Forces in One and Two Dimensions

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The structured problem solving approach

Identify

What object is moving?

How is the position, $x(t)$, measured? (Origin and axes of coordinate system).

Find initial conditions, $x(t_0)$ and $v(t_0)$.

→

Model

Find the forces acting on the object.

Introduce models for the forces.

Apply Newton's second law of motion to find the acceleration, $a = a(x, v, t)$.

→

Solve

Solve the equation:

$$\frac{d^2x}{dt^2} = a(x, v, t) \, ,$$

with the initial conditions $x(t_0) = x_0$ and $v(t_0) = v_0$ using analytical or numerical techniques.

The solution gives the position and velocity as a function of time, $x(t)$, and $v(t)$.

→

Analyse

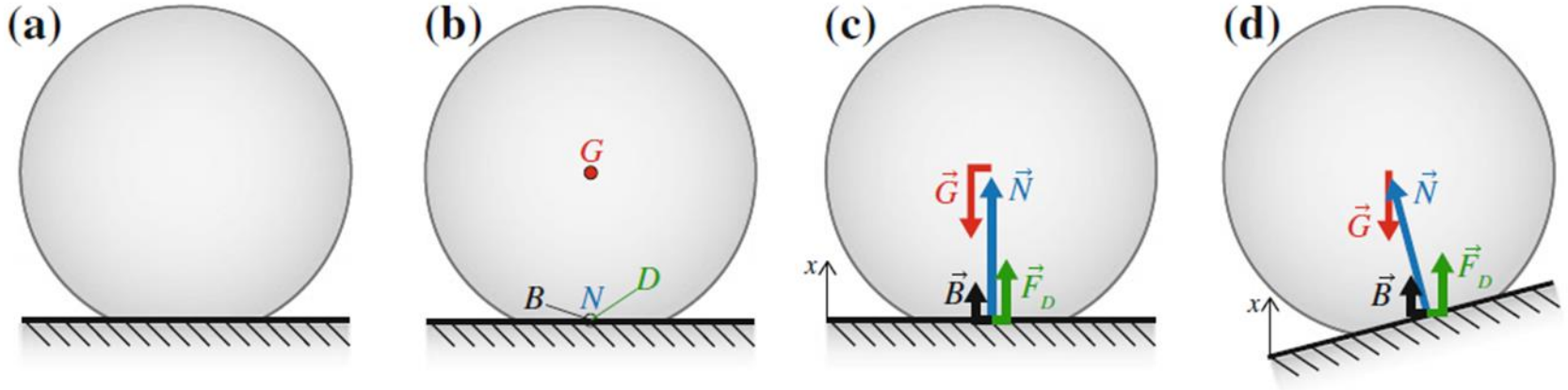
Check validity of $x(t)$ and $v(t)$.

Use $x(t)$ and $v(t)$ the answer questions posed.

Evaluate the answers.

Identifying Forces

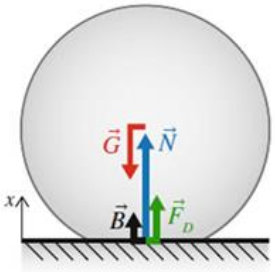
First, we need to discern between the object, also called the system, and the environment, which is everything else. In this case, the system is the ball, and the environment is everything else, such as the floor, the air surrounding the ball, and the Earth.



Newton's Second Law of Motion

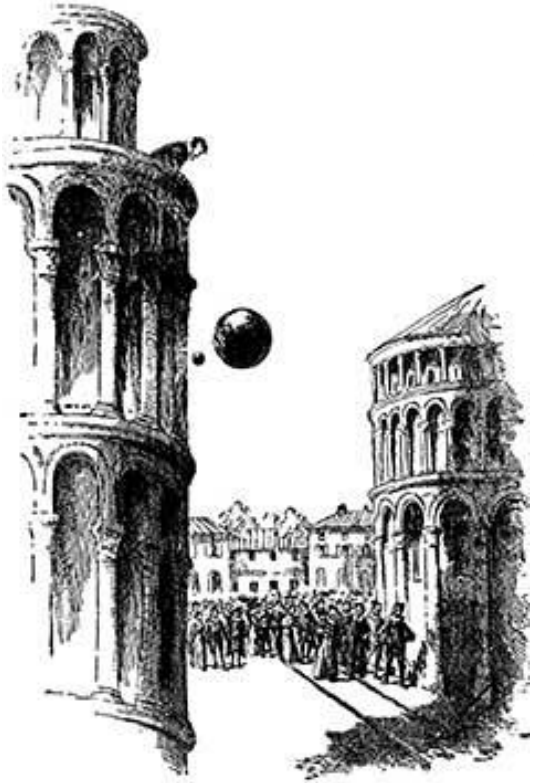
The force, F, in Newton's second law is the net external force acting on the object. By external we mean that the force has a cause outside the system, as we insisted when we drew a free-body diagram of an object. By net force we mean that if there are several forces acting on an object, it is the sum of all the external forces that causes the acceleration. We call this sum the net force:

$$\vec{F}_{net} = \sum_j \vec{F}_j = m\vec{a}$$

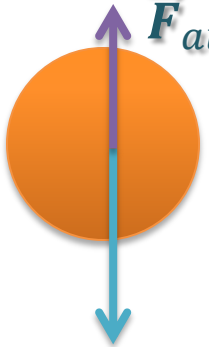


$$\vec{F}_{net} = \sum_j \vec{F}_j = \vec{G} + \vec{N} + \vec{F}_D + \vec{B} = m\vec{a}$$

Free Fall of an Object: An Experiment by Galileo



Galileo didn't know calculus (because Newton and Leibniz hadn't discovered it yet) so he couldn't derive the equation mathematically. Since we do know calculus we know that acceleration is the variation of velocity with time.

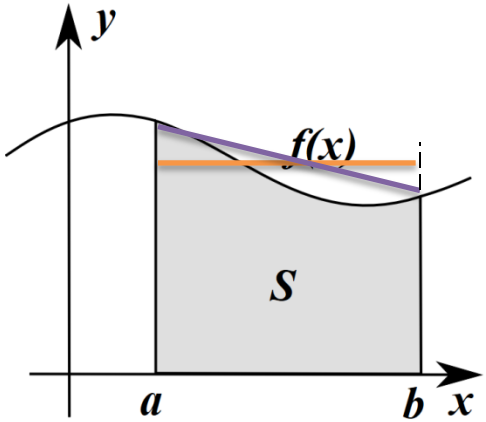


$$\vec{F}_{air\ drag} = \frac{1}{2}\rho A C |\vec{v}|^2 \hat{v}$$

$$\vec{F}_{net} = \sum_j \vec{F}_j = m\vec{a}.$$

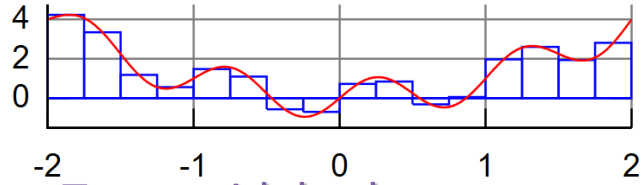
$$\vec{F}_{gravity} = m\vec{g}.$$

Numerical integration



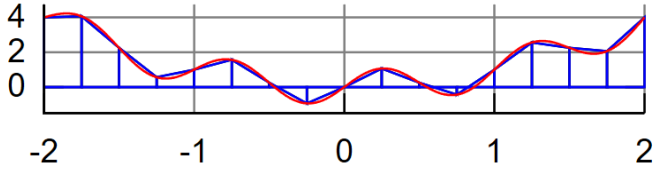
$$\int_a^b f(x)dx$$

Rectangle rule



$$\int_a^b f(x)dx \approx (b-a) f\left(\frac{a+b}{2}\right)$$

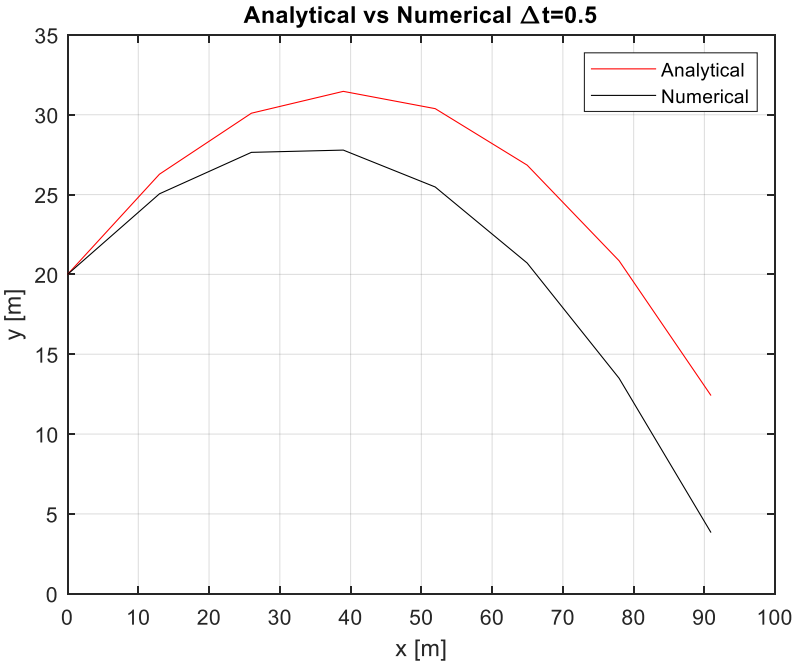
Trapezoidal rule



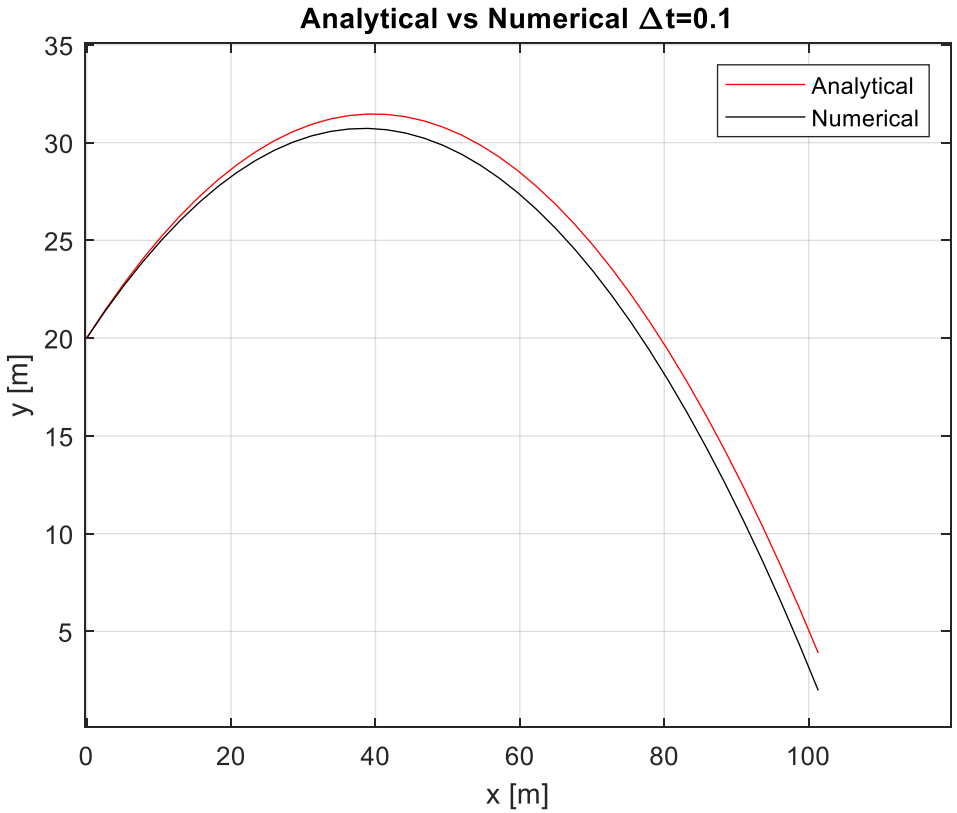
$$\int_a^b f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2}$$

Comparison of Analytical vs Numerical Methods

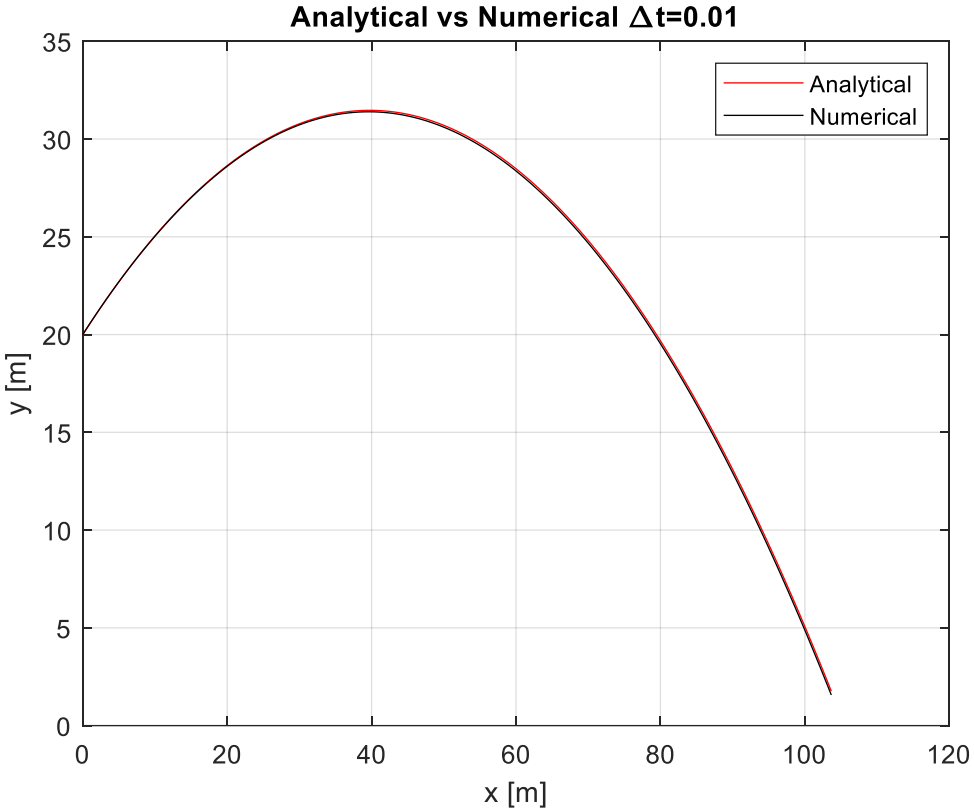
$$y_t = y_0 + v_0 sin(\alpha)t - \frac{1}{2}\vec{g}t^2 \qquad x_t = x_0 + v_0 \cos(\alpha)t$$



Comparison of Analytical vs Numerical Methods



Comparison of Analytical vs Numerical Methods

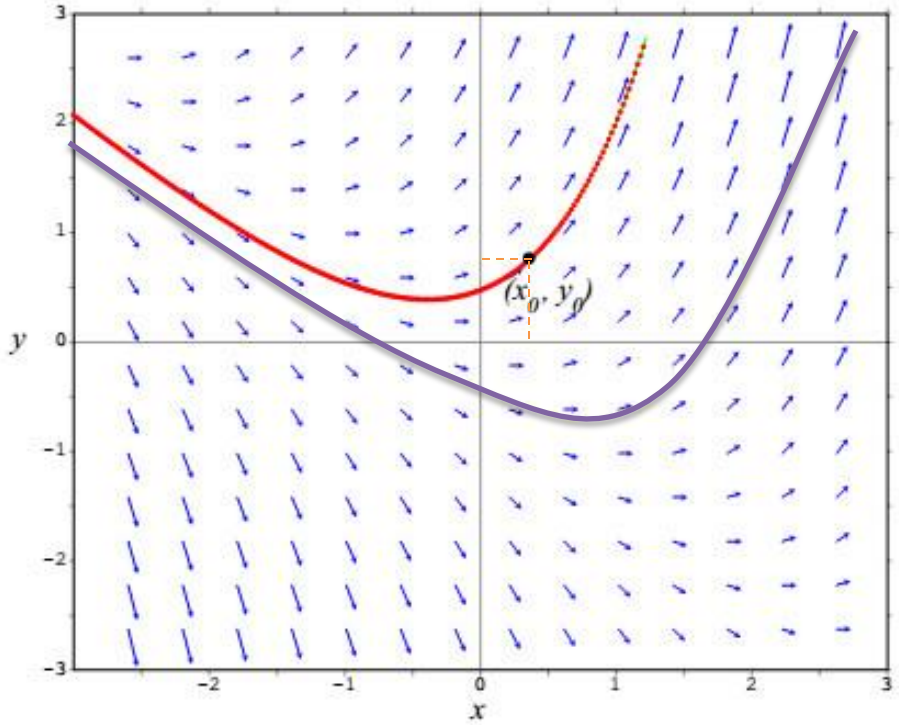


The vector field plot for the differential equation

$$\frac{dy}{dx} = y + x$$

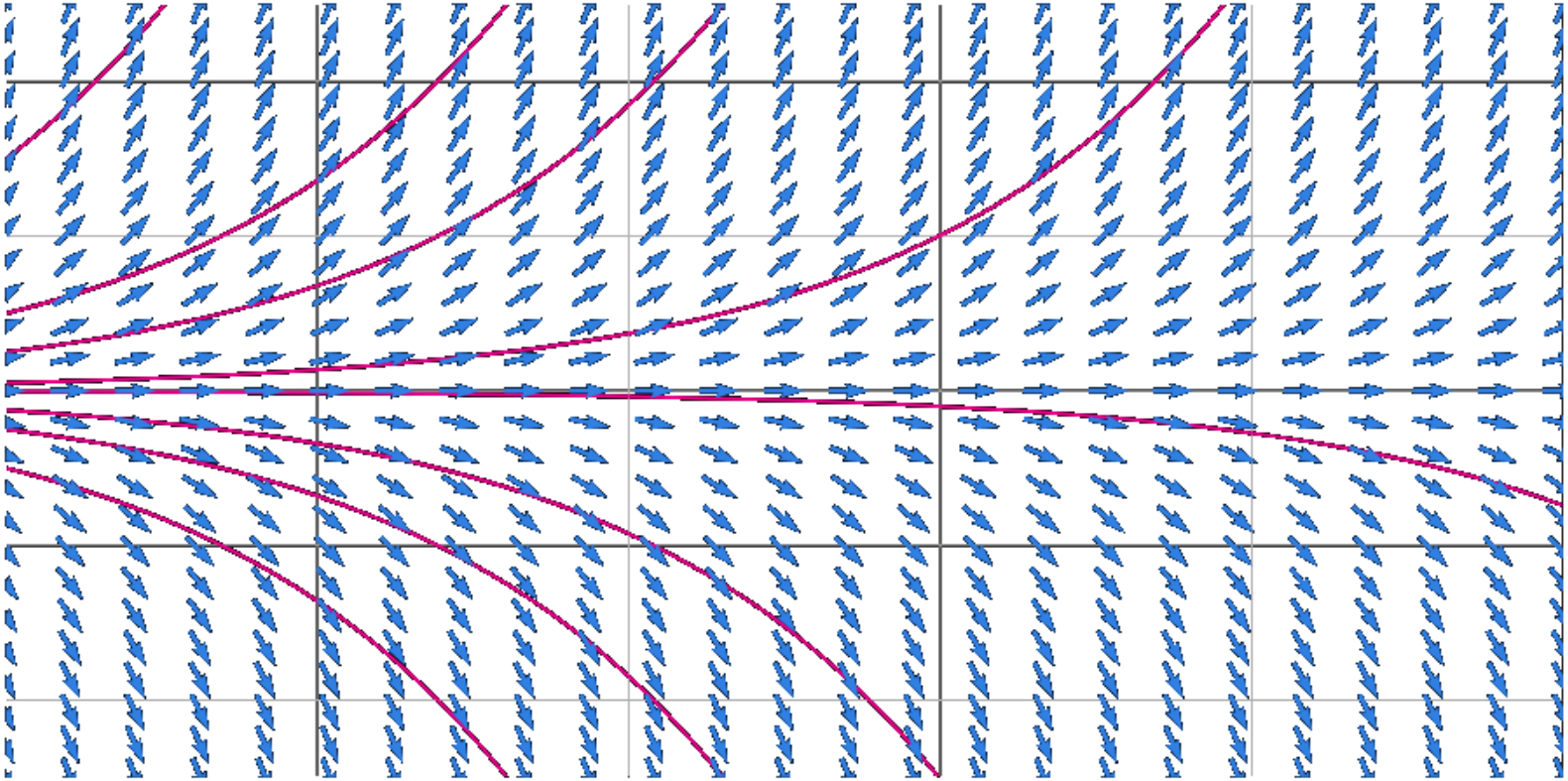
$$x_0 = 0.4$$

$$y_0 = 0.8$$



Initial-Value Problem : An IVP is a differential equation together with a place for a solution to start.

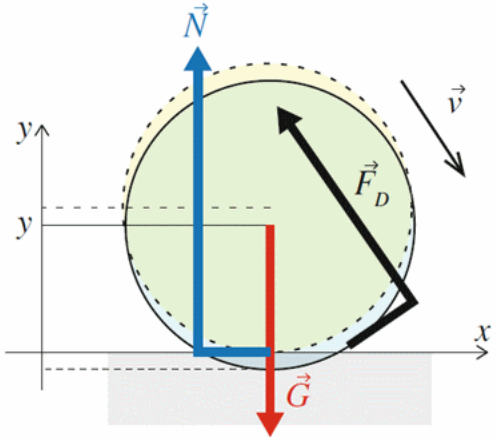
There are infinitely many integral curves, each corresponding to an integral curve.



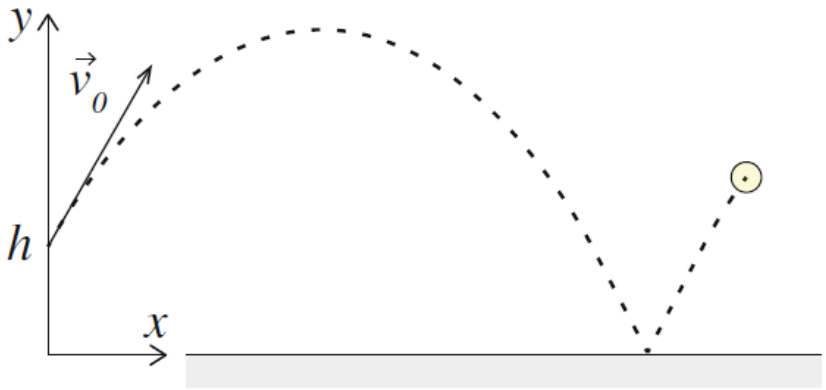
Motion of a Bouncing Ball with Air Resistance

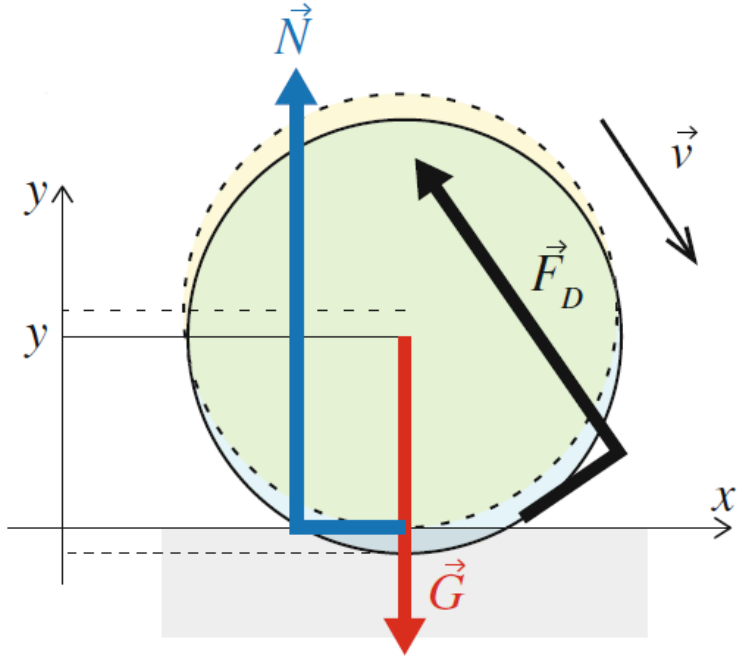
A ball is thrown from a height h above the ground with an initial velocity v_0. Find the velocity and position of the ball as a function of time t. Include the normal force from the floor while the ball is in contact with the floor. We describe the position of the ball by $\vec{\mathbf{r}}(t)$, measured in a coordinate system with origin at the floor. The initial position and velocity of the projectile is $\vec{\mathbf{r}}(t_0) = h\,\mathbf{j}$ and $\vec{\mathbf{v}}(t_0) = v_{x_0}\,\mathbf{i} + v_{y_0}\,\mathbf{j}$.

The motion of the ball is determined by the forces acting: air resistance, $\vec{F}_D$, the normal force $\vec{N}$ from the floor, and gravity, $\vec{\mathbf{G}} = -mg\mathbf{j}$, as illustrated in the free-body diagram. We use a square law for air resistance: $\vec{F}_D = -Dv^2$

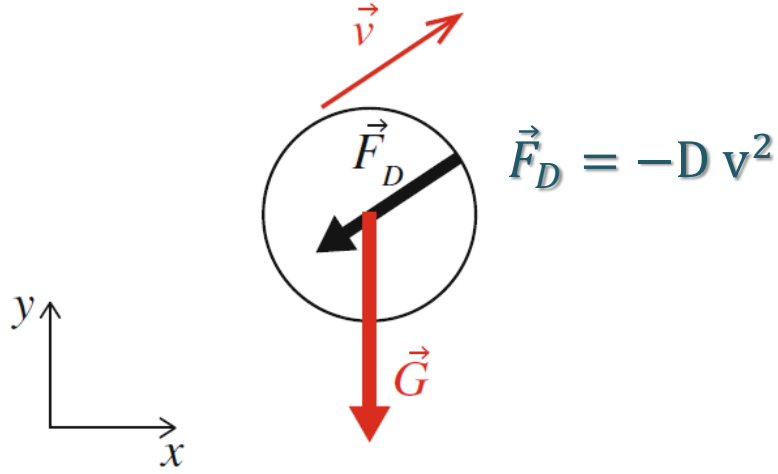


The normal force from the floor on the ball is represented by a spring force. This is a strong simplification of the actual deformation process occurring at the contact between the ball and the floor due to the deformation of both the ball and the floor.





Since the projectile will be moving fast, we use the square-law force model for the air resistance. For **a spherical object** we have that the **pre-factor** is $\boldsymbol{D \cong 3.0\rho d^2}$, where ***d*** is the diameter of the sphere and $\boldsymbol{\rho}$ is the density of the surrounding air. At sea level and at 15 °C air has a density of approximately 1.225 kg/m^3. Here, we will assume that the density of the surrounding air does not change significantly. We will use $\boldsymbol{\rho}$ = 1.225 kg/m^3. Let us also assume that the projectile has a diameter of **d** = 0.02 m, and that its mass is **m** = 0.2 kg.



$$\vec{F}_D = -\mathrm{D}\,\mathrm{v}^2$$

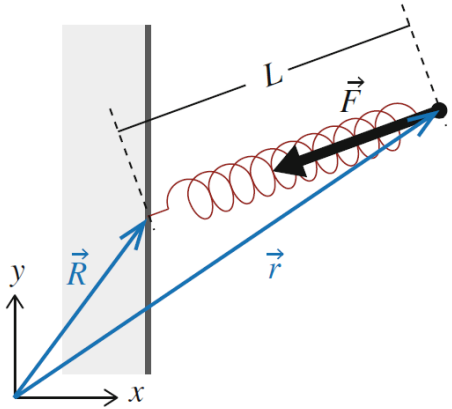
Values of Air Density As a Function of Altitude

Altitude (*m*)	Altitude (*ft*)	Density (kg/m^3)
0.0	0.0	1.225
305	1000	1.189
610	2000	1.154
914	3000	1.121
1219	4000	1.088
1524	5000	1.055
2134	7000	0.992
3048	10,000	0.905

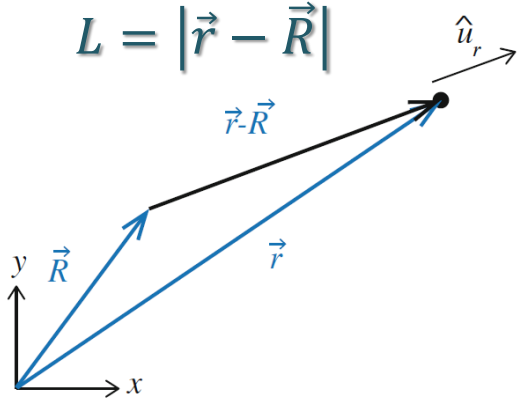
From one-dimensional experiments, we expect the force from a spring on the object attached to the spring to depend on the elongation of the spring and act in the direction of the spring. A spring is characterized by its equilibrium length, $\mathbf{L_0}$, and its spring constant, **k**. The force from the spring on the object is:

$$\vec{\boldsymbol{F}} = -\mathrm{k}(\mathrm{L} - L_0)\hat{\boldsymbol{u}}_r$$

where **L is the length of the spring**, and the unit vector $\hat{\boldsymbol{u}}_r$ points from the spring towards the object.

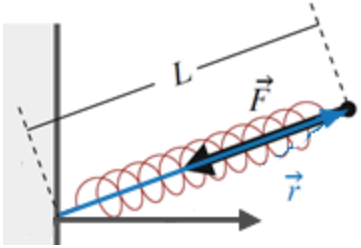


$$L = |\vec{r} - \vec{R}|$$



$$\hat{u}_r = \frac{\vec{r} - \vec{R}}{|\vec{r} - \vec{R}|}$$

Often, **we will place the origin at the attachment point of the spring**, so that R = 0, and the full model simplifies to:



$$\vec{F} = -k(r - L_0)\frac{\vec{r}}{r}$$

where the length of the spring, L = r = |r|, is the distance from the origin to the particle. We have named this model the "full model" because it most closely represents the behavior of a real, physical spring. This force model is versatile and general and can be widely applied. For example, it can be used to model the deformation of an elastic body. This model will be our preferred model for contact forces such as forces due to deformation in two- and three-dimensional systems.

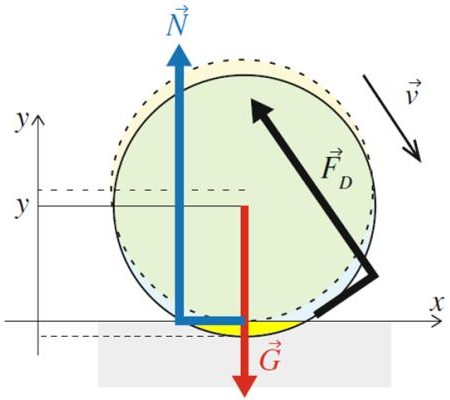
Notice that the force model has a spherical symmetry: When we choose the origin at the attachment point (R = 0), the force from the spring on the attached object always acts along a line through the origin, and the magnitude of the force depends on the distance r to the origin. This means that force on the particle from the spring in the x-direction is:

$$\vec{F}_x = -\mathrm{k}(\mathrm{r} - L_0)\frac{x}{r} = -\mathrm{k}\left(\sqrt{x^2 + y^2} - L_0\right)\frac{x}{\sqrt{x^2 + y^2}}$$

That is, the force in the x-direction, depends not only on the x-position, but also on the y coordinate. If we apply Newton's second law of motion to such a system, the acceleration of the object in the x-direction, will depend on the x, y, and z coordinates of the object: We call such a system **coupled**.

The deformed region corresponds roughly to the region of "***overlap***" between the ball and the floor. The depth of this region is $\Delta y = R - y(t)$, where R is the radius of the ball, which corresponds to the compression ΔL of the spring:

$$\vec{\mathbf{N}} = -\mathrm{k}\big(\mathrm{R} - \mathrm{y(t)}\big)\,\mathbf{j}$$



we must also ensure that the normal force only acts when the ball is in contact with the floor, otherwise the normal force is zero.

$$\vec{\mathbf{N}} = \begin{cases} -\mathrm{k}\big(\mathrm{R} - \mathrm{y(t)}\big)\,\mathbf{j} & \text{when } \mathrm{y}(t) < R \\ 0 & \text{when } \mathrm{y}(t) \geq R \end{cases}$$

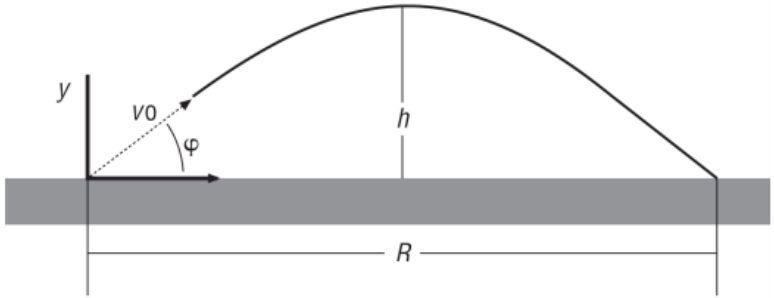
Newton's second law: Newton's second law is now

$$\sum_{\mathbf{j}} \vec{\mathbf{F}}_{\mathbf{j}} = \vec{\boldsymbol{G}} + \vec{\boldsymbol{F}}_D + \vec{\mathbf{N}} = \vec{\mathrm{G}} = -\mathrm{mg}\mathbf{j} - \mathrm{D}\,\mathrm{v}^2 + \vec{\mathbf{N}} = \mathrm{m}\vec{\boldsymbol{a}}$$

which gives

$$\vec{\boldsymbol{a}} = -\left(\frac{\mathrm{D}}{\mathrm{m}}\right)\mathrm{v}^2 - \mathrm{g}\,\mathbf{j} + \vec{\mathbf{N}}/\mathrm{m}$$

with the initial conditions: $\mathbf{r}(t_0) = \mathbf{r}(0) = \mathbf{r}_0$ and $\mathbf{v}(t_0) = \mathbf{v}(0) = \mathbf{v}_0$. While it is difficult to determine the motion analytically, we may be able to find analytical solutions for parts of the motion. However, we can determine the motion numerically by integrating using Euler's method.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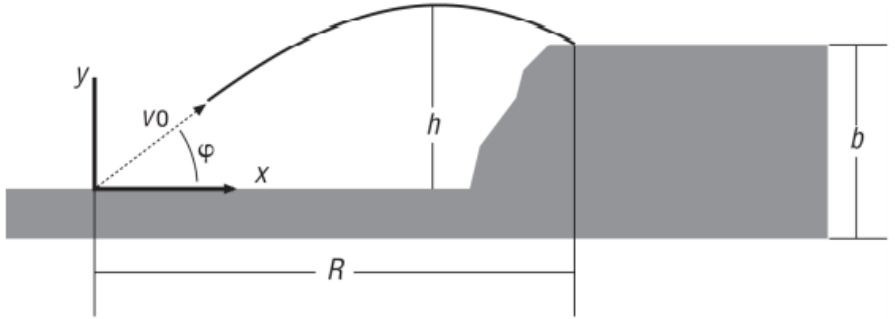
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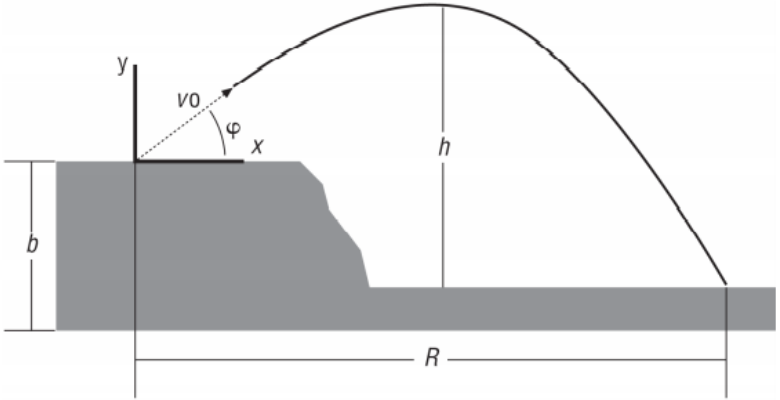
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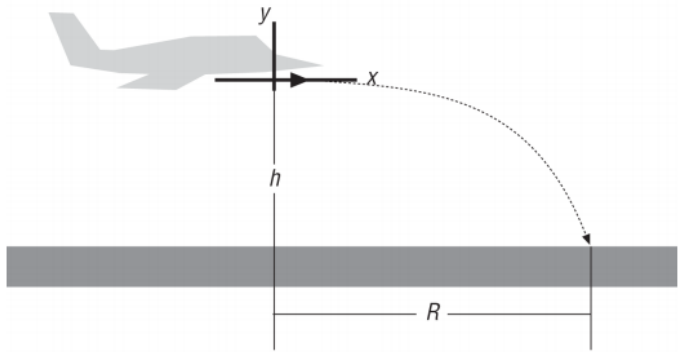
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$$\mathbf{v}(t_i + \Delta t) \cong \mathbf{v}(t_i) + \Delta t\, \mathbf{a}(t_i, \mathbf{r}(t_i), \mathbf{v}(t_i))$$

$$\mathbf{r}(t_i + \Delta t) \cong \mathbf{r}(t_i) + \Delta t\, \mathbf{v}(t_i + \Delta t)$$

The implementation is straight-forward:

may be able to find analytical solutions for parts of the motion. However, we can determine the motion numerically by integrating using Euler's method.

Normalization using **NumPy norm**

Normalization of a vector or a matrix is a common operation performed in a variety of scientific, mathematical, and programming applications. NumPy has a dedicated submodule called **linalg** for functions related to Linear Algebra.

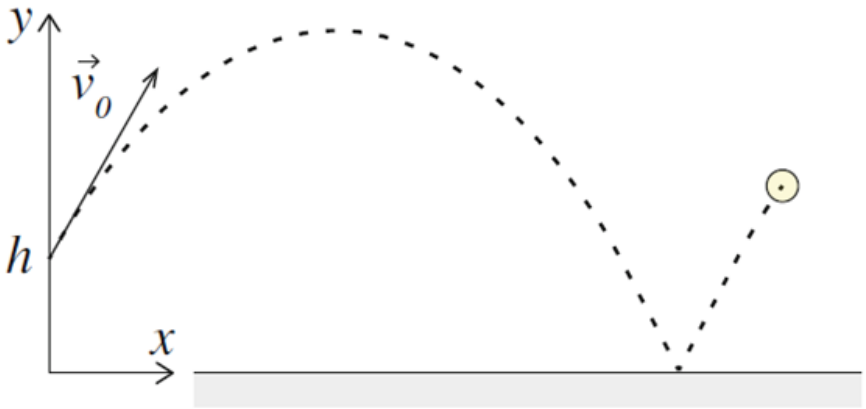
```
import numpy as np

a = np.array([1, 2, 3, 4, 5])

a_norm = np.linalg.norm(a)

print(a_norm)
```

$$||a||_2 = \sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2} = 7.4162$$

PHYSICS in COMPUTER ANIMATIONS and GAMES

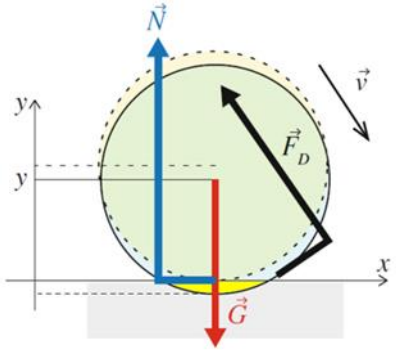
```
import matplotlib.pyplot as plt
import numpy as np
# Physical variables
m = 0.2                  # mass of the ball in kg
g = 9.81                 # gravitational acceleration of the Earth in m/s^2
alpha = np.radians(75)   # Initial angle in radian
v0 = 30                  # Initial velocity in m/s
y0 = 0                   # Initial position in meters
diam = 0.02              # diameter of the ball in meters
rho = 1.225              # air density in kg/m^3
D = 3.0*rho*diam**2      # drag coefficient
R = (diam/2)             # ball contact starts - equilibrium length of Spring
k = 1000.0               # stiffnes of the spring N/m
time = 10.0              # simulation time
dt = 0.01                # time step
```

```
# Numerical initialization
n = int(np.ceil(time/dt))
a = np.zeros((n, 2),float)
v = np.zeros((n, 2),float)
r = np.zeros((n, 2),float)
t = np.zeros((n, 2),float)
# Set initial values
r[0,1] = y0                                  # initial position of the ball
v[0,:] = v0*np.cos(alpha), v0*np.sin(alpha)  # initial velocity in i, j
```

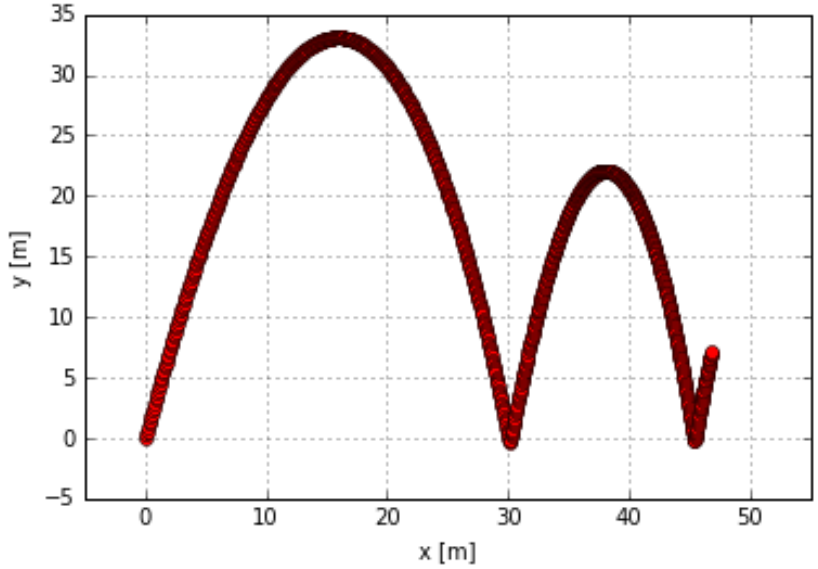


```
# Integration loop
for i in range(n-1):
    if (r[i,1] < R):
        N = k*(R-r[i,1])*np.array([0,1])
    else:
        N = np.array([0,0])
    FD = - D*np.linalg.norm(v[i,:])*v[i,:]
    G = -m*g*np.array([0,1])
    Fnet = N + FD + G
    a = Fnet/m
    v[i+1,:] = v[i] + a*dt
    r[i+1,:] = r[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
```

$$\vec{\mathbf{N}} = \begin{matrix} -\mathrm{k}\big(\mathrm{R}-\mathrm{y}(\mathrm{t})\big)\,\mathbf{j} & \text{when } \mathrm{y}(t) < R \\ 0 & \text{when } \mathrm{y}(t) \geq R \end{matrix}$$



```
# Plotting the results,
fig, ax = plt.subplots()
ax.plot(r[:,0],r[:,1], 'ro')
ax.axis([0, 50, 0, 50]), ax.axis('equal')
ax.set_xlabel('x [m]'), ax.set_ylabel('y [m]'), ax.grid(True)
```



Classworks

Use different values for;

mass – **`m`** [kg],
diameter – **`diam`** [m],
air density – **`rho`** [kg/m^3],
gravity – g [m/s^2]

and plot the results.

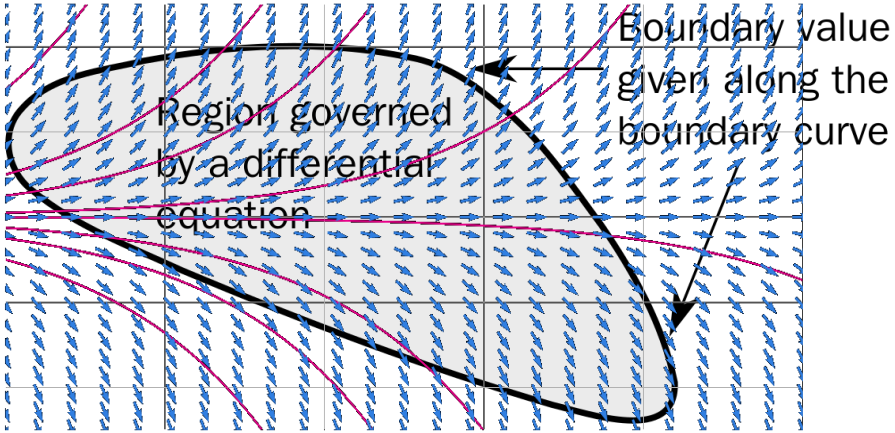
PHYSICS in COMPUTER ANIMATIONS and GAMES

Flock of Birds with the mass and the diameters

mass	12	8	7	5	5	4	4	2	1
size	Large	Large Medium	Large Medium	Medium	Medium	Medium	Medium Small	Small	Small

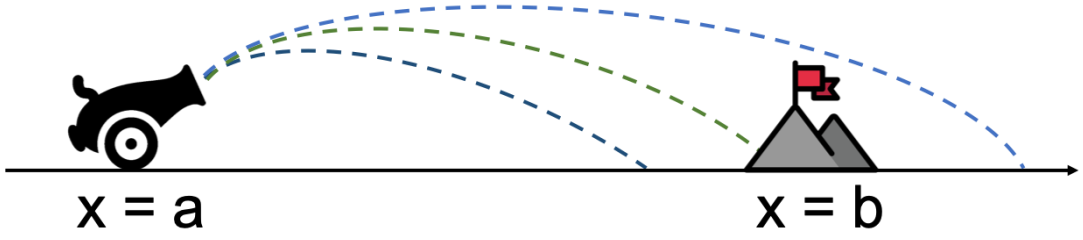
Boundary Value Problems

In mathematics, in the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.



Boundary Value Problems - The Shooting Methods

The shooting methods are developed with the goal of transforming the ODE boundary value problems to an equivalent **initial value problems** (**IVP**), then we can solve it using the methods that we learned. In the IVP, we can start at **the initial value** and **march forward to get the solution**. But this method is not working for the boundary value problems, because there are not enough initial value conditions to solve the ODE to get a unique solution. Therefore, the shooting methods was developed to overcome this difficulty.



The Shooting Methods

We are going out to launch a rocket, and let **y(t)** is the altitude (meters from the surface) of the rocket at time **t**. We know the gravity **g = 9.8 m/s²**. If we want to have the rocket at **50 m** off the ground after **5** seconds after launching, what should be the velocity at launching? (we ignore the drag of the air resistance). The problem is a boundary value problem for a second-order ODE. The ODE is:

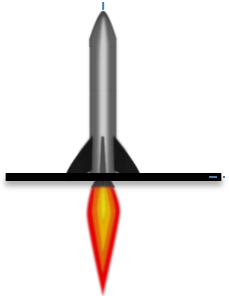
$$\frac{d^2y}{dt^2} = -g$$

with the two **boundary conditions** are: $y(0)=0$ and $y(5)=50$

y(5)=50 m

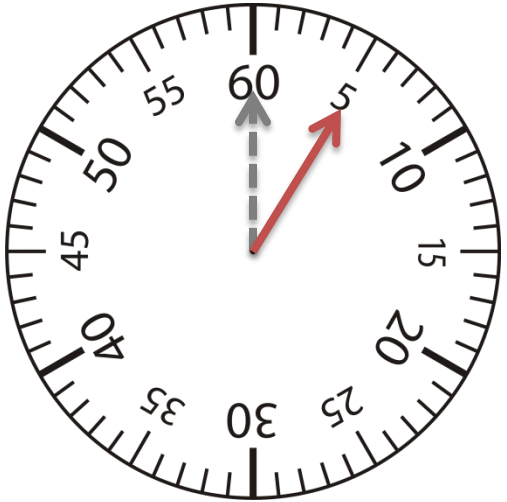
Equation of Motion

$$\frac{d^2y}{dt^2} = -g$$



y(0)=0 m

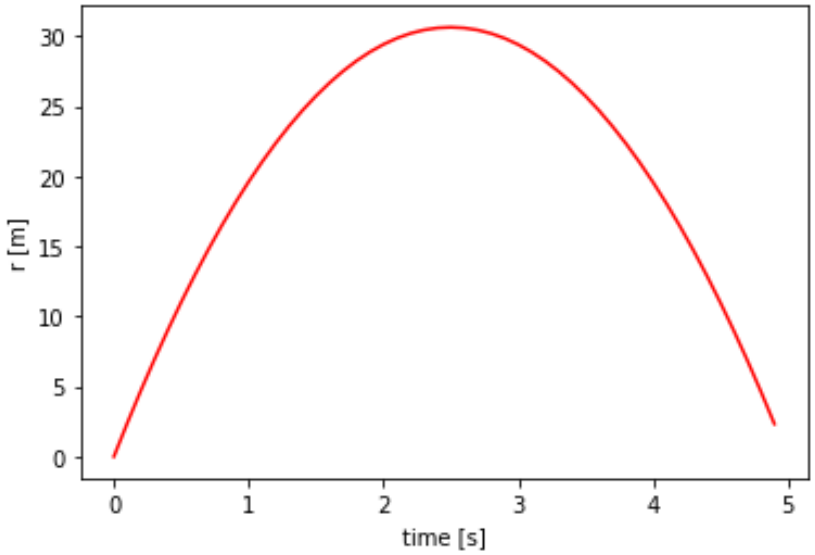
V_0= ?



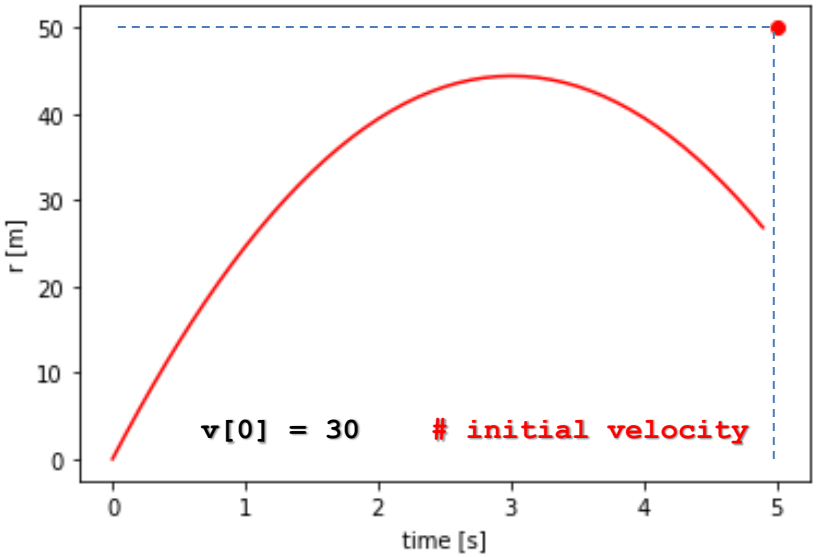
boundary conditions are: y(0)=0 and y(5)=50

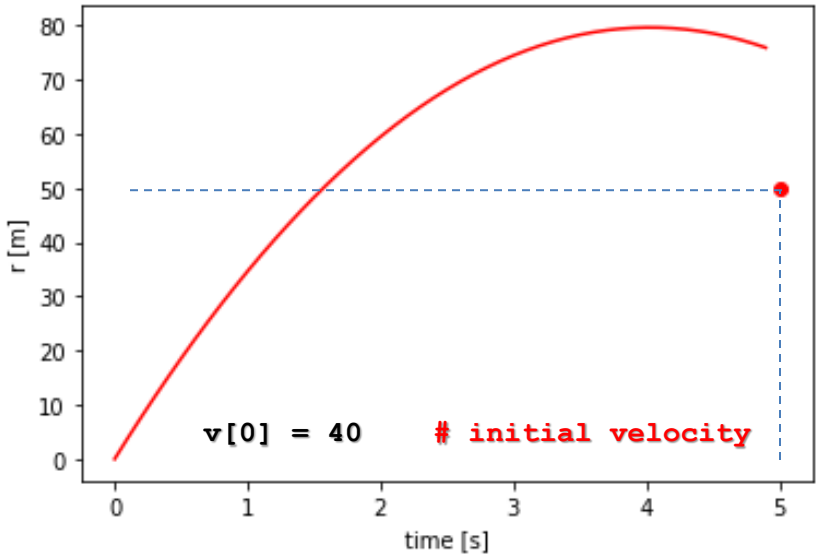
PHYSICS in COMPUTER ANIMATIONS and GAMES

```
import numpy as np
import matplotlib.pyplot as plt
# Physical variables
g = 9.81          # gravity
time = 5.0        # Simulation Time
dt = 0.1          # (h) time step
# Numerical initialization
n = int(np.ceil(time/dt))
a = np.zeros(n, float)
v = np.zeros(n, float)
r = np.zeros(n, float)
t = np.zeros(n, float)
# Set initial values
r[0] = 0      # meters
v[0] = 25     # initial velocity
a[:] = -g     # constant acceleretion
# Integration loop
for i in range(n-1):
    v[i+1] = v[i] + a[i]*dt
    r[i+1] = r[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
# Position Plotting
fig, ax = plt.subplots()
ax.plot(t, r, '-r')
ax.set_xlabel('time [s]')
ax.set_ylabel('r [m]')
plt.show()
```



PHYSICS in COMPUTER ANIMATIONS and GAMES



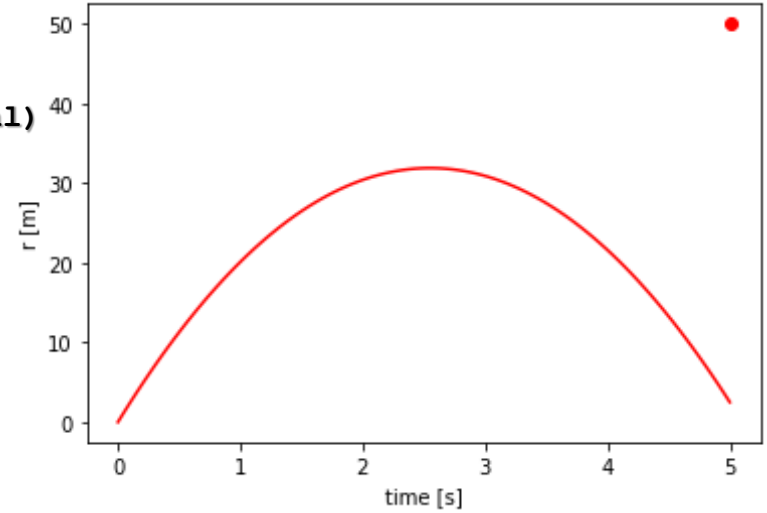


scipy.integrate.solve_ivp

Solve an initial value problem for a system of ODEs.

This function numerically integrates a system of ordinary differential equations given an initial value

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

F = lambda t, s: np.dot(np.array([[0,1],[0,-9.8/s[1]]]), s)

t_span = np.linspace(0, 5, 100)
y0 = 0
v0 = 25
t_eval = np.linspace(0, 5, 100)
sol = solve_ivp(F, [0, 5], [y0, v0], t_eval = t_eval)

# Position Plotting
fig, ax = plt.subplots()
ax.plot(sol.t, sol.y[0], '-r')
ax.plot(5, 50, 'ro')
ax.set_xlabel('time [s]')
ax.set_ylabel('r [m]')
plt.show()
```



PHYSICS in COMPUTER ANIMATIONS and GAMES

```
import numpy as np
from scipy.integrate import solve_ivp
from scipy.optimize import fsolve

f = lambda t, s: np.dot(np.array([[0,1],[0,-9.8/s[1]]]),s)

y0 = 0
v0 = 40
t_eval = np.linspace(0, 5, 100)

def objective(v0):
    sol = solve_ivp(f, [0, 5], [y0, v0], t_eval = t_eval)
    y = sol.y[0]
    return y[-1] - 50

v0, = fsolve(objective, 10)
print(v0)
```

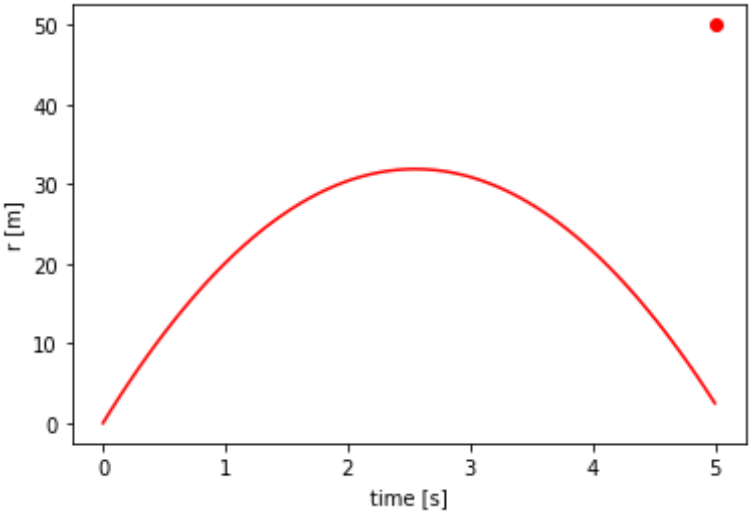
```
>>> 34.49999999999999
```

PHYSICS in COMPUTER ANIMATIONS and GAMES

scipy.integrate.solve_ivp

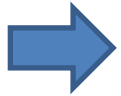
Solve an initial value problem for a system of ODEs.
This function numerically integrates a system of ordinary differential equations given an initial value

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

f = lambda t, s: np.dot(np.array([[0,1],[0,-9.8/s[1]]]), s)

t_span = np.linspace(0, 5, 100)
y0 = 0
v0 = 25
t_eval = np.linspace(0, 5, 100)
sol = solve_ivp(f, [0, 5], \
                [y0, v0], t_eval = t_eval)

# Position Plotting
fig, ax = plt.subplots()
ax.plot(sol.t, sol.y[0], '-r')
ax.plot(5, 50, 'ro')
ax.set_xlabel('time [s]')
ax.set_ylabel('r [m]')
plt.show()
```



Find a solution to the system of equations:

$x_0 * \cos(x_1) = 4$
$x_1 * x_0 - x_1 = 5$



$x_0 * \cos(x_1) - 4 = 0$
$x_1 * x_0 - x_1 - 5 = 0$

scipy.optimize.fsolve
Find the roots of a function.
Return the roots of the (non-linear) equations defined by func(x) = 0 given a starting estimate.

```
import numpy as np
from scipy.optimize import fsolve

def func(x):
    return [x[0] * np.cos(x[1]) - 4,
            x[1] * x[0] - x[1] - 5]

# The starting estimate [1, 1] for the roots of func(x) = 0.
root = fsolve(func, [1, 1])
print(root)
print(np.isclose(func(root), [0.0, 0.0]))  # func(root) should be 0.0
```

```
[6.50409711 0.90841421]
[ True  True]
```

numpy.isclose(a, b, rtol=1.0000000000000001e-05, atol=1e-08, equal_nan=False)

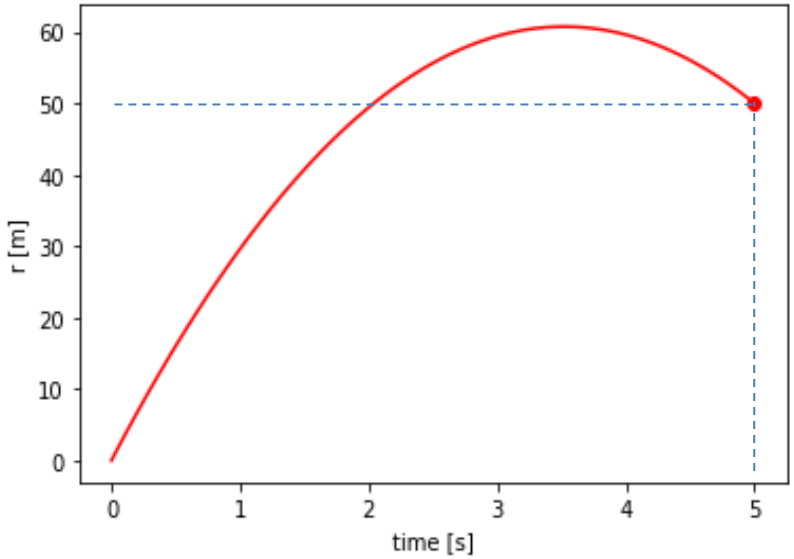
Returns a boolean array where two arrays are element-wise equal within a tolerance.

```
import numpy as np
from scipy.integrate import solve_ivp
from scipy.optimize import fsolve

f = lambda t, s: np.dot(np.array([[0,1],[0,-9.8/s[1]]]),s)

y0 = 0
v0 = 40
t_eval = np.linspace(0, 5, 100)

def objective(v0):
    sol = solve_ivp(f, [0, 5], [y0, v0], t_eval = t_eval)
    y = sol.y[0]
    return y[-1] - 50

v0, = fsolve(objective, 10)
print(v0)
```

```
>>> 34.49999999999999
```

PHYSICS in COMPUTER ANIMATIONS and GAMES

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

f = lambda t, s: np.dot(np.array([[0,1],[0,-9.8/s[1]]]), s)

t_span = np.linspace(0, 5, 100)
y0 = 0
v0 = 34.49999999999999
t_eval = np.linspace(0, 5, 100)
sol = solve_ivp(f, [0, 5], \
                [y0, v0], t_eval = t_eval)

# Position Plotting
fig, ax = plt.subplots()
ax.plot(sol.t, sol.y[0], '-r')
ax.plot(5, 50, 'ro')
ax.set_xlabel('time [s]')
ax.set_ylabel('r [m]')
plt.show()
```



PHYSICS in COMPUTER ANIMATIONS and GAMES

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

f = lambda t, s: np.dot(np.array([[0,1],[0,-9.8/s[1]]]), s)

y0 = 0
t_eval = np.linspace(0, 5, 100)

def objective(v0):
    sol = solve_ivp(F, [0, 5], \
            [y0, v0], t_eval = t_eval)
    y = sol.y[0]
    return y[-1] - 50

for v0_guess in range(1, 100, 10):
    v0, = fsolve(objective, v0_guess)
    print(f"Init: {v0_guess}, Result: {v0}")
```

```
Init: 1, Result: 34.499999999999986
Init: 11, Result: 34.499999999999986
Init: 21, Result: 34.49999999999999
Init: 31, Result: 34.49999999999998
Init: 41, Result: 34.49999999999999
Init: 51, Result: 34.499999999999986
Init: 61, Result: 34.499999999999986
Init: 71, Result: 34.499999999999986
Init: 81, Result: 34.499999999999986
Init: 91, Result: 34.499999999999986
```

Note that changing the initial guesses does not change the result, which means that this method is stable

Non-Linear Equations

A Non-linear equation is a type of equation. The degree in non-linear equations is two or more than two. The general equation of a linear equation is Ax + By + C = 0 is a linear equation. Other than that are a non-linear equation. The general equation is :

$Ax^2 + By^2 = C \Rightarrow Ax^2 + By^2 - C = 0$

Where A, B, and C are constants, x and y are variables. It forms a curve when it is plotted on a graph.

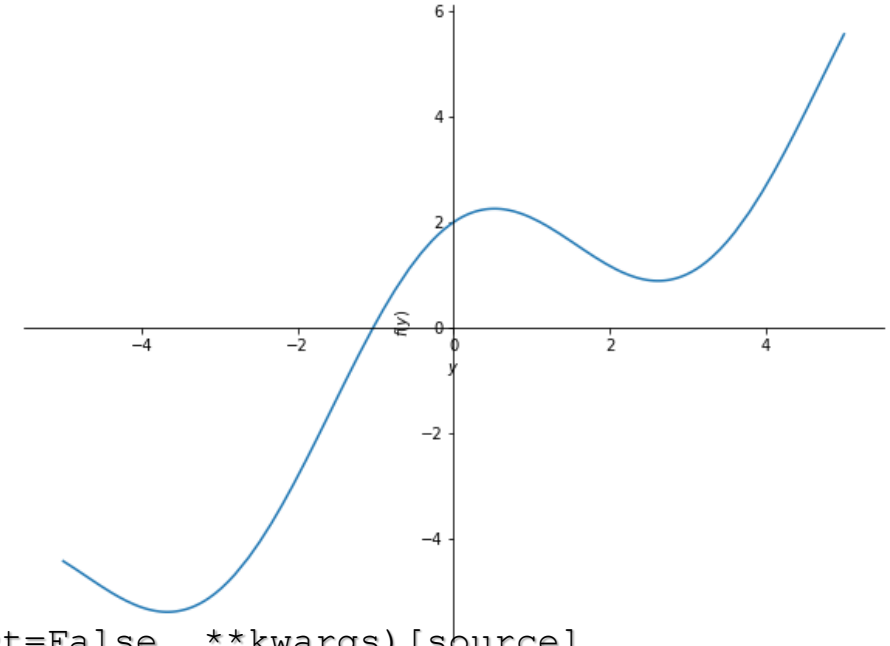
What is fsolve?

It is a function in a scipy module that returns the roots of non-linear equations:

```
scipy.optimize.fsolve (func, x0, args=(), fprime=None,
full_output=0, col_deriv=0, xtol=1.49012e-08, maxfev=0, band=None,
epsfcn=None, factor=100, diag=None)

Parameters
    func: It is a function that takes an argument and returns the value.
    x0: ndarray, It is a starting estimate for the root of fun(x)=0.
    args: Tuple, it is an extra argument to the function, optional.

Returns
    x: ndarray, It is a solution.
```

To find the roots of an equation y+2cos(y) starting point – 0.2?

```
from sympy import *

init_printing(use_unicode=True)

y = symbols('y')
plot(y+2*cos(y), (y, -5, 5))
nsolve(y+2*cos(y), y, 0.2)
-1.0298665293226
```

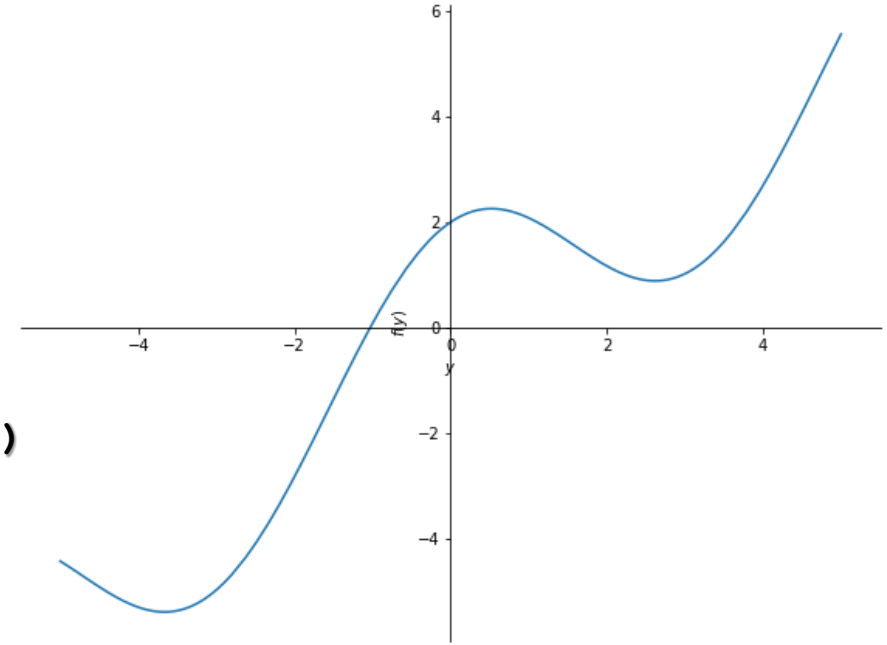


sympy.solvers.solvers.nsolve(*args, dict=False, **kwargs)[source]

Solve a **nonlinear** equation system **numerically**:

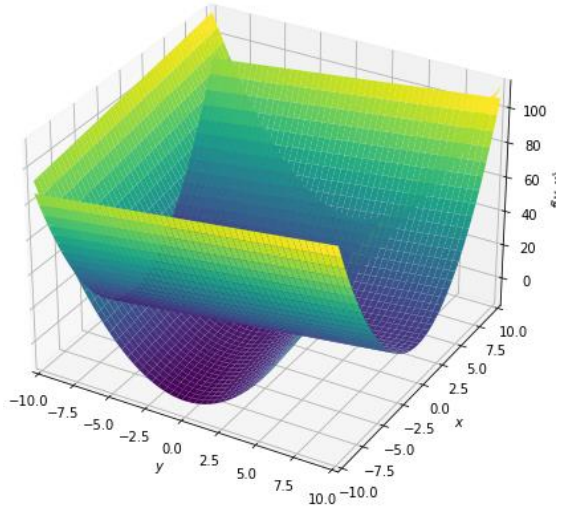
To find the roots of an equation y+2cos(y) starting point – 0.2?

```
from math import cos
import scipy.optimize

def fun(y):
    x = y + 2*cos(y)
    return x

x = scipy.optimize.fsolve(fun, 0.2)

print (x)

[-1.02986653]
```

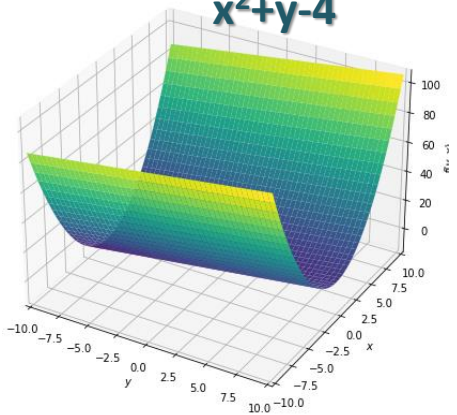


To solve an equations for x^2+y-4 and $x+ y^2+3$?
First plot the equation $x^2+y-4 = 0$ then $x+ y^2+3$

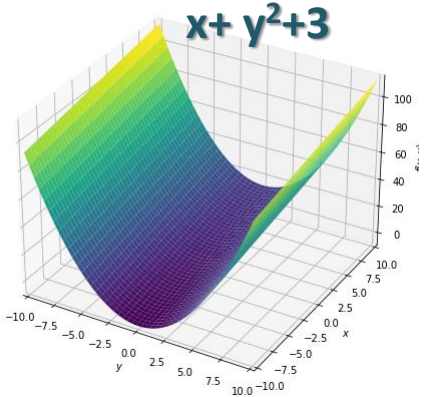
```
from sympy import symbols, nsolve
from sympy.plotting import plot3d

x, y = symbols('x y')

e1 = x**2+y-4
e2 = x+y**2+3

plot3d(e1)
plot3d(e2)
plot3d(e1, e2)
```



x^2+y-4



$x+ y^2+3$



PHYSICS in COMPUTER ANIMATIONS and GAMES

To solve an equations for x^2+y-4 and $x+ y^2+3$?

```
from sympy import symbols, nsolve
from sympy.plotting import plot3d

x, y = symbols('x y')

e1 = x**2+y-4
e2 = x+y**2+3

nsolve((e1, e2), (x, y), (0.1, 1))
```

```
ValueError:  Could  not  find  root  within  given  tolerance.
(0.953441320283561482907 > 2.168404344971008868 01e-19)
Try another starting point or tweak arguments.
```

PHYSICS in COMPUTER ANIMATIONS and GAMES

To solve an equations for x^2+y-4 and $x+ y^2+3$?

```
import scipy.optimize

def fun(variables) :
    (x,y)= variables
    eqn_1 = x**2+y-4
    eqn_2 = x+y**2+3
    return [eqn_1,eqn_2]

result = scipy.optimize.fsolve(fun, (0.1, 1))
print(result)

[-2.08470396 -0.12127194]
```

PHYSICS in COMPUTER ANIMATIONS and GAMES

To solve an equations for $x^2+y^2+z^2-1$ and $x-2y+3z-0.5$ and $x+y+z$?

```
import numpy as np
import scipy.optimize as opt

def fun(var):
    x = var[0]
    y = var[1]
    z = var[2]
    Func= np.empty((3))
    Func[0] = x**2 + y**2 + z**2 - 1
    Func[1] = x - 2*y + 3*z - 0.5
    Func[2] = x + y + z
    return Func

a= np.array([2,1,3])
b= opt.fsolve(fun, a)
print(b)
>>>[-0.78990497  0.21596199  0.57394298]
```