Course Outline

1. Power Quality (PQ) Definitions and Objectives
2. Power System Modeling and Simulation for PQ Analyses
3. Voltage Quality
4. Harmonics and Harmonic Elimination
5. Reactive Power Compensation
6. EMI, Grounding and Wiring
Power System Harmonics

- Definitions and Objectives
- Harmonic Analysis
- Harmonic Sources
- Effects of Harmonic Distortion
- Computation of Harmonic Flows
- Harmonic Monitoring
- Harmonic Elimination
Definitions and Objectives
(Waveform) Distortion

- Any deviation from the normal sine wave of an AC quantity
- Common Types:
  - DC offset
  - Harmonics
  - Interharmonics
  - Notching
  - Noise
A voltage or current is periodic if the value of the function at time $t$ is equal to the value at time $t + T$, where $T$ is the period of the function.
Fundamental

• The component of order 1 (50Hz or 60Hz for power systems) of the Fourier Series of a periodic quantity.
Harmonic (Component)

- A component of order greater than 1 of the fourier series of a periodic quantity
- Content of the function whose frequency is an integer multiple of the system fundamental frequency
Interharmonic

- A frequency component of a periodic quantity that is not an integer multiple of the supply frequency
- Static frequency converters, cycloconverters, induction furnaces, arc and ladle furnaces
If Voltage is not distorted and harmonics appear in the current:

Case 1: Load is non-linear
Case 2: Load is time-varying
Case 3: Active load (which stores or produces electrical energy) that can be modelled as a voltage or current source of other than fundamental frequency

Current distortion caused by **nonlinear** resistances
Harmonic Power Flow

At fundamental frequency

At the harmonic frequency

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Harmonic Power Generation

\[ V_A = V_{A1} + V_{Ah} + V_{A0} \]

**Fundamental:**

\[ V_{A1} = \sqrt{2}V_{A1} \sin(\omega t + \theta_1) \]

\[ i_1 = \sqrt{2}I_1 \sin(\omega t + \xi_1) \]

**Harmonics:**

\[ V_{Ah} = \sum_{h=2}^{n} \sqrt{2}V_{Ah} \sin(h\omega t + \theta_h) \]

\[ i_h = \sum_{h=2}^{n} \sqrt{2}I_h \sin(h\omega t + \xi_h) \]

**DC Component:**

\[ V_{A0} = \frac{1}{T} \int_{0}^{T} V_A \, dt = V_{dc} \]

\[ I_0 = \frac{V_{dc} - E}{R} \]
Harmonic Power Generation

EXAMPLE CIRCUIT

- Voltage source: $v_T$
- Voltage across the thyristor: $v_A$
- Load voltage: $L$
- Current: $i$

(a) Voltage vs. time graph
(b) Thyristor voltage vs. time graph
(c) Load voltage vs. time graph
(d) Current vs. time graph

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EXAMPLE CIRCUIT

LOAD VOLTAGE COMPONENTS

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POWER GENERATED:

\[ P_G = V_1 I_1 \cos \xi_1 \]

POWER APPLIED TO THE LOAD:

\[ P_A = P_{A1} + P_{Ah} + P_{A0} \]

\[ P_{A1} = V_{A1} I_1 \cos (\theta_1 - \xi_1) = I_1^2 R \]

\[ P_{Ah} = \sum_{h=2}^{n} V_{Ah} I_h \cos (\theta_h - \xi_h) = \sum_{h=2}^{n} I_h^2 R \]

\[ P_{A0} = V_{dc} I_0 = EI_0 + I_0^2 R \]

Neglecting thyristor losses:

\[ P_G = P_A = I^2 R + EI_0 \]

\[ I = \sqrt{I_0^2 + I_1^2 + \sum_{h=2}^{n} I_h^2} \]
Harmonic Power Generation

\[ V_{\text{rms}} = \sqrt{\sum_{h=1}^{h_{\text{max}}} \left( \frac{1}{\sqrt{2}} V_h \right)^2} = \frac{1}{\sqrt{2}} \sqrt{V_1^2 + V_2^2 + V_3^2 + \ldots + V_{h_{\text{max}}}^2} \]

\[ I_{\text{rms}} = \sqrt{\sum_{h=1}^{h_{\text{max}}} \left( \frac{1}{\sqrt{2}} I_h \right)^2} = \frac{1}{\sqrt{2}} \sqrt{I_1^2 + I_2^2 + I_3^2 + \ldots + I_{h_{\text{max}}}^2} \]

**GENERAL HARMONIC INDICES:**

\[ \text{THD} = \sqrt{\sum_{n=2}^{N} V_n^2} \text{ over } V_1 \]

\[ \text{TDD} = \sqrt{\sum_{n=2}^{N} I_n^2} \text{ over } I_R \]

- THD: Total Harmonic Distortion
- TDD: Total Distortion Factor

Rated or maximum over a period of time
Harmonic Power Generation

\[ P = \frac{1}{T} \int_0^T v(t) i(t) \, dt \]

\[ P = \frac{V_1 I_1}{2} \cos \theta_1 = V_{1\text{rms}} I_{1\text{rms}} \cos \theta_1 = S \cos \theta_1 \]

\[ Q = S \sin \theta_1 = \frac{V_1 I_1}{2} \sin \theta_1 = V_{1\text{rms}} I_{1\text{rms}} \sin \theta_1 \quad Q = \sum_k V_k I_k \sin \theta_k \]

\[ S = \sqrt{P^2 + Q^2 + D^2} \]

\[ D = \sqrt{S^2 - P^2 - Q^2} \]

\( D \) represents all cross products of voltage and current at different frequencies, which yield no average power
Harmonic Phase and Sequence

\[ i_{a1} = I_{a1} \sin \omega t \]
\[ i_{b1} = I_{b1} \sin (\omega t - 120^\circ) \]
\[ i_{c1} = I_{c1} \sin (\omega t - 240^\circ) \]
Harmonic Phase and Sequence

\[ i_{a3} = I_{a3} \sin 3\omega t \]

\[ i_{b3} = I_{b3} \sin (3\omega t - 120^\circ) = I_{b3} \sin (3\omega t - 360^\circ) = I_{b3} \sin 3\omega t \]

\[ i_{c3} = I_{c3} \sin (3\omega t - 240^\circ) = I_{c3} \sin (3\omega t - 720^\circ) = I_{c3} \sin 3\omega t \]

\[ i_{a5} = I_{a5} \sin 5\omega t \]

\[ i_{b5} = I_{b5} \sin (5\omega t - 120^\circ) = I_{b5} \sin (5\omega t - 600^\circ) = I_{b5} \sin (5\omega t - 240^\circ) \]

\[ i_{c5} = I_{c5} \sin (5\omega t - 240^\circ) = I_{c5} \sin (5\omega t - 1200^\circ) = I_{c5} \sin (5\omega t - 120^\circ) \]

\[ i_{a7} = I_{a7} \sin 7\omega t \]

\[ i_{b7} = I_{b7} \sin (7\omega t - 120^\circ) = I_{b7} \sin (7\omega t - 840^\circ) = I_{b7} \sin (7\omega t - 120^\circ) \]

\[ i_{c7} = I_{c7} \sin (7\omega t - 240^\circ) = I_{c7} \sin (7\omega t - 1680^\circ) = I_{c7} \sin (7\omega t - 240^\circ) \]
Harmonic Phase and Sequence

Harmonic Order vs. Phase Sequence

<table>
<thead>
<tr>
<th>Harmonic Order</th>
<th>Sequence</th>
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<tbody>
<tr>
<td>1, 4, 7, 10, 13, 16, 19</td>
<td>Positive</td>
</tr>
<tr>
<td>2, 5, 8, 11, 14, 17, 20</td>
<td>Negative</td>
</tr>
<tr>
<td>3, 6, 9, 12, 15, 18, 21</td>
<td>Zero</td>
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Voltage Distortion

Electrical Power Quality

Harmonics  Voltage Quality

Pure Sinusoid  (Voltage Drop)  Distorted Load Current  Distorted Voltage
Existing Harmonic Standards

- IEEE 519-1992:
  - Identifies sources, effects, analysis methods and measurements, compensation, voltage distortion

- IEC 61000 Series:
  - Harmonics and interharmonics

- EN 50160
  - Up to 35kV

- IEC 61000-X-X
  - 1-4: Harmonic limiting
  - 2-1: Sources of harmonics
  - 2-2: Compatibility levels, LV dist.
  - 3-6: LV and MV harm. Voltage capability
  - 4-7, 4-13: Testing and measurement techniques
Harmonic Analysis
Harmonic analysis

• The process of calculating the magnitudes and phases of the fundamental and higher-order harmonics of the periodic waveform.

• Fourier series establishes a relationship between a time-domain function and that function in the frequency domain.

• Fourier series is the special case of the Fourier Trans. Applied to a periodic signal.
Fourier Series

\[ x(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{2\pi nt}{T} \right) + b_n \sin \left( \frac{2\pi nt}{T} \right) \right) \]

The corresponding \( n^{th} \) harmonic vector:

\[ A_n \angle \phi_n = a_n + jb_n \quad A_n = \sqrt{a_n^2 + b_n^2} \quad \phi_n = \tan^{-1} \left( \frac{b_n}{a_n} \right) \]
Fourier Coefficients

Evaluating for $a_0$

$$a_0 = 1/T \int_{-T/2}^{T/2} x(t) \, dt$$

Evaluating for $a_n$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos \left( \frac{2\pi nt}{T} \right) \, dt \quad \text{for } n = 1 \to \infty$$

Evaluating for $b_n$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin \left( \frac{2\pi nt}{T} \right) \, dt \quad \text{for } n = 1 \to \infty$$
Expressed in terms of the angular frequency:

\[ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(\omega t) \, d(\omega t) \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x(\omega t) \cos(n\omega t) \, d(\omega t) \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x(\omega t) \sin(n\omega t) \, d(\omega t) \]

\[ x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)] \]
Waveform Symmetry

\[ a_n = \frac{2}{T} \int_0^{T/2} x(t) \cos \left( \frac{2\pi nt}{T} \right) dt + \frac{2}{T} \int_{+T/2}^0 x(-t) \cos \left( -\frac{2\pi nt}{T} \right) d(-t) \]

\[ = \frac{2}{T} \int_0^{T/2} [x(t) + x(-t)] \cos \left( \frac{2\pi nt}{T} \right) dt \]

\[ b_n = \frac{2}{T} \int_0^{T/2} [x(t) - x(-t)] \sin \left( \frac{2\pi nt}{T} \right) dt \]
The waveform has odd symmetry if \( x(t) = -x(-t) \). Then the \( a_n \) terms become zero for all \( n \), while the \( b_n \) terms stay the same. The waveform has even symmetry if \( x(t) = x(-t) \). In this case \( b_n = 0 \) for all \( n \) and \( a_n \) stay the same.
Halfwave Symmetry

Fourier Coefficients

Fourier Coefficients for $x(t)$
The Fourier series is most generally used to approximate a periodic function by truncation of the series, in the sense that it minimises the square error between the function and the truncated series.

\[ \cos(r \omega t) = \frac{e^{jn\omega t} + e^{-jn\omega t}}{2} \]

\[ \sin(r \omega t) = \frac{e^{jn\omega t} - e^{-jn\omega t}}{2j} \]

\[ x(t) = \sum c_n e^{jn\omega t} \]

\[ c_n = \frac{1}{2} (a_n - jb_n), \quad n > 0 \]

\[ c_{-n} = c_n \]

\[ c_0 = a_0 \]

\[ c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x(\omega t)e^{-jn\omega t} \, d(\omega t) \]

\[ c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(\omega t) \, d(\omega t) \]
Complex Form of the Fourier Series

\[ a_n \cos n\omega t + b_n \sin n\omega t = d_n \sin(n\omega t + \psi_n) \]

\[ d_n = \sqrt{a_n^2 + b_n^2} \]

\[ \psi_n = \tan^{-1} \frac{b_n}{a_n} \]

\[ \Psi_n = d_n e^{j\psi_n} \]

\[ d_n \sin(n\omega t + \psi_n) = \mathcal{I}\{\Psi_n e^{jn\omega t}\} \]

\[ = |\Psi_n| \sin(n\omega t + \angle \Psi_n) \]

The harmonic phasor Fourier series:

\[ f(t) = \sum_{n=0}^{\infty} \mathcal{I}\{\Psi_n e^{jn\omega t}\} \]
Convolution of Harmonic Phasors

\[ \mathbf{F}_A \otimes \mathbf{F}_B = \sum_{k=0}^{n_h} \sum_{m=0}^{n_h} A_k \otimes B_m \]

\[ f_a(t) f_b(t) = \sum_{n=-n_h}^{n_h} c_{an} e^{jn \omega t} \sum_{l=-n_h}^{n_h} c_{bl} e^{jl \omega t} \]

\[ = \sum_{n=-n_h}^{n_h} \sum_{l=-n_h}^{n_h} c_{an} c_{bl} e^{j(l+n)\omega t} \]
Fourier Transformation

\[ X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f T} \, dt \]

\[ x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f T} \, df \]

\( X(f) \) is known as the spectral density function of \( x(t) \).
Fourier Transformation

\[ X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt \]

\[ = \int_{-T/2}^{T/2} K e^{-j2\pi ft} \, dt \]

\[ = \frac{-K}{\pi f} \cdot \frac{1}{2j} [e^{-j\pi fT} - e^{j\pi fT}] \]

\[ \sin \phi = \frac{1}{2j} (e^{j\phi} - e^{-j\phi}) \]

\[ X(f) = \frac{K}{\pi f} \sin(\pi fT) \]

\[ = KT \left[ \frac{\sin(\pi fT)}{\pi fT} \right] \]
Sampled Time Functions

Sampled time-domain function

\[ x(t) = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} X(f) e^{j2\pi ft_1} df \]

Frequency spectrum for discrete time-domain function

\[ X(f) = \sum_{n=-\infty}^{\infty} x(nt_1) e^{-j2\pi fn t_1} \]
Discrete Fourier Transform (DFT)

In the case where the frequency domain spectrum is a sampled function, as well as the time domain function, we obtain a Fourier transform pair made up of discrete components:

\[ X(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \]

Both the time domain function and the frequency domain spectrum are assumed periodic.
Discrete Fourier Transform (DFT)

\[ W = e^{-j\frac{2\pi}{N}} \]

Rewriting the equation:

\[ X(f_k) = \frac{1}{N} \sum_{n=0}^{N-1} x(t_n) W^{kn} \]

In a condensed form:

\[ [X(f_k)] = \frac{1}{N}[W^k][x(t_n)] \]

\[ X(f_0) \]
\[ X(f_1) \]
\[ \vdots \]
\[ X(f_N - 1) \]

\[ = \frac{1}{N} \]

\[ 1 \quad 1 \quad \ldots \quad 1 \quad \ldots \quad 1 \]
\[ 1 \quad W \quad \ldots \quad W^k \quad \ldots \quad W^{N-1} \]
\[ \vdots \quad \vdots \quad \ldots \quad \vdots \quad \ldots \quad \vdots \]
\[ 1 \quad W^k \quad \ldots \quad W^{k^2} \quad \ldots \quad W^{k(N-1)} \]
\[ \vdots \quad \vdots \quad \ldots \quad \vdots \quad \ldots \quad \vdots \]
\[ 1 \quad W^{N-1} \quad \ldots \quad W^{(N-1)k} \quad \ldots \quad W^{(N-1)^2} \]

\[ x(t_0) \]
\[ x(t_1) \]
\[ \vdots \]
\[ x(t_k) \]
\[ \vdots \]
\[ x(t_{N-1}) \]
The Discrete Fourier Transform (DFT) is a mathematical technique for converting a signal from its original domain (often time or space) to a representation in the frequency domain. It is widely used in signal processing and data analysis to decompose signals into their constituent frequencies.

1. **Calculation of the $N$ frequency components from the $N$ time samples**: The DFT requires computing the frequency components from the time samples. This is done by multiplying each sample by a complex exponential and summing the results. The total number of multiplications required to implement the DFT is $N^2$.

2. **Matrix Representation**: Each element in the matrix $[W_{kn}]$ represents a unit vector with a clockwise rotation of $\frac{2\pi n}{N}$ ($n = 0, 1, 2, \ldots, (N - 1)$) introduced between successive components.

3. **Dependence on $N$**: Depending on the value of $N$, a number of these elements are the same, which can simplify the computation.

The DFT is a powerful tool in signal processing, enabling the analysis of signals in the frequency domain, which can provide insights into the periodicity, frequency content, and spectral characteristics of the signal.
DFT

i.e. If N=8:

\[ W = e^{-j2\pi/8} \]
\[ = \cos \frac{\pi}{4} - j \sin \frac{\pi}{4} \]

\[ W^0 = -W^4 = 1 \]
\[ W^1 = -W^5 = (1/\sqrt{2} - j1/\sqrt{2}) \]
\[ W^2 = -W^6 = -j \]
\[ W^3 = -W^7 = -(1/\sqrt{2} + j1/\sqrt{2}) \]
DFT

i.e. If $N=8$:

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</table>
• for the DFT and the matrix \([W_{kn}]\), it can be observed that for the rows \(N/2\) to \(N\), the rotations applied to each time sample are the negative of those in rows \(N/2\) to 1.

• Frequency components above \(k = N/2\) can be considered as negative frequencies, since the unit vector is being rotated through increments greater than \(\pi\) between successive components.
• Hence, $-X(k)$ corresponds to $X(N - k)$ for $k = 1$ to $N/2$
Nyquist Frequency and Aliasing

- the sampling frequency must be at least twice the highest frequency contained in the original signal for a correct transfer of information to the sampled system.

- The frequency component at half the sampling frequency is referred to as the Nyquist frequency.
Nyquist Frequency and Aliasing

\[ x(t) = k \]

\[ x(t) = k \cos 2\pi nt \]

\[ x_1(t) = k \cos 4\pi nt \]
\[ x_2(t) = k \cos 2\pi nt \]
\[ 2m > n \]
Fast Fourier Transform (FFT)

• For large values of $N$, the computational time and cost of executing the $N^2$ complex multiplications of the DFT can become prohibitive.

• Instead, a calculation procedure known as the FFT, which takes advantage of the similarity of many of the elements in the matrix $[W_{kn}]$

• Using only $N/2 \log_2 N$ multiplications to execute the solution of the equation
The reduction in the number of multiplications required, to \((N/2)\log_2 N\), is obtained by recognising that:

\[
W^{N/2} = -W^0
\]

\[
W^{(N+2)/2} = -W^1 \text{ etc.}
\]
DFT to FFT

For the example where $N = 8$, row 5, represented as 100 in binary (row 1 is 000), now becomes row 2, or 001 in binary. Thus, rows 2 and 5 are interchanged. Similarly, rows 4 and 7, represented as 011 and 110, respectively, are also interchanged. Rows 1, 3, 6 and 8 have binary representations which are symmetrical with respect to bit reversal and hence remain unchanged.

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</table>
This new matrix can be separated into \(\log_2 8 (=3)\) factor matrices:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -j & -1 & j & 1 & -j & -1 & j \\
1 & j & -1 & -j & 1 & j & -1 & -j \\
1 & W & -j & W^3 & -1 & -W & j & -W^3 \\
1 & -W & -j & -W^3 & -1 & W & j & W^3 \\
1 & W^3 & j & W & -1 & -W^3 & -j & -W \\
1 & -W^3 & j & -W & -1 & W^3 & -j & W \\
\end{array}
\]

there are only two non-zero elements in each row of these matrices.
FFT

• The re-ordering of the $[W_{kn}]$ matrix results in a frequency spectrum which is also re-ordered. To obtain the natural order of frequencies, it is necessary to reverse the previous bit reversal.

• Using $N = 2^m$, it is possible to represent $n$ and $k$ by $m$ bit binary numbers such that:

$$n = n_{m-1}2^{m-1} + n_{m-2}2^{m-2} + \cdots + 4n_2 + 2n_1 + n_0$$

$$k = k_{m-1}2^{m-1} + k_{m-2}2^{m-2} + \cdots + 4k_2 + 2k_1 + k_0$$

$n_i = 0,1$ and $k_i = 0,1$
$$X(k_2, k_1, k_0) = \sum_{n_2=0}^{1} \sum_{n_1=0}^{1} \sum_{n_0=0}^{1} \frac{1}{N} x(n_2, n_1, n_0) W$$

$$A_1(k_0, n_1, n_0) = \sum_{n_2=0}^{1} \frac{1}{N} x(n_2, n_1, n_0) W^{4k_0 n_2}$$

$$A_2(k_0, k_1, n_0) = \sum_{n_1=0}^{1} A_1(k_0, n_1, n_0) W^{2(k_0+2k_1)n_1}$$

$$A_3(k_0, k_1, k_2) = \sum_{n_0=0}^{1} A_2(k_0, k_1, n_0) W^{(k_0+2k_1+4k_2)n_0}$$

**Bit Reversal**

<table>
<thead>
<tr>
<th>Binary</th>
<th>Reversed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_3(3)$</td>
<td>$A_3(011)$</td>
</tr>
<tr>
<td>$A_3(4)$</td>
<td>$A_3(100)$</td>
</tr>
<tr>
<td>$A_3(5)$</td>
<td>$A_3(101)$</td>
</tr>
</tbody>
</table>
FFT Algorithms

- Split-radix FFT algorithm
- Prime-factor FFT algorithm
- Bruun's FFT algorithm
- Rader's FFT algorithm
- Bluestein's FFT algorithm
Windowing

• It is common to limit sampling interval in data acquisition.

• Windowing: a time domain function which lies within finite time limits. Outside of these, the function is zero.

• The simplest window function is the rectangular window.
Windowing

rectangular window function

periodic function

\[ x(t) = A \cos(4\pi t/T) \]

infinite periodic function viewed through a rectangular time window
Windowing

Case (b) viewed as a discrete frequency spectrum

time window is an exact multiple of the period of the waveform

time window is a \((2n + 1)/2\) multiple of the period of the waveform
Choice of Window Function

The rectangular window function, defined by

\[ W(t) = \begin{cases} 
1 & \text{for } -T/2 < t < T/2 \\
0 & \text{otherwise} 
\end{cases} \]

The triangular window, defined by

\[ W(t) = \begin{cases} 
1 + 2t/T & \text{for } -T/2 < t < 0 \\
1 - 2t/T & \text{for } 0 < t < -T/2 \\
0 & \text{otherwise} 
\end{cases} \]
Window

rectangular;

Hamming;

triangular;

Gaussian;

cosine squared (Hanning);

Dolph-Chebyshev
Interharmonics

Waveform with harmonic and inter-harmonic components

Result of Hanning window
Interharmonics

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>1.0</td>
</tr>
<tr>
<td>104</td>
<td>0.3</td>
</tr>
<tr>
<td>117</td>
<td>0.4</td>
</tr>
<tr>
<td>134</td>
<td>0.2</td>
</tr>
<tr>
<td>147</td>
<td>0.2</td>
</tr>
<tr>
<td>250</td>
<td>0.5</td>
</tr>
</tbody>
</table>

FFT Spectrum

FFT Spectrum after four fold zero padding technique
Alternative Transformations

Wavelet Transformation

\[ g'(a, b, t) = \frac{1}{\sqrt{a}} g \left( \frac{t - b}{a} \right) \]

Hartley Transformation

\[ F(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos(vt) \, dt \]

\[ \cos(vt) = \cos(vt) + \sin(vt) \]

Walsh Transformation

\[ \text{WT} (a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) g \left( \frac{t - b}{a} \right) \, dt \]
For Further Information:

- http://yunus.hacettepe.edu.tr/~bmutluer/