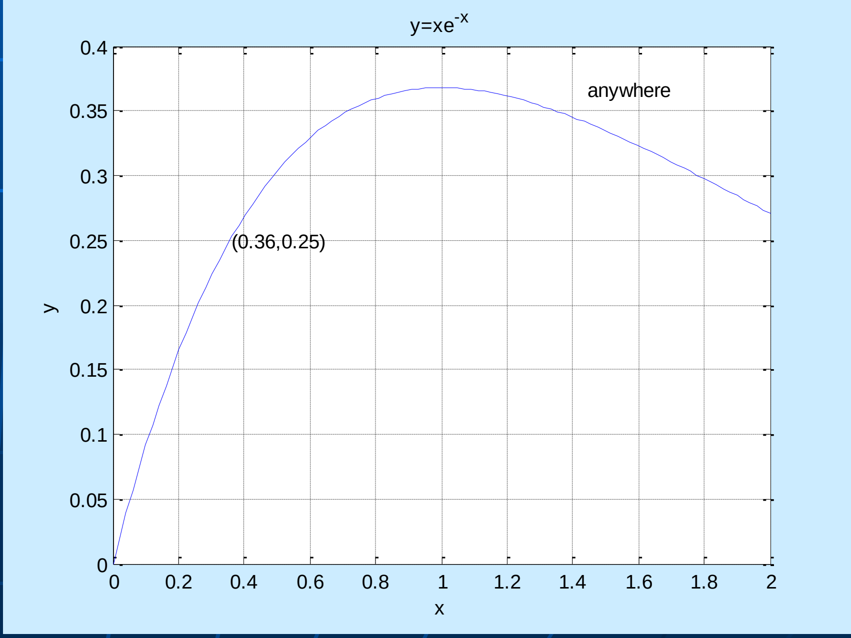
2-D Graphs

```
>> x=linspace(0,2);
% linspace generates a row vector of 100 linearly
% equally spaced points between x1 and x2
>> y=x.*exp(-x);
>> plot(y)
% plots y versus their index
```



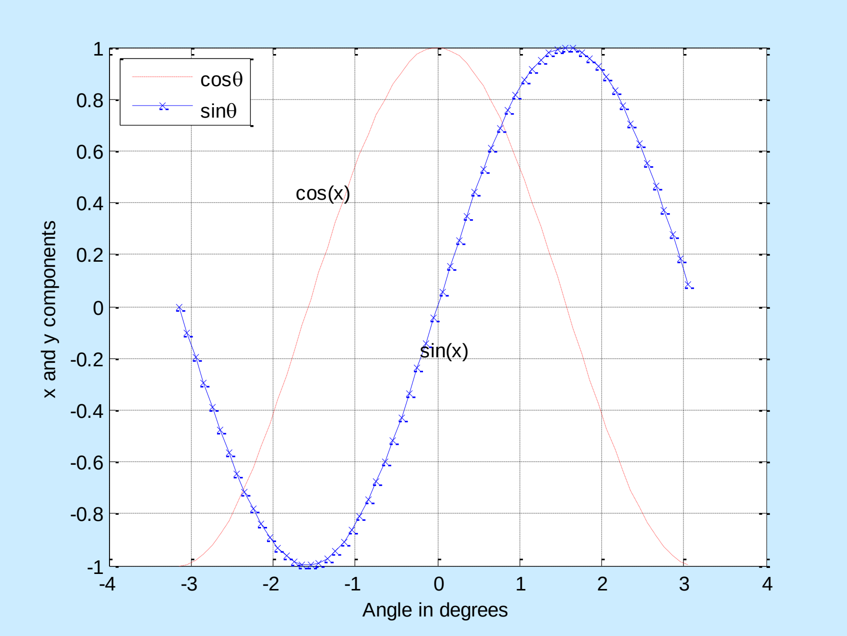
2-D Graphs

```
% ExPlot2DGraph.m
x=linspace(0,2);
% linspace generates a row vector of 100 linearly
% equally spaced points between x1 and x2
y=x.*exp(-x);
plot(x,y)          % plots y agains x
grid on            % adds grid lines to the current axes
xlabel('x')        % adds text below the x-axis
ylabel('y')
% adds text besides the y-axis
title('y=xe^{-x}')
% adds text at the top of graph
gtext('anywhere')
%places text with mouse
text(0.36,0.25,'(0.36,0.25)')
% places text at the specific point
```



2-D Graphs

```
% ExPlot.m: This program draws a graph of sin(x) and cos(x) % where
    0 <= x <= 3.14
angle=-pi:0.1:pi;          % Create array
xcomp=cos(angle);          % Create array
plot(angle,xcomp,'r:');    % Plot using dots(:) with red(r)
hold on                    % Add another plot on the same graph
ycomp=sin(angle);
% Create array
plot(angle,ycomp,'b-x');
% Plot using lines(-) and the symbol x
% at each data point with blue(b)
grid on
xlabel('Angle in degrees');
ylabel('x and y components');
legend('cos{\theta}','sin{\theta}',2)
gtext('cos(x)');
gtext('sin(x)');
% Display mouse-movable text
```



2-D Graphs: Other Commands

- More than one function can be plotted on one graph:
  >>plot(x,X.*exp(-x),'.',x,x*sin(x),'-.')
- More than one graph can be shown in different frames
  >>subplot(2,1,1), plot(x,x.*cos(x))
  >>subplot(2,1,2), plot(x,x.*sin(x)
- Axis limits can be seen or modified
  >>axis
  >>axis([0,1.5,0,1.5])
- Figure window can be cleared
  >>clf

2-D Graphs: Other Commands

- Comet like trajectory of the function

  >> shg, comet(x,y)

% shg brings up the current graphic window

- By using figure(n) command, it is possible to use more than one graphic window, n: positive integer
- Another easy way to plot a function:

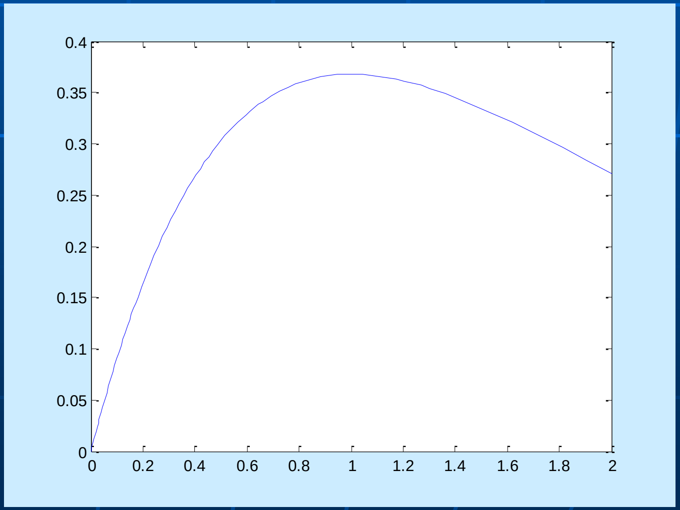
  >>fplot(x*)

2-D Graphs: Other Commands

Another easy way to plot a function: fplot

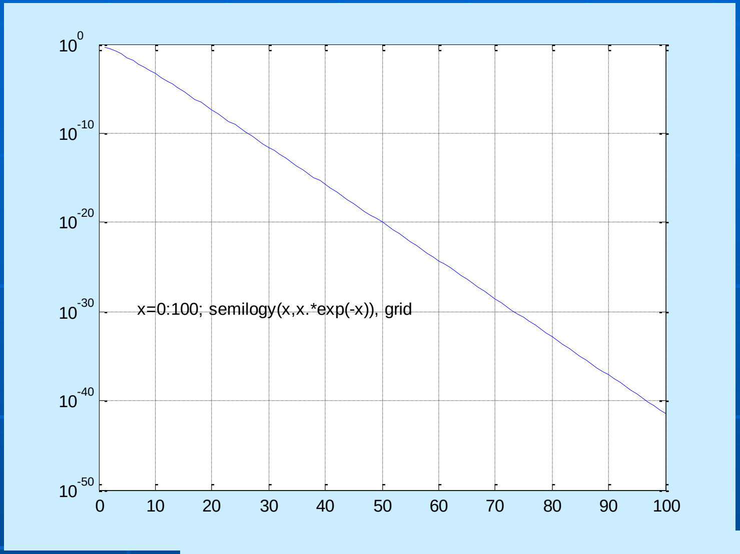
- fplot(@humps,[0 1])
- fplot(@(x)[tan(x),sin(x),cos(x)], 2*pi*[-1 1 -1 1])
- fplot(@(x) sin(1./x), [0.01 0.1], 1e-3)
- f = @(x,n)abs(exp(-1j*x*(0:n-1))*ones(n,1));
  fplot(@(x)f(x,10),[0 2*pi])

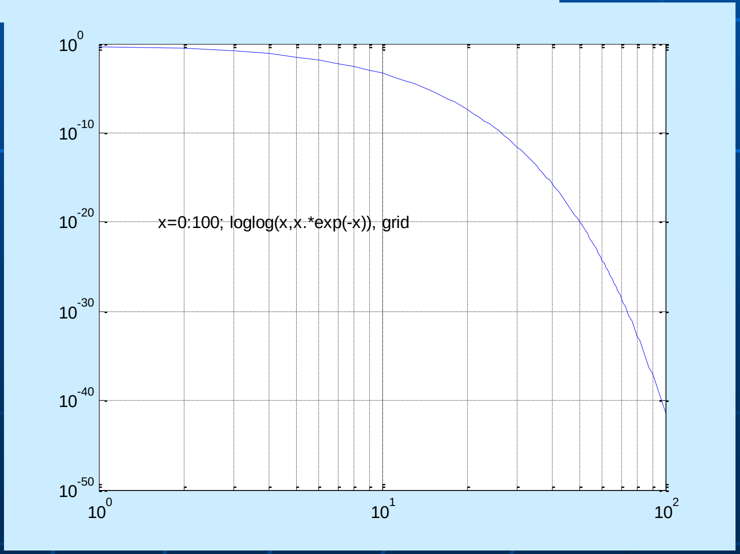
>> fplot('x*exp(-x)',[0,2])



2-D Graphs: Other Commands

semilogx(x,y)
% semilogarithmic plot
% log scale x-axis
semilogy(x,y)
% log scale y-axis
loglog(x,y)
% log scale x- and y-axis





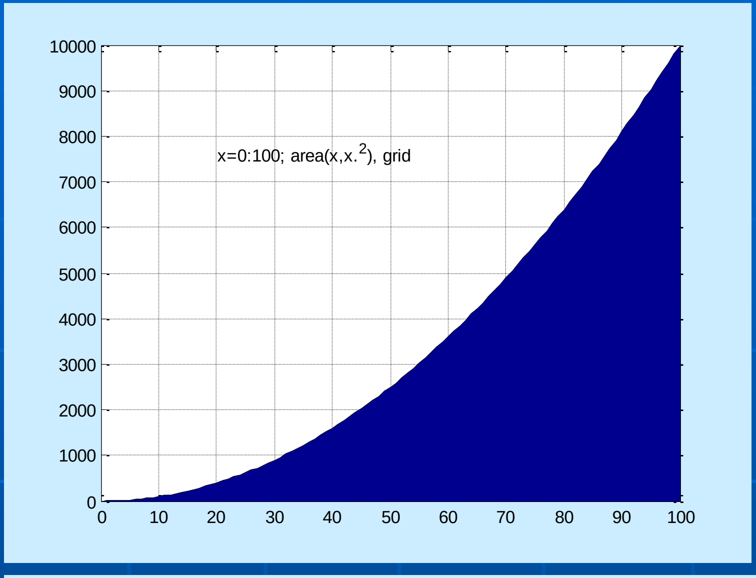
2-D Graphs: Other Commands

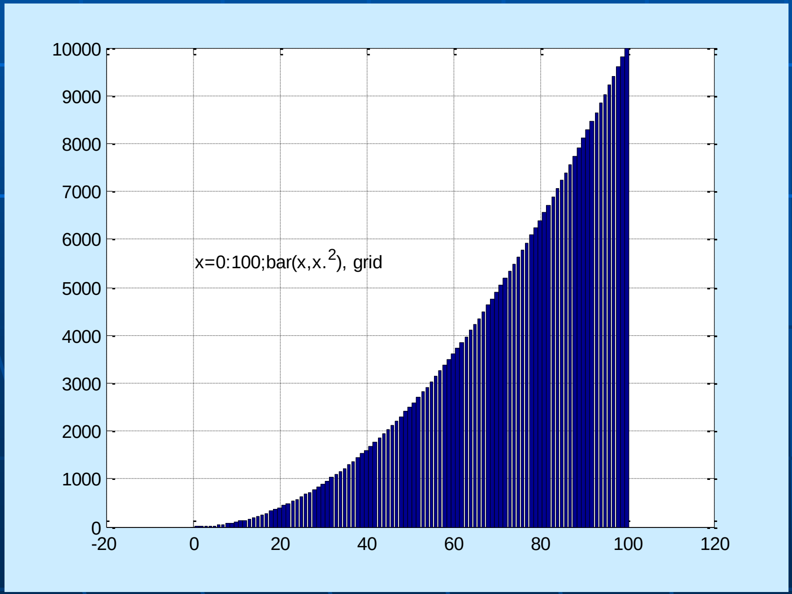
area(x,y) % filled area plot

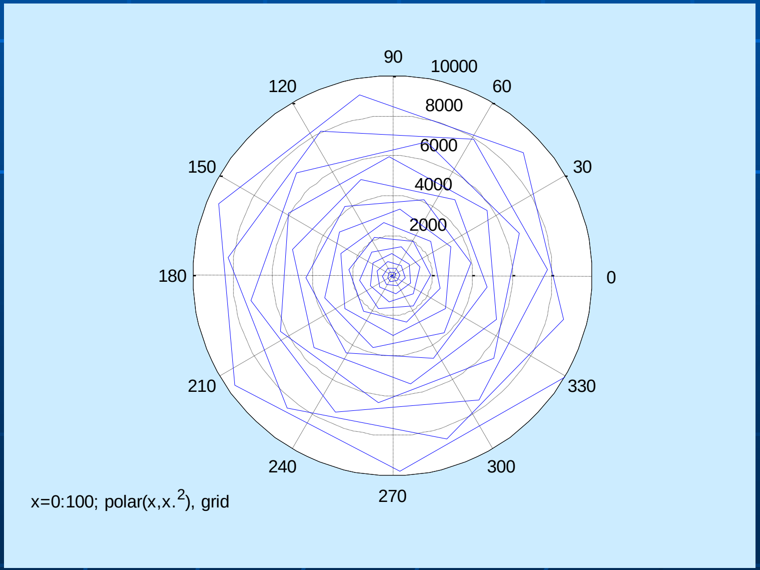
polar(x,y)

% polar coordinate plot

bar(x,y)   % bar graph

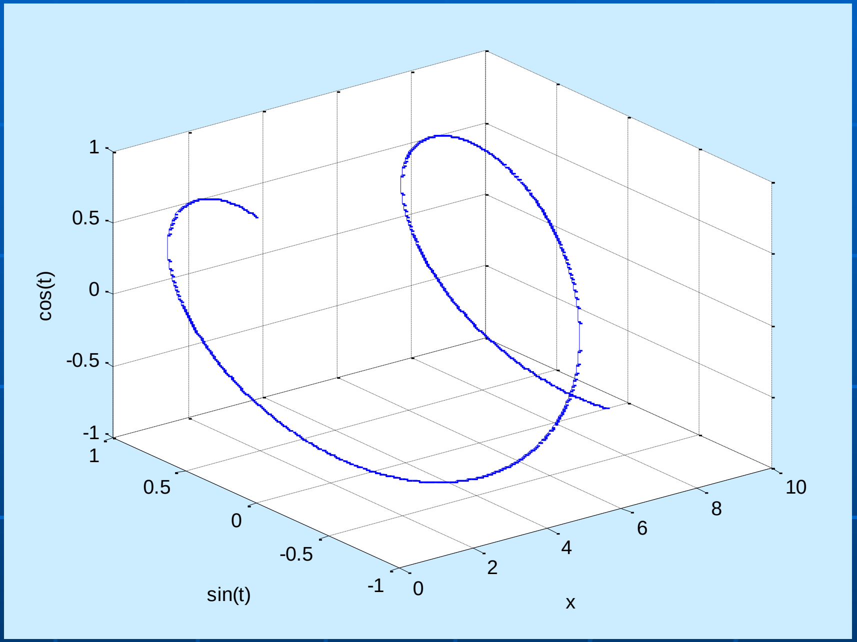






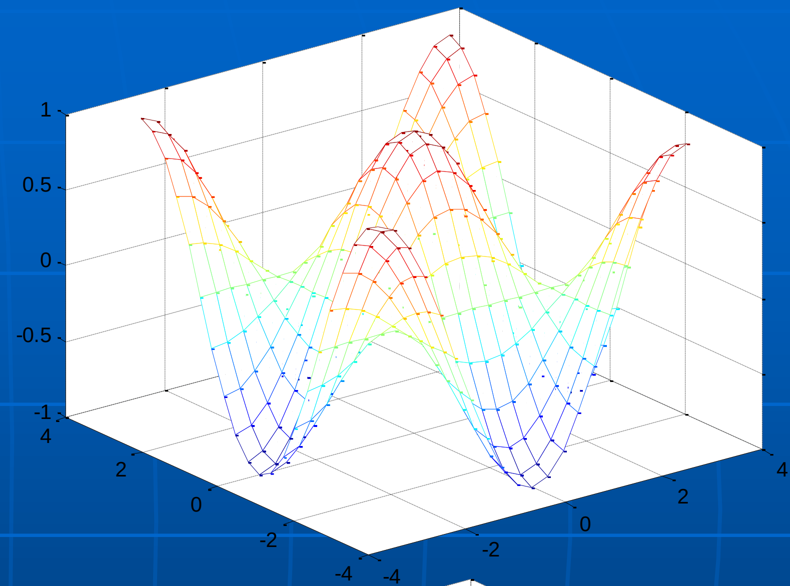
3-D Graphs

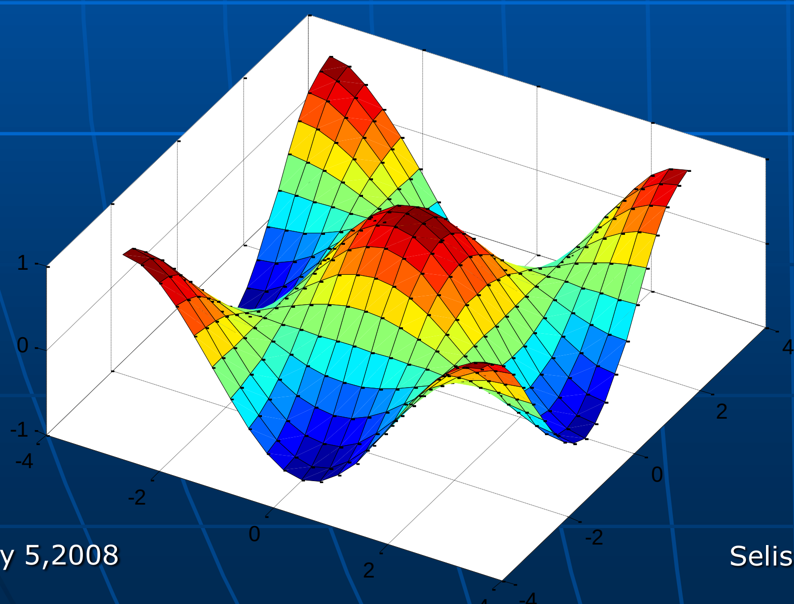
```
% ExPlot3DGraph.m
t=0:0.01:3*pi;
plot3(t,sin(t),cos(t))
xlabel('x'),
ylabel('sin(t)'),
zlabel('cos(t)')
grid
```

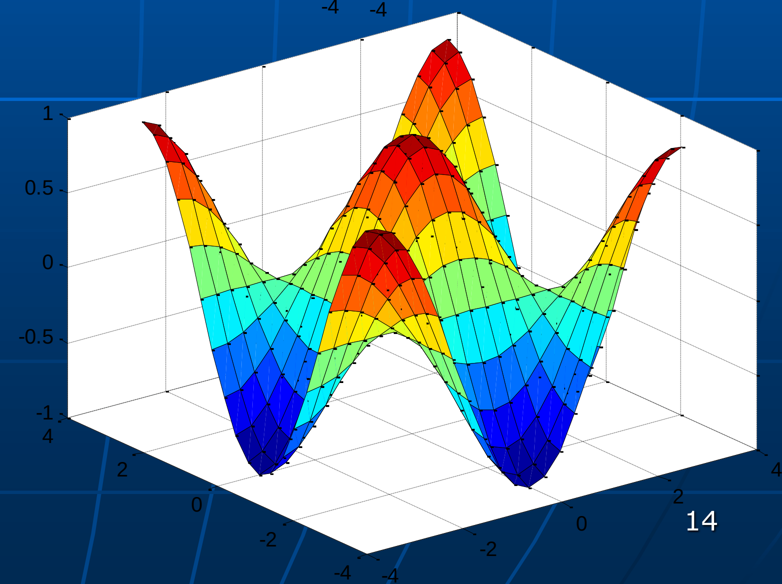


3-D Graphs: Surfaces

```
% ExPlot3DGraphSurface.m
[x,y]=meshgrid(-pi:pi/10:pi, -
    pi:pi/10:pi);
z=cos(x).*cos(y);
figure(1), mesh(x,y,z)
figure(2), surf(x,y,z)
figure(3), surf(x,y,z), view(30,60)
```







3-D Graphs: Surfaces

shading controls the color shading of **surface** and **patch** objects
surface and **patch** objects are created by the functions **surf**, **mesh**, **pcolor, fill**, and **fill3**

Options: shading flat, interp, faceted (default)

Flat shading is piecewise constant; each mesh line segment or surface patch has a constant color determined by the color value at the end point of the segment or the corner of the patch which has the smallest index or indices.

Interpolated shading, which is also known as Gouraud shading, is piecewise bilinear; the color in each segment or patch varies linearly and interpolates the end or corner values.

Faceted shading is flat shading with superimposed black mesh lines. This is often the most effective and is the default.

3-D Graphs: Surfaces

COLORBAR Display color bar (color scale)
COLORBAR appends a colorbar to the current axes in the default (right) location

COLORBAR('peer',AX) creates a colorbar associated with axes AX instead of the current axes.

COLORBAR(...,LOCATION) appends a colorbar in the specified location relative to the axes. LOCATION may be any one of the following strings:

'North'	inside plot box near top
'South'	inside bottom
'East'	inside right
'West'	inside left
'NorthOutside'	outside plot box near top
'SouthOutside'	outside bottom
'EastOutside'	outside right
'WestOutside'	outside left

COLORBAR(...,P/V Pairs) specifies additional property name/value pairs for colorbar

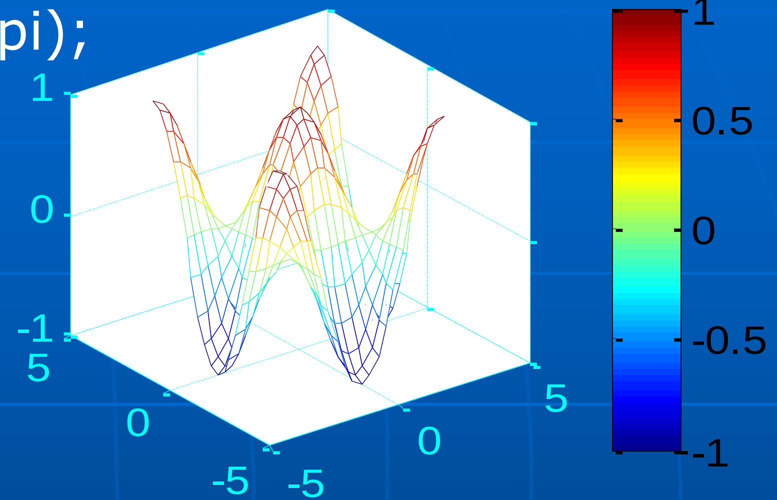
H = COLORBAR(...) returns a handle to the colorbar axes

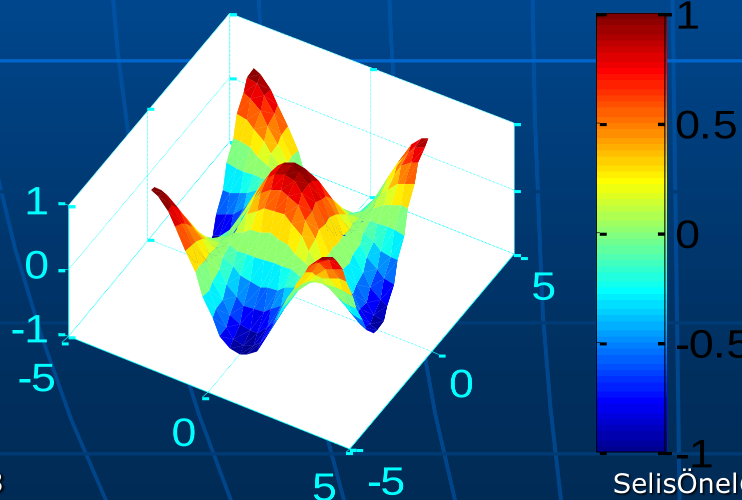
MATLAB® Plotting Command: **surf**

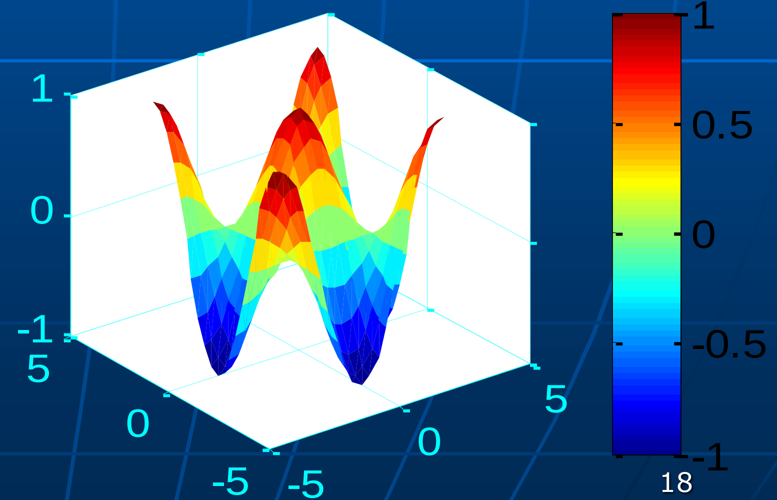
- **Plots 3-D surface**
- **surf**(x,y,z,c) plots the colored parametric surface defined by four matrix arguments.
  - The view point is specified by **view.** The axis labels are determined by the range of X, Y and Z, or by the current setting of **axis**
  - The color scaling is determined by the range of C, or by the current setting of **caxis**.
  - The scaled color values are used as indices into the current **colormap**
  - The shading model is set by **shading**
- **surf**(x,y,z) uses c=z, so color is proportional to surface height

3-D Graphs: Surfaces

```
% ExPlot3DGraphSurface.m
[x,y]=meshgrid(-pi:pi/10:pi,-pi:pi/10:pi);
z=cos(x).*cos(y);
subplot(2,2,2)
mesh(x,y,z), shading interp, colorbar
subplot(2,2,4)
surf(x,y,z), shading interp, colorbar
subplot(2,2,3), surf(x,y,z)
view(30,60), shading interp, colorbar
```







MATLAB® Plotting Command: **meshgrid**

- 2-D arrays, x and y, can be generated from 1-D arrays x1 and y1 as:

  [x,y]=**meshgrid**(x1,y1)

  - x1 and y1 represent x_i and y_j
  - x and y represent $x_{i,j}$ and $y_{i,j}$

Plot the grid using:

mesh(x,y,0*x);

view([0,0,10000]);

xlabel(‘x’); ylabel(‘y’)

MATLAB® Plotting Command: **mesh**

- 2-D function $z_{i,j}=f(x_{i,j},y_{i,j})$ can be plotted using the mesh command

Ex: $x_{i,j}=x_i=-2+0.2(i-1),\ 1\le i\le 21$

$y_{i,j}=y_j=-2+0.2(j-1),\ 1\le j\le 21$

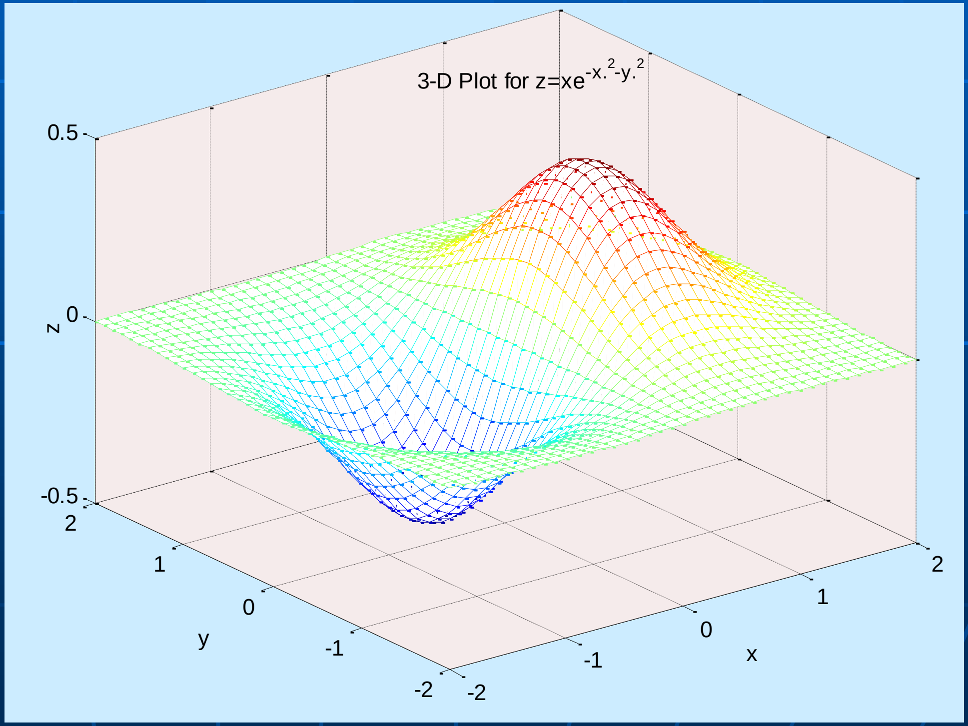
The function is defined by

$z_{i,j}=x_{i,j}e\text{^}(-x_{i,j}^2-y_{i,j}^2)$

Plot the grid using mesh…

MATLAB® Plotting Command: **mesh**

z=x.*exp(-x.^2-y.^2); mesh(x,y,z),
title('3-D Plot for z=xe^{-x.^{2}-y.^{2}}'),
xlabel('x');
ylabel('y');
zlabel('z');



2.5-D Graphs: Surface

Used for visualizing a 3-D graph on a 2-D system of coordinates, i.e.,

showing different z-levels on an x-y system of coordinates by its contour lines

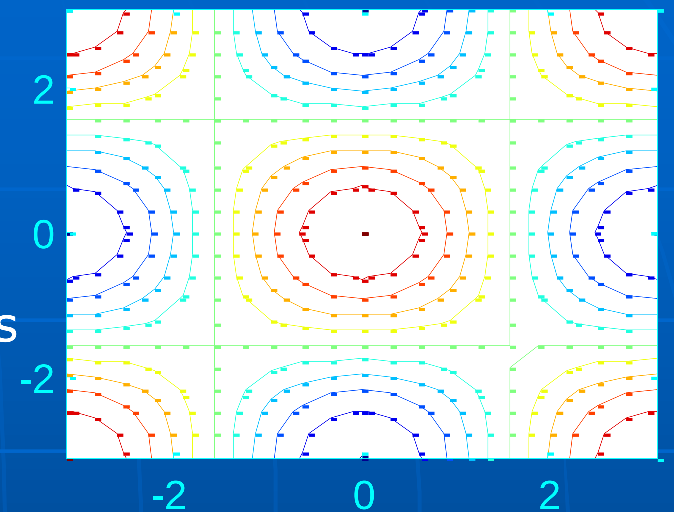
Ex: contour(x,y,z)

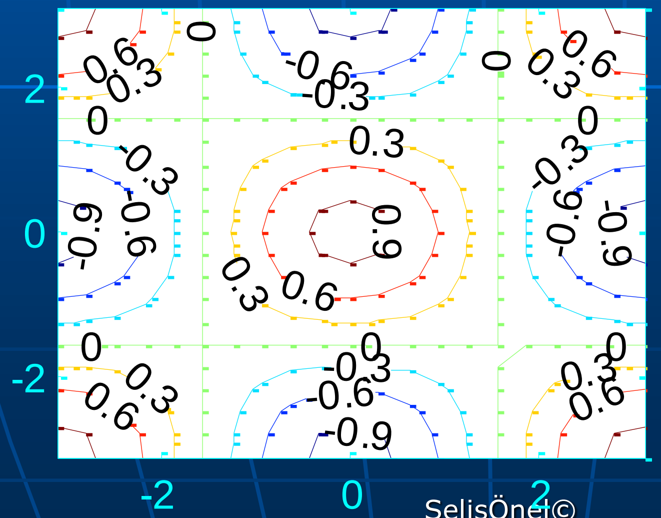
MATLAB® Plotting Command: **contour**

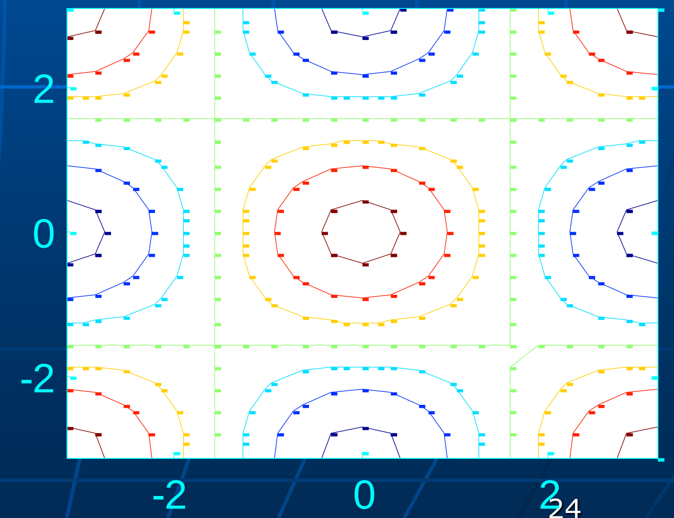
- **contour**(Z) is a contour plot of matrix Z treating the values in Z as heights above a plane. A contour plot are the level curves of Z for some values V. The values V are chosen automatically
- **contour**(X,Y,Z) X and Y specify the (x,y) coordinates of the surface as for **surf**
- **contour**(Z,N) and **contour**(X,Y,Z,N) draw N contour lines, overriding the automatic value
- **contour**(Z,V) and **contour**(X,Y,Z,V) draw **length**(V) contour lines at the values specified in vector V
- Use **contour**(Z,[v v]) or **contour**(X,Y,Z,[v v]) to compute a single contour at the level v.

2.5-D Graphs: Surface

```
figure(5)
subplot(2,2,2), contour(x,y,z)
% Show only the levels required
subplot(2,2,4), contour(x,y,z,[-0.9:.3:.9])
% Show the level of values on the contour lines
subplot(2,2,3),
[c,h]=contour(x,y,z,[-0.9:.3:.9])
clabel(c,h)
```







Other Plot Commands

waterfall(...): Same as mesh(...) except that the column lines of the mesh are not drawn - thus producing a "waterfall" plot. For column-oriented data analysis, use waterfall(z') or waterfall(x',y',z')

ribbon: Draws 2-D lines as ribbons in 3-D

ribbon(x,y) is the same as plot(x,y) except that the columns of y are plotted as separated ribbons in 3-D. ribbon(y) uses the default value of x=1:size(y,1).

ribbon(x,y,width) specifies the width of the ribbons to be width. The default value is width = 0.75;

Other Plot Commands

spy : Zero/nonzero values, i.e., it visualizes sparsity pattern.

spy(S) plots the sparsity pattern of the matrix S

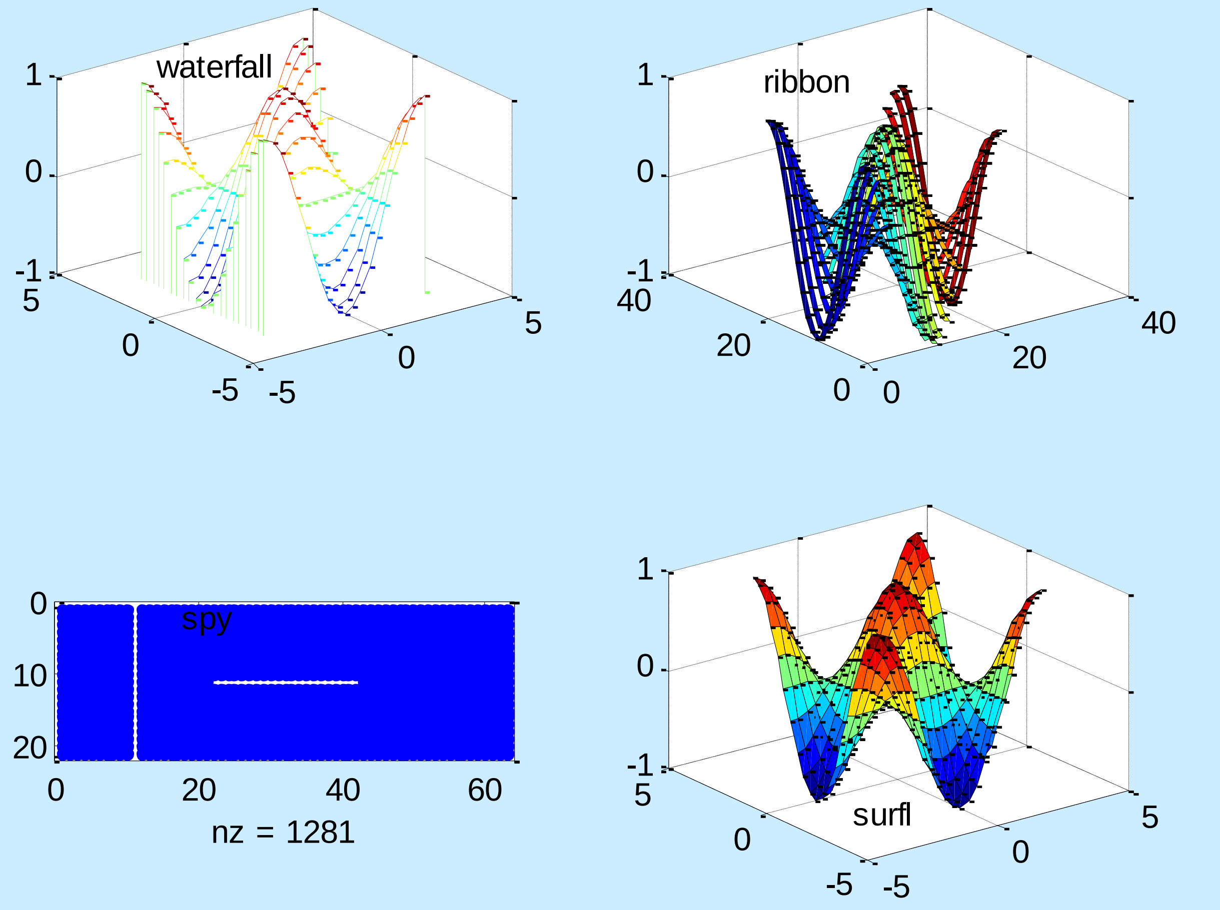
spy(S,'LineSpec') uses the color and marker from the line specification string 'LineSpec'

surfl : 3-D shaded surface plot with light effects

same as surf(...) except that it draws the surface with highlights from a light source

Other Plot Commands

```
figure(6)
subplot(2,2,1)
waterfall(x,y,z),
text(-5,-5,-1,'waterfall')
subplot(2,2,2)
ribbon(z);
text(0,0,-1,'ribbon');
subplot(2,2,3)
S=[x y z]; spy(S);
gtext('spy');
subplot(2,2,4)
surfl(x,y,z,'light');
text(-5,-5,-1,'surfl');
```



Data Export and Import

```
A=magic(3); B=magic(4);
% save all variables in MATLAB® workspace
save f1
% to save only certain variables
save f2 A
%These files have .mat extension and can only be
   retrieved by MATLAB®
% to load use:
load f1
% To save data as text
save f3 B –ascii
```

Data Export and Import

fprintf : Writes formatted data to file

count = fprintf(fid, format, A, ...) formats the data in the real part of matrix A (and in any additional matrix arguments) under control of the specified format string, and writes it to the file associated with file identifier fid

fid = fopen(filename, mode)

```
>> x = 0:.1:1; y = [x; exp(x)];
 fid = fopen('exp.txt','wt');
 fprintf(fid,'%6.2f  %12.8f\n',y);
 fclose(fid);
```

```
>>   type exp.txt
  0.00    1.00000000
  0.10    1.10517092
  0.20    1.22140276
  0.30    1.34985881
  0.40    1.49182470
  0.50    1.64872127
  0.60    1.82211880
  0.70    2.01375271
  0.80    2.22554093
  0.90    2.45960311
  1.00    2.71828183
```

Data Export and Import

fread : Reads binary data from file.

A = fread(FID) reads binary data from the specified file and writes it into matrix A. FID is an integer file identifier obtained from fopen.

A = fread(FID,SIZE,PRECISION) reads the file according to the data format specified by the string PRECISION. Valid entries for SIZE are:

N read N elements into a column vector.

inf read to the end of the file.

[M,N] read elements to fill an M-by-N matrix, in column order. N can be inf, but M can't.

Successive Substitution Iteration

- Nonlinear system of equations usually can be written in the same form as linear equations → Ax=y
- Then, the coefficient matrix **A** and the inhomogeneous term **y** may be dependent on the solution
- So, an iterative solution for a nonlinear system can be written as:

$$A_{n-1}x_n=y_{n-1}$$

Successive Substitution Iteration

$A_{n-1}x_n=y_{n-1}$ can be solved using successive substitution iteration, where

A_{n-1} : is computed using the most recent calculation result for x_n

x_n : is the nth iterative solution

y_{n-1} : is an inhomogeneous term assumed to be a function of x_n

Successive Substitution Iteration

Start the iteration with an initial guess for the solution x

Determine the coefficient matrix

Solve the system as a linear system

Get the solution x

Rearrange the coefficient matrix

Solve the system again

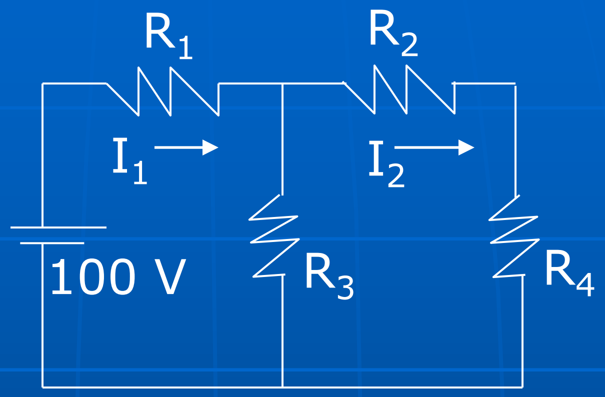
In case of instability:

Add the (under)relaxation parameter, i.e.

$x_n=\omega inv(A_{n-1})y_{n-1}+(1-\omega)x_{n-1}$ where $0<\omega<1$

Ex: Successive Substitution Iteration

Electric circuit between heating elements can be shown schematically as:



The resistance of the j^{th} heating element is a function of temperature:

$$R_j = a_j + b_j T_j + c_j T_j^2$$,where

a_j, b_j, c_j : constants

T_j :temperature of the j^{th} element

Temperature of each heating element is determined by:

$$I_j^2 R_j = A_j \sigma (T_j^4 - T_\infty^4) + A_j h (T_j - T_\infty)$$, where

T_∞ : temperature of the surrounding environment

A_j : the surface area of the j^{th} element

Ex: Successive Substitution Iteration

Electric currents (I_i) should satisfy:

$$(R_1 + R_3)I_1 - R_3I_2 = 100$$

$$-R_3I_1 + (R_2 + R_4 + R_3)I_2 = 0$$

These equations are in fact nonlinear as

I=f(T) from $R_j = a_j + b_jT_j + c_jT_j^2$ and T=f(R,I)

If T is low, nonlinear effects vanish and system becomes linear

Otherwise, we need to solve the nonlinear system…

Ex: Successive Substitution Iteration

Solution Algorithm:

First Solve

$$(R_1 + R_3)I_1 - R_3 I_2 = 100$$
$$-R_3 I_1 + (R_2 + R_4 + R_3) I_2 = 0$$

As a simultaneous linear system with initial (cold) values of resistances

Solve

$$I_j^2 R_j = A_j \sigma (T_j^4 - T_\infty^4) + A_j h (T_j - T_\infty)$$

for temperature

Calculate each resistance as a function of T

$$R_j = a_j + b_j T_j + c_j T_j^2$$

Convergence

$|R_k - R_{k-1}| < tol$

Resolve with updated values of resistances

$$(R_1 + R_3)I_1 - R_3 I_2 = 100$$
$$-R_3 I_1 + (R_2 + R_4 + R_3) I_2 = 0$$

END

Newton's Method

- Newton's method used for finding the root of a nonlinear equation can be extended to solving a system of nonlinear equations
- Ex: Intersection of a circle with a parabola

$$f(x,y)=x^2+y^2-1 \quad \text{and} \quad g(x,y)=x^2-y$$

Start with an initial estimate of a common solution, (x_0,y_0)

The plane tangent to function $z=f(x,y)$ at $(x_0,y_0,f(x_0,y_0))$ is given by (using 1st order truncated Taylor's expansion):

$$z-f(x_0,y_0)=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

$f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ are partial derivatives of $f(x,y)$ wrt x and y, evaluated at (x_0,y_0)

The plane tangent to function $z=g(x,y)$ at $(x_0,y_0,g(x_0,y_0))$ is given by:

$$z-g(x_0,y_0)=g_x(x_0,y_0)(x-x_0)+g_y(x_0,y_0)(y-y_0)$$

Newton's Method

- To find an approximation to the desired solution, the intersection of these two planes
  $z1-f(x_0,y_0)=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$
  $z2-g(x_0,y_0)=g_x(x_0,y_0)(x-x_0)+g_y(x_0,y_0)(y-y_0)$
  with the xy plane at z=0 need to be found

- Let $r=x-x_0$ and $s=y-y_0$ and solve the linear system

  $f_x(x_0,y_0)r+f_y(x_0,y_0)s=-f(x_0,y_0)$
  $g_x(x_0,y_0)r+g_y(x_0,y_0)s=-g(x_0,y_0)$

- $x=r+x_0$ and $y=s+y_0$ (note that r and s give the amount of displacement the location of the intersection point (x,y) from the point (x_0,y_0)
  $(x,y) = (r+x_0, s+y_0)$

Newton's Method

Ex contd.: $f(x,y)=x^2+y^2-1$ and $g(x,y)=x^2-y$

$f_x=2x$ $f_y=2y$ $g_x=2x$ $g_y=-1$

- Choosing initial estimate: $(x_0,y_0)=(1/2,1/2)$ gives $f=-1/2$, $g=-1/4$, $f_x=1$, $f_y=1$, $g_x=1$, $g_y=-1$, with the xy plane at z=0 need to be found
- Then the linear system
  $f_x(x_0,y_0)r+f_y(x_0,y_0)s=-f(x_0,y_0)$
  $g_x(x_0,y_0)r+g_y(x_0,y_0)s=-g(x_0,y_0)$ becomes

  $1*r+1*s=-(-1/2)$ → $r+s=1/2$
  $1*r -1*s=-(-1/4)$ → $r-s=1/4$ which gives $r=3/8$ and $s=1/8$
- $x=r+x_0$ and $y=s+y_0$, so the second approximate solution (x_1,y_1) is:
  $x_1=x_0+r= ½+3/8=7/8$
  $y_1=y_0+s= ½+1/8=5/8$
- Using the new approximate solution (x_1,y_1)→ f,g, and f_x,f_y,g_x,g_y are calculated
  Iterations are continued until $|\Delta x|<$tolerance

Quiz: Newton's Method

Find the roots of the following nonlinear system of equations using Newton's method.

Start with initial guess values of [3/4 2/3]

Show three consecutive iterations and check convergence for a tolerance of 0.0001

$$f(x,y)=x^2+y^2-1$$

$$g(x,y)=x^2-y$$

Newton's Method

At each iteration, an updated approximate solution vector x_{new} is found using the current approximate solution x from:

$$x_{new}=x-J^{-1}(x)f(x)$$

Where the Jacobian J is:

$$J(x_1,\ldots,x_n)= \begin{matrix} \partial f_1/x_1 & \ldots & \partial f_1/\partial x_n \\ \ldots & & \ldots \\ \partial f_n/\partial x_n & \ldots & \partial f_n/\partial x_n \end{matrix}$$

$J(x)y=-f(x)$

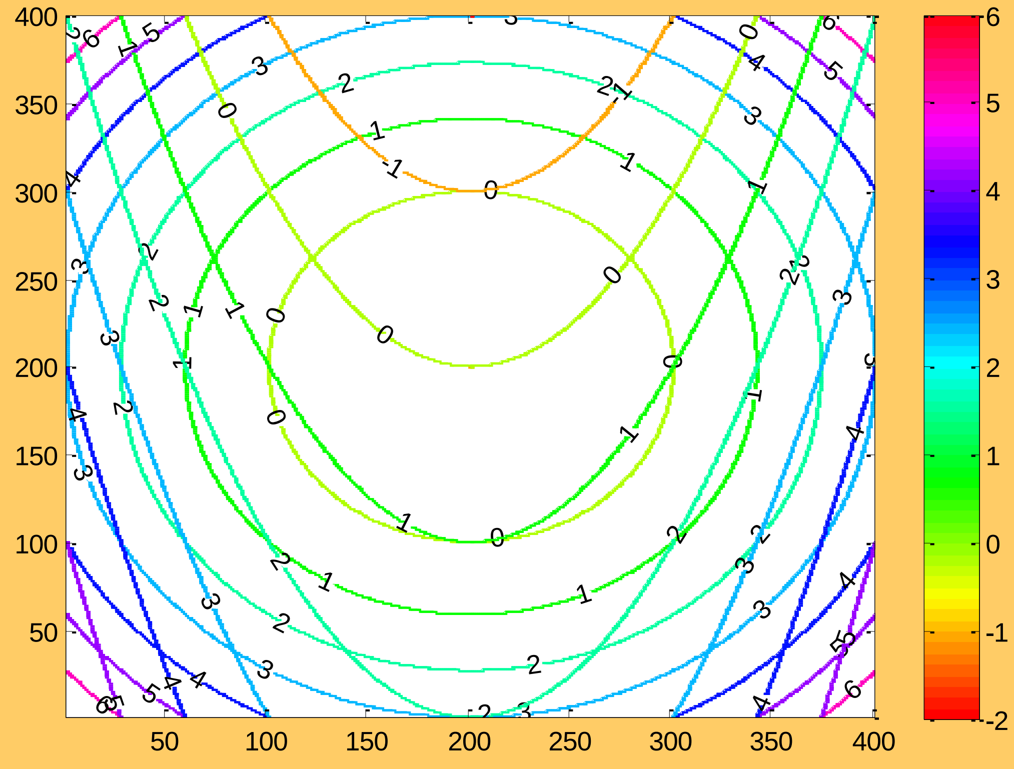
$x_{new}= x+y$

Newton's Method in MATLAB®

```
function x=funNewtonMulti(fun,Jac,x0,tol,kmax)
%Adopted from Fausett, 2nd Ed.,p.256
xold=x0; it=1;
while  (it<=kmax)
   y=-feval(Jac, xold)\feval(fun,xold);
   xnew=xold+y';        dif=norm(xnew-xold);
   disp([it    xnew     dif]);
   if   dif<=tol
      x=xnew;  disp('Newton method has converged');
   return,
   else
      xold=xnew;
   end
   it=it+1;
end
disp('Newton method did not converge');
x=xnew;
```

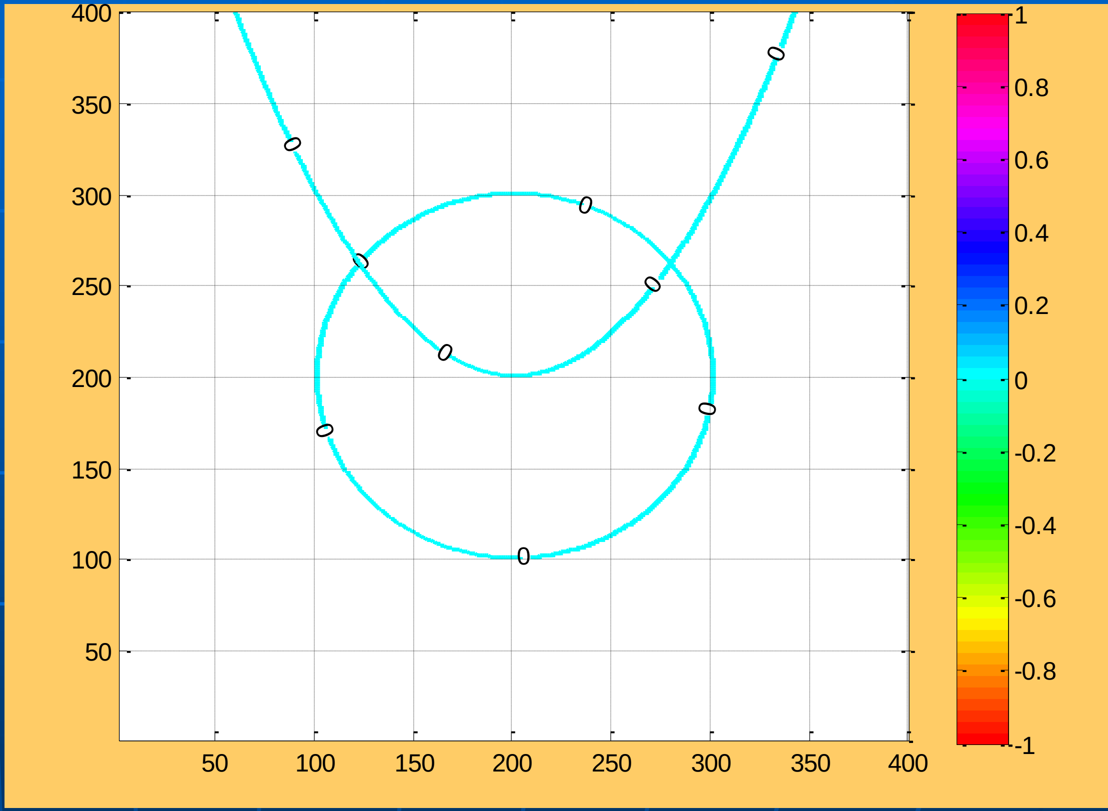
2.5-D Graph in MATLAB®

```
>> x1=-2:0.01:2;
>> x2=-2:0.01:2;
>> [x,y]=meshgrid(x1,x2);
>> f1=x.^2+y.^2-1 ;
>> f2=x.^2-y;
>> [c,h] = contour(f1);
    clabel(c,h),
   colormap hsv, hold on,
>> [c,h] = contour(f2);
    clabel(c,h),
   colorbar, hold off
```



2.5-D Graph in MATLAB®

```
>> x1=-2:0.01:2;
>> x2=-2:0.01:2;
>> [x,y]=meshgrid(x1,x2);
>> f1=x.^2+y.^2-1 ;
>> f2=x.^2-y;
>>  [c,h] = contour(f1,[.0000 .0000],'linewidth',2);
    clabel(c,h),
    colormap hsv,
    hold on,
>> [c,h] = contour(f2,[.0000 .0000],'linewidth',2);
    clabel(c,h),
    colorbar,
    hold off,
    grid on,
```



Newton's Method in MATLAB®

```
>> fun=inline('[x(1).^2+x(2).^2-1;
                              x(1).^2-x(2)]');
   Jac=inline('[2*x(1), 2*x(2); 2*x(1), -1]');
>> x=funNewtonMulti(fun,Jac,[1/2 1/2],0.0001,30)

    1.0000    0.8750    0.6250    0.3953
    2.0000    0.7907    0.6181    0.0846
    3.0000    0.7862    0.6180    0.0045
    4.0000    0.7862    0.6180    0.0000
Newton method has converged
x =    0.7862    0.6180
```

Checking the result:

```
>> x =0.7862; y=0.6180;
>> f1=x.^2+y.^2-1;  f2=x.^2-y;    f1,f2,
f1 =  3.4440e-005
f2 =  1.1044e-004
```

Summary of Newton's Method

$f_i(x_1, x_2, \ldots, x_n) = 0$ for $i = 1, 2, \ldots, n$
where f_i is a nonlinear function of x_js

If an initial guess of the solution is known, solution can be written as:

$$\mathbf{x_j^{(k+1)} = x_j^{(k)} + \Delta x_j}$$

where Δx_j is an unknown correction

Linearizing nonlinear equations:

Writing the 1st order truncated Taylor's expansion about $x_j^{(k)}$ gives…

Summary of Newton's Method

Writing the 1st order truncated Taylor's expansion about $x_j^{(k)}$ gives:

$$\sum_j \frac{\partial f_i}{\partial x_j} \Delta x_j = -f_i(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$$

where the partial derivatives are evaluated with the initial guesses.

This equation can be shown in matrix form as:

$$[J][\Delta x] = -[f], \quad \text{where} \quad [J] = \left[\frac{\partial f_i}{\partial x_j}\right], \quad [\Delta x] = \begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{pmatrix}, \quad [f] = \begin{pmatrix} f_1(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \\ \vdots \\ f_n(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \end{pmatrix}$$

The partial derivatives may be evaluated using difference approximation:

$$\frac{\partial f_i}{\partial x_j} \approx \frac{f_i(x_1^{(0)}, \ldots, x_j^{(0)} + \delta x_j, \ldots, x_n^{(0)}) - f_i(x_1^{(0)}, \ldots, x_j^{(0)}, \ldots, x_n^{(0)})}{\partial x_j},$$

where ∂x_j is an arbitrarily chosen small value

Ex: Plotting two functions in MATLAB®

```
clear, clf, hold off
x1=0:0.1:2;   y1=-2:0.1:2;   [x,y]=meshgrid(x1,y1);   [f1,f2]=funnonlin(x,y);
figure(1)
subplot(1,2,1)
mesh(f1,'linewidth',2),  hold on,   mesh(f2,'linewidth',2),
axis([min(x1) max(x1) min(y1) max(y1)  -10 10]);
xlabel('x'); ylabel('y'); zlabel('z');  grid on;    hold off,
subplot(1,2,2)
[c,h]=contour(x,y,f1,'-r','linewidth',2); clabel(c,h);
hold on
[c,h]=contour(x,y,f2,'linewidth',2); clabel(c,h);
hold off
axis([min(x1) max(x1) min(y1) max(y1)]);  xlabel('x'); ylabel('y'); grid on;  legend('f1','f2‘,2)

x2=0:0.1:20; y2=-2:0.1:20;  [x,y]=meshgrid(x2,y2);   [f1,f2]=funnonlin(x,y);
figure(2)
subplot(1,2,1)
mesh(f1),   hold on,    mesh(f2),  axis([min(x2) max(x2) min(y2) max(y2)  -10 10]);
xlabel('x'); ylabel('y'); zlabel('z');   grid on;    hold off,
subplot(1,2,2)
[c,h]=contour(x,y,f1,'-r','linewidth',2); clabel(c,h);
hold on
[c,h]=contour(x,y,f2,'linewidth',2); clabel(c,h);
hold off
axis([min(x1) max(x1) min(y1) max(y1)]); xlabel('x'); ylabel('y'); grid on;  legend('f1','f2')
```

Plotting two functions in MATLAB®