Modules Having $\ast$-Radical

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Abstract

ABSTRACT. Let $R$ be a ring with identity and $M$ a right $R$-module. Let $E(M)$ denote the injective hull of $M$ and $Z^\ast(M) := M \cap \mathrm{Rad}E(M)$. We say $M$ has $\ast$-radical if $Z^\ast(M) = \mathrm{Rad}M$. In this note we characterize rings in terms of modules having $\ast$-radical. First we prove that $R$ is a right V-ring (GV-ring) if and only if every (singular) right $R$-module has $\ast$-radical. After that we show that $R$ is a right H-ring if and only if every right $R$-module that has $\ast$-radical is lifting and, $R$ is a semiprimary QF-3 ring if and only if $R$ is right perfect and every projective right $R$-module that has $\ast$-radical is injective (extending). Finally we obtain that $R$ is a QF-ring if and only if every right $R$-module that has $\ast$-radical is projective if and only if $Z^\ast(R) = J(R)$ and every projective right $R$-module that has $\ast$-radical is injective (extending).

1 Preliminaries

Throughout this paper we assume that $R$ is an associative ring with unit and all $R$-modules considered are unitary right $R$-modules. Let $M$ be an $R$-module. We write $E(M)$, $\mathrm{Rad}M$, $\mathrm{Soc}(M)$ and $Z(M)$ for the injective envelope, the Jacobson radical, the socle and the singular submodule of $M$, respectively. $J(R)$ is the Jacobson radical of $R$. A submodule $N$ of $M$ is indicated by writing $N \leq M$. The notation $N \leq_e M$ is reserved for essential submodules.

DEFINITION. A ring $R$ is called a right V-ring if every right ideal of $R$ is an intersection of maximal right ideals. $R$ is called a right GV-ring if every simple right $R$-module is injective [12]. $R$ is a right V-ring iff every simple right $R$-module is injective iff $\mathrm{Rad}M = 0$ for every right $R$-module $M$. [7]

DEFINITION. A module $M$ is called extending if every submodule of $M$ is essential in a summand of $M$. A module $M$ is called quasi-continuous if it is extending and for summands $M_1$ and $M_2$ of $M$ such that $M_1 \cap M_2 = 0$, $M_1 \oplus M_2$ is a summand of $M$. $M$ is called continuous if it is extending and for a submodule $A$ of $M$ which is isomorphic to a summand of $M$, $A$ is a summand of $M$. Note that quasi-injective modules are continuous (see, for example [15]).

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$M$ is called $\Sigma$-extending (-injective) if every direct sum of copies of $M$ is extending (-injective) (see for example [6] or [8]).

**DEFINITION.** Let $N$ be a submodule of a module $M$. $N$ is called a small submodule if whenever $N + L = M$ for some submodule $L$ of $M$ we have $L = M$ and in this case we write $N << M$. $M$ is called lifting if for every submodule $N$ of $M$ there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 << M$ (see, for example [15]). Oshiro [18] called a ring $R$ a right $H$-ring if every injective right $R$-module is lifting. He also called a ring $R$ a right $co-H$-ring if every projective right $R$-module is extending.

A ring $R$ is called semilocal if $R/J(R)$ satisfies the minimum condition on right ideals. A ring $R$ is semiprimary if $R$ is semilocal and $J(R)$ is nilpotent. A ring $R$ is called a right $QF$-$3$ ring if $R$ has injective projective faithful right ideal. We call $R$ is a right $QF$-$3^+$ ring if $E(R_R)$ is projective. Jans [13] showed that among rings with minimal condition on right ideals, the classes of $QF$-3 and $QF$-$3^+$ rings coincide.

A ring $R$ is a semiprimary $QF$-3 ring when $R$ is a semiprimary left and right $QF$-3 ring. The class of semiprimary $QF$-3 rings is a generalization of the class of $QF$-rings (Quasi-Frobenius rings). The class of $H$-rings and $co-H$-rings are generalizations of semiprimary $QF$-3 rings. Tachikawa [23, Proposition 3.3] proved that a semiprimary $QF$-3 ring is a right and left $QF$-$3^+$-ring.

**DEFINITION.** An $R$-module $M$ is said to be small if it is a small submodule of some $R$-module and it is said to be non-small if it is not a small module. $M$ is a small module if and only if $M$ is small in its injective hull [14]. We put

$$Z^*(M) = \{ m \in M : mR \text{ is small} \}$$

[11]. Since $\text{Rad}(M)$ is the union of all small submodules in $M$, $\text{Rad}M \leq Z^*(M)$, and

$$Z^*(M) = M \cap \text{Rad} E(M) = M \cap \text{Rad} E'$$

for every injective module $E' \supseteq M$. Note that simple modules are either injective or small. If $M$ is a small module then $Z^*(M) = M$.

In this note we say a module $M$ has $\ast$-radical if $Z^*(M) = \text{Rad}(M)$. A ring $R$ has $\ast$-radical if $R_R$ has $\ast$-radical. Clearly injective modules have $\ast$-radical. But modules that have $\ast$-radical are not injective in general (Example 4.1). In the light of this result we define the following properties in this note.

(T1) Every module has $\ast$-radical.
(T2) Every singular module has $\ast$-radical.
(T3) Every projective module has $\ast$-radical.
(T4) Every module that has $\ast$-radical is projective.
(T5) Every module that has $\ast$-radical is injective.
(T6) Every projective module that has $\ast$-radical is projective.
(T7) Every projective module that has $\ast$-radical is extending.

At once it can be easily seen that (T1) $\implies$ (T2) and (T3); (T5) $\implies$ (T6) $\implies$ (T7).

In the second part of this note we prove that $R$ is a right $V$-ring $\iff$ (T1) holds $\iff$ Every quasi-injective module has $\ast$-radical $\iff$ Every quasi-projective module has $\ast$-radical $\iff$ (T3) holds and $R$ is a right GV-ring. And (T2) holds $\iff$ $R$ is a right GV-ring.
In the third part we prove that (T4) holds \( \iff \) \( R \) is a QF-ring. Also we give some other results about (T3).

In the last part of this study we prove that \( R \) is a right H-ring if and only if every module that has \(*\)-radical is lifting if and only if \( R \) is a right perfect ring and (T5) holds. After that we show that (T7) holds \( \iff \) Every projective module that has \(*\)-radical is quasi-injective \( \iff \) Every projective module that has \(*\)-radical is continuous \( \iff \) Every projective module that has \(*\)-radical is quasi-continuous. If \( R \) is a right QF-3 \( + \) ring, (T6) \( \iff \) (T7). And \( R \) is a semiprimary QF-3 ring \( \iff \) (T6) holds and \( R \) is right perfect \( \iff \) (T7) holds and \( R \) is right perfect. Finally we give a characterization of QF-rings by using these properties.

### 2 Properties (T1) and (T2)

First we give the following useful lemmas.

**Lemma 2.1** Let \( R \) be a ring and let \( \varphi : M \to M' \) be a homomorphism of \( R \)-modules \( M, M' \). Then \( \varphi(Z^*(M)) \leq Z^*(M') \).

**Proof** If \( i : M' \to E(M') \) is the inclusion mapping, then the homomorphism \( i \varphi : M \to E(M') \) can be lifted to a homomorphism \( \theta : E(M) \to E(M') \). Now \( \theta(\text{Rad} E(M)) \leq \text{Rad} E(M') \) by [1, Proposition 9.14]. Then \( \varphi(Z^*(M)) \leq Z^*(M') \).

**Lemma 2.2** Any direct summand of a module that has \(*\)-radical has \(*\)-radical.

**Proof** Let \( M \) be a module that has \(*\)-radical and \( N \) a direct summand of \( M \). Let \( x \in Z^*(N) \). Then \( xR << E(N) \leq E(M) \). It follows that \( x \in Z^*(M) = \text{Rad}(M) \) and then \( xR << M \). Since \( N \) is a direct summand of \( M \), \( xR << N \). Hence \( Z^*(N) = \text{Rad}(N) \).

**Proposition 2.3** The following are equivalent for any ring \( R \).

(i) \( R \) is a right V-ring,
(ii) \( R \) satisfies (T1),
(iii) Every quasi-injective right \( R \)-module has \(*\)-radical,
(iv) Every quasi-projective right \( R \)-module has \(*\)-radical,
(v) \( R \) satisfies (T3) and is a right GV-ring.

**Proof** We first note that \( R \) is a right V-ring \( \iff \) for every right \( R \)-module \( M \), \( Z^*(M) = 0 \) [19, Theorem 12].

(i)\( \implies \) (ii) As \( \text{Rad} M \leq Z^*(M) \) for any \( R \)-module \( M \), it is clear. (ii)\( \implies \) (iii) Clear.

(iii)\( \implies \) (i) Let \( M \) be a simple \( R \)-module. Then \( \text{Rad} M = Z^*(M) = 0 \), i.e. \( M \) is injective. (i)\( \implies \) (iv) Clear. (iv)\( \implies \) (v) Let \( M \) be a simple singular \( R \)-module. Since \( M \) is quasi-projective, \( \text{Rad} M = Z^*(M) = 0 \). Then \( M \) is injective. (v)\( \implies \) (i) Let \( M \) be a simple \( R \)-module. If \( M \) is singular \( M \) is injective. If \( M \) is projective, by (T3), \( \text{Rad} M = Z^*(M) = 0 \). Again \( M \) is injective.

**Proposition 2.4** The following are equivalent for any ring \( R \).

(i) \( R \) is a right GV-ring,
(ii) \( R \) satisfies (T2).
Proof  \( R \) is a right GV-ring \( \iff Z(M) \cap Z^*(M) = 0 \) for any right \( R \)-module \( M \) [19, Theorem 10].

(i) \( \implies \) (ii) Let \( M \) be a singular \( R \)-module. Then \( Z^*(M) = 0 \). Hence \( Z^*(M) = \text{Rad}M \).

(ii) \( \implies \) (i) Let \( M \) be a simple singular \( R \)-module. By hypothesis, \( Z^*(M) = \text{Rad}M = 0 \). Since \( M \) is simple, \( M \) is injective. \( \Box \)

Example 2.5 There exists a ring \( R \) with \( * \)-radical, but \( R \) has a right \( R \)-module which does not have \( * \)-radical. Let \( R \) be the endomorphism ring of an infinite dimensional (left) vector space \( V \) over a field \( F \). Then \( R \) is a von Neumann regular right self-injective ring but not a right \( V \)-ring, because \( VR \) is a simple small module (see [25, 23.6]). Then \( Z^*(R_R) = J(R) = 0 \) but \( 0 = J(V_R) \neq Z^*(V_R) = VR \).

3 Properties (T3) and (T4)

Example 3.1 Every projective module does not have \( * \)-radical in general.

Proof  Let \( R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix} \) be lower triangular matrices over a field \( F \). Then \( J(R) = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix} \) and \( \text{Soc}(R_R) = \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix} \). By [19, Example 11], \( \text{Soc}(R_R) = Z^*(R_R) \neq J(R) \). \( \Box \)

By Proposition 2.3, V-rings satisfy (T3). Also QF-rings satisfy (T3) because over a QF-ring \( R \), every projective right \( R \)-module is injective [8, 24.8]. If \( R \) satisfies (T3), then \( R \) is not necessarily a V-ring nor a QF-ring. Because there are many examples of QF-rings which are not V-rings and V-rings which are not QF-rings.

Note that any projective module that has \( * \)-radical is non-small. Because projective modules do not equal to their radicals. Hence small rings, for example commutative domains (see [22]), do not satisfy (T3).

In [21], Rayar showed that \( R \) is a QF-ring iff every \( R \)-module is a direct sum of an injective and a singular module iff every \( R \)-module is a direct sum of a projective and a small module. Now,

Proposition 3.2 Let \( R \) be a right Noetherian or a semilocal ring. If \( R \) satisfies (T3) then every semisimple right \( R \)-module is a direct sum of an injective module and a singular module.

Proof  Let \( M \) be a semisimple module. As any simple module is projective or singular then \( M \) has a decomposition \( M = N \oplus K \) where \( N \) is the direct sum of projective simples and \( K \) is the direct sum of singular simples. Then \( K \) is singular. Also by (T3), \( Z^*(N) = \text{Rad}N = 0 \). Hence \( N \) is the direct sum of injectives. If \( R \) is right Noetherian, by [8, 20.1 Theorem], \( N \) is injective. If \( R \) is semilocal then \( N \) is also injective by [20, Theorem 4]. \( \Box \)

For the converse of the Proposition 3.2 we give the following example.

Example 3.3 [2, Example 12.18] Let \( S \) be \( \mathbb{Z} \) localised at \( 2\mathbb{Z} \) and set

\[
R = \left\{ \begin{bmatrix} a & 2b \\ c & d \end{bmatrix} : a, b, c, d \in S, a - d \in 2S \right\}
\]
with the usual matrix operations, then $R$ is a prime left and right Noetherian local ring which is not an integral domain. $J = J(R) = 2S_{e_{11}} + 2S_{e_{12}} + S_{e_{21}} + 2S_{e_{22}}$ then $R/J \cong \mathbb{Z}/2\mathbb{Z}$.

Let $M$ be a semisimple $R$-module and $N$ a simple submodule of $M$. As $R$ is local, $N \cong R/J$; and as $Z$ is uniform, $N$ is singular. This implies that $M$ is singular.

On the other hand since $R$ is a prime right Goldie ring which is not primitive, $Z^*(M) = M$ for every right $R$-module $M$ [19]. So $R$ does not satisfy (T3) because $Z^*(R_R) = R$.

Harada proved that over a right perfect ring $R$, $R$ is a right QF-3+ ring if and only if any non-small indecomposable projective $R$-module is injective [11, Theorem 1.3]. He also proved that if $R$ is a right Artinian right QF-3+ ring with $Z^*(R) = J(R)$ then it is a QF-ring. Now we give the following result over a right perfect ring.

**Theorem 3.4** Let $R$ be a right perfect right QF-3+ ring and assume that $R$ satisfies (T3). Then $R$ is a QF-ring.

**Proof** Let $R = e_1R \oplus \ldots \oplus e_nR$ where $\{e_1, \ldots, e_n\}$ is an orthogonal set of idempotents with each $e_iR$ is local indecomposable projective (see [1] and [15]). By (T3), $Z^*(e_iR) = J(e_iR)$ for all $i$. Then each $e_iR$ is non-small. Hence each $e_iR$ is injective by [11, Theorem 1.3]. This implies that $R$ is right self-injective.

Now we claim that $R$ is a semiprimary ring. Since $R$ is extending and has no infinite set of orthogonal idempotents, $R$ has acc on right annihilator ideals. $Z(R)$ and hence $J(R)$ is nilpotent by [10, Theorem 3.31]. This implies that $R$ is a semiprimary ring.

Since $R$ is semiprimary and a right QF-3+ ring $R$ is a semiprimary QF-3 ring. Then $E(R) = R$ is $\sum$-injective by [5], i.e. $R$ is a QF-ring. $\square$

Note that a ring $R$ is a QF-ring if and only if every injective right $R$-module is projective by [8, 24.8].

**Theorem 3.5** The following are equivalent for any ring $R$.

(i) $R$ is a QF-ring,

(ii) $R$ satisfies (T4).

**Proof** (ii)⇒ (i) Let $M$ be an injective $R$-module. Then $Z^*(M) = \text{Rad}M$. Hence $M$ is projective. This implies that $R$ is a QF-ring.

(i)⇒ (ii) Let $M$ be an $R$-module with $Z^*(M) = \text{Rad}M$. By [21], $M$ has a decomposition $M = P \oplus S$ where $P$ is projective and $S$ is small. Then $Z^*(S) = \text{Rad}S = S$. Since $R$ is right perfect, $S = 0$. Hence $M$ is projective. $\square$

**Corollary 3.6** (T4)⇒ (T3).

4 Properties (T5), (T6) and (T7)

In this section we characterize QF-rings, H-rings and semiprimary QF-3 rings.

**Example 4.1** Every module that has $*$-radical need not be injective.
Proof  Let \( R \) be the ring of polynomials in countably many indeterminates \( \{x_i\} \) over \( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \) where we impose the following relations:
(i) \( x_k^3 = 0 \) for all \( k \),
(ii) \( x_k x_j = 0 \) for all \( k \neq j \) and,
(iii) \( x_k^2 = x_j^2 \) for all \( k, j \).

\( R \) is commutative, semiprimary, local, continuous but not self-injective by [17].
\( J(\mathbb{R}) = (x_1, x_2, \ldots) \) is the unique maximal ideal in \( \mathbb{R} \). Since \( J(\mathbb{R}) \leq \mathbb{Z}^*(\mathbb{R}) = J(\mathbb{R}) \) or \( \mathbb{Z}^*(\mathbb{R}) = \mathbb{R} \).

If \( \mathbb{Z}^*(\mathbb{R}) = \mathbb{R} \) then for any injective module \( M \), \( \mathbb{Z}^*(\mathbb{M}) = \text{Rad}(\mathbb{M}) = \mathbb{M} \). This contradicts that \( R \) is a perfect ring. Hence \( \mathbb{Z}^*(\mathbb{R}) = J(\mathbb{R}) \) but \( R \) is not self-injective.

**Theorem 4.2** [18, Theorem 2.11] The following statements are equivalent for any ring \( \mathbb{R} \).
(i) \( \mathbb{R} \) is a right \( H \)-ring,
(ii) \( \mathbb{R} \) is right Artinian and every non-small \( \mathbb{R} \)-module contains a non-zero injective submodule,
(iii) \( \mathbb{R} \) is right perfect and for any exact sequence \( \phi : P \rightarrow E \rightarrow 0 \) where \( E \) injective and \( \ker \phi \) is small in \( P \), \( P \) is injective,
(iv) Every \( \mathbb{R} \)-module is a direct sum of an injective module and a small module.

When this is so, then \( \mathbb{R} \) is a semiprimary QF-3 ring.

**Lemma 4.3** Let \( \mathbb{R} \) be a ring which satisfies \( (T5) \). Then for any exact sequence \( \phi : P \rightarrow E \rightarrow 0 \) where \( E \) is injective and \( \ker \phi << P \), \( P \) is injective.

**Proof** Let \( \phi : P \rightarrow E \rightarrow 0 \) be an exact sequence where \( E \) is injective and \( \ker \phi << P \). Then \( \phi(\text{Rad}P) = \text{Rad}E \leq \phi(\mathbb{Z}^*(P)) \leq \mathbb{Z}^*(E) = \text{Rad}E \) by [1, Proposition 9.15] and Lemma 2.1, and so \( \phi(\text{Rad}P) = \phi(\mathbb{Z}^*(P)) \). Since \( \ker \phi \leq \text{Rad}P, \text{Rad}P = \mathbb{Z}^*(P) \). By hypothesis, \( P \) is injective.

**Theorem 4.4** The following statements are equivalent for any ring \( \mathbb{R} \).
(i) \( \mathbb{R} \) is a right \( H \)-ring,
(ii) \( \mathbb{R} \) is right perfect and satisfies \( (T5) \),
(iii) Every right \( \mathbb{R} \)-module that has \( \ast \)-radical is lifting.

**Proof** (i)\( \Rightarrow \) (ii) \( \mathbb{R} \) is right perfect by Theorem 4.2. Let \( M \) be a module that has \( \ast \)-radical. \( M = N \oplus K \) where \( N \) is injective and \( K \) is small by Theorem 4.2. Then \( K = \mathbb{Z}^*(K) \leq \mathbb{Z}^*(M) = \text{Rad}M \). Since \( R \) is right perfect, \( \text{Rad}M << M \). It follows that \( K << M \). So \( M = N \) is injective.

(ii)\( \Rightarrow \) (i) By Lemma 4.3 and Theorem 4.2.

(iii)\( \Rightarrow \) (ii) Let \( M \) be a right \( \mathbb{R} \)-module that has \( \ast \)-radical. By (ii), \( M \) is injective. Then \( M \) is lifting by Theorem 4.2.

(iii)\( \Rightarrow \) (i) It is clear.

**Lemma 4.5** \( \mathbb{R} \) satisfies \( (T7) \) if and only if for every \( \mathbb{R} \)-module \( M \) that has \( \ast \)-radical and has a projective cover \( P \), \( P \) is \( \sum \)-extending.

**Proof** (\( \Leftarrow \)) It is clear.

(\( \Rightarrow \)) Let \( M \) be a module that has \( \ast \)-radical and \( f : P \rightarrow M \) an epimorphism with
ker f \ll P. Then by the proof of Lemma 4.3, Z^*(P) = \text{Rad}P. Hence Z^*(P^{(\Lambda)}) = \text{Rad}(P^{(\Lambda)}) for any index set \Lambda. Since any direct sum of projective modules is projective, P^{(\Lambda)} is projective. By (T7), P is \sum\text{-extending}.

\section*{Proposition 4.6} The following are equivalent for any ring R.

(i) R satisfies (T7),

(ii) Every projective R-module that has \(*\)-radical is quasi-continuous,

(iii) Every projective R-module that has \(*\)-radical is continuous,

(iv) Every projective R-module that has \(*\)-radical is quasi-injective.

\textbf{Proof} (iv) \implies (iii) \implies (ii) \implies (i) Clear.

(i) \implies (iv) Let M be a projective R-module that has \(*\)-radical. Then M is \sum\text{-extending by Lemma 4.5. By [4, 3.6], M has a decomposition } M = \oplus M_i (i \in I) \text{ where each } M_i \text{ is finitely generated, quasi-injective and indecomposable. In addition, } M_i \text{'s have local endomorphism ring by [25, 19.9] and then } M_i \text{'s are local by [25, 19.7]. Since } M_i \text{'s are non-small and local, every monomorphism } M_i \rightarrow M_j (i \neq j) \text{ is an isomorphism. Hence by [6, Corollary 8.9], M is quasi-injective.} \qed

Now we deal with the relationship between (T6) and (T7).

\section*{Proposition 4.7} Assume that R is a right QF-3\textsuperscript{+} ring and satisfies (T7). Then R satisfies (T6).

\textbf{Proof} Let M be a projective R-module that has \(*\)-radical. Then M is \sum\text{-extending by Lemma 4.5. By [4, 3.6], M has a decomposition } M = \oplus M_i (i \in I) \text{ where each } M_i \text{ is finitely generated, quasi-injective and indecomposable. In addition, } M_i \text{'s have local endomorphism ring by [25, 19.9] and then } M_i \text{'s are local by [25, 19.7]. Since } M_i \text{'s are non-small and local, every monomorphism } M_i \rightarrow M_j (i \neq j) \text{ is an isomorphism. Hence by [6, Corollary 8.9], M is quasi-injective.} \qed

\section*{Example 4.8} If R is (right and left) perfect right QF-3\textsuperscript{+} then R need not satisfy (T7).

\textbf{Proof} Let R be any (right and left) perfect ring such that E(R_R) is projective but E(R_R) is not (for the existence of such a ring see [16]). Let M be a direct sum of countably many copies of E(R_R). Then M is not quasi-injective by [26, Lemma 3.1]. But M is projective and has \(*\)-radical. Hence R_R does not satisfy (T7) by Proposition 4.6. \qed

We do not know whether (T7) is equivalent to (T6) for any ring R. Now we give some results over a perfect ring.

Colby and Rutter [5, Theorem 1.3] proved that a ring R is semiprimary QF-3 if and only if R is right perfect and the projective cover of every injective R-module is injective if and only if R is right perfect and injective envelope of every projective R-module is projective. After that Vanaja [24, Theorem 1.5] showed that R is semiprimary QF-3 if and only if R is right perfect and any projective R-module whose indecomposable direct summands are non-small is extending.

Now, let R be a semiperfect ring and M a projective R-module that has \(*\)-radical. Then M has a decomposition M \cong \oplus \alpha M_\alpha (\alpha \in \Lambda) where each M_\alpha is indecomposable local (see [1, 27.11], [1, 27.6] and [25, 19.7]). By Lemma 2.2, Z^*(M_\alpha) = \text{Rad}(M_\alpha) and then M_\alpha is non-small for all \alpha.
Theorem 4.9 The following are equivalent for any ring $R$.

(i) $R$ is a semiprimary QF-3 ring,

(ii) $R$ satisfies (T6) and is right perfect,

(iii) $R$ satisfies (T7) and is right perfect.

Proof (ii) $\implies$ (iii) It is clear.

(i) $\implies$ (ii) Let $M$ be a projective module that has $\ast$-radical. By above remark, $M \cong \bigoplus M_{\alpha}$ ($\alpha \in \Lambda$) where each $M_{\alpha}$ is indecomposable and non-small. Since $R$ is a right QF-3$^+$ ring, all $M_{\alpha}$ is injective. $M \cong \bigoplus M_{\alpha}$ is a direct summand of $E(RR)^{(\Lambda)}$. Then as $E(RR)$ is $\sum$-injective $M$ is injective.

(iii) $\implies$ (i) Let $M$ be a projective module which every indecomposable summands are non-small. Then $M \cong \bigoplus M_{\alpha}$ ($\alpha \in \Lambda$) where each $M_{\alpha}$ is indecomposable non-small and local. Then $Z^*(M_{\alpha}) = \text{Rad}(M_{\alpha})$ ($\alpha \in \Lambda$). This implies that $Z^*(M) = \text{Rad}(M)$. By (T7), $M$ is extending. Thus by [24, Theorem 1.5], we get the result.

\[ \square \]

Example 4.10 If $R$ satisfies (T6), $R$ need not satisfy (T5).

Proof Let $R = \begin{bmatrix} R & 0 & 0 \\ R & Q & 0 \\ R & R & R \end{bmatrix}$ where $R$ is the real numbers and $Q$ is the rational numbers. $R$ is a semiprimary QF-3 ring but not right Noetherian [5, 1.4 Remarks]. By Theorem 4.9, $R$ satisfies (T6) and by Theorem 4.2 and Theorem 4.4, $R$ does not satisfy (T5).

\[ \square \]

Proposition 4.11 Assume that $R$ is semiperfect. If $R$ satisfies (T6) then any non-small indecomposable projective $R$-module is injective. The converse holds when, in addition, $R$ is right Noetherian.

Proof Let $M$ be a non-small indecomposable projective $R$-module. Since $R$ is semiperfect, $M$ is local. This implies that $Z^*(M) = \text{Rad}(M)$. By (T6), $M$ is injective.

For the converse, let $M$ be a projective $R$-module that has $\ast$-radical. Again $M \cong \bigoplus M_{\alpha}$ ($\alpha \in \Lambda$) where each $M_{\alpha}$ is non-small indecomposable projective. By assumption, $M_{\alpha}$’s are injective. As $R$ is right Noetherian, $M$ is injective.

Another relationship between (T6) and "any non-small indecomposable projective module is injective" is given over a right GV-ring. In [19, Theorem 10] it is also proved that $R$ is a right GV-ring if and only if every small module is projective.

Proposition 4.12 If $R$ is a right GV-ring and satisfies (T6) then any non-small indecomposable projective module is injective.

Proof Let $M$ be a non-small indecomposable projective module. We claim that $Z^*(M) = \text{Rad}(M)$. If not, let $x \in Z^*(M) - \text{Rad}(M)$. Then there exists a maximal submodule $B$ of $xR$ such that $xR/B \leq_d M/B$. Then $M/B = xR/B \oplus L/B$ for some $L$. Since $xR$ is small, then $xR/B$ is small. By [19, Theorem 10], $xR/B$ is projective. This implies that $M/L$ is simple projective. Hence $L \leq_d M$. If $L = 0$, $M/B = xR/B$ and then $B \leq_d M$. If $B = 0$, $M = xR$ which is contradicted by $M$ is non-small. If $B = M$, $xR = B$, a contradiction. If $L = M$, again $xR = B$, a contradiction. Hence $Z^*(M) = \text{Rad}(M)$. By (T6), $M$ is injective.

\[ \square \]
Theorem 4.13 [18, Theorem 3.18], [6, 11.13] The following are equivalent for any ring $R$.

(i) $R$ is a right co-H-ring,
(ii) Every $R$-module is expressed as a direct sum of a projective module and a singular module,
(iii) The family of all projective $R$-modules is closed under taking essential extensions,
(iv) $R$ is right $\sum$-extending.

When this is so, then $R$ is a semiprimary QF-3 ring.

Theorem 4.14 [18, Theorem 4.3] The following are equivalent for any ring $R$.

(i) $R$ is a QF-ring,
(ii) $R$ is a right H-ring with $Z(R) = J(R)$,
(iii) $R$ is a right co-H-ring with $Z(R) = J(R)$.

Lemma 4.15 Let $R$ be a semiperfect ring. If $Z^*(R_R) = Z(R_R)$ then $Z^*(R_R) = J(R)$. The converse holds when $R$ is right or left perfect right quasi-continuous.

Proof Let $R$ be a semiperfect ring and assume $Z^*(R_R) = Z(R_R)$. Then there exists an idempotent $e$ of $R$ such that $eR \leq Z(R_R)$ and $(1-e)R \cap Z(R_R)$ is small in $R$ by [15, Corollary 4.42]. Since $Z(R_R)$ does not contain any non-zero idempotents, it follows that $Z(R_R) \leq J(R)$. Hence $Z^*(R_R) = J(R)$.

For converse, assume that $Z^*(R_R) = J(R)$. Since $R$ is right or left perfect right quasi-continuous $Z(R_R) = J(R)$ by [3, Lemma 6]. Hence $Z^*(R_R) = Z(R_R)$.

Theorem 4.16 The following are equivalent for any ring $R$.

(1) $R$ is a QF-ring,
(2) $Z^*(R_R) = J(R)$ and
   (a) $R$ satisfies (T5) or
   (b) $R$ satisfies (T6) or
   (c) $R$ satisfies (T7) or
   (d) $R$ is a right co-H-ring or
   (e) $R$ is a right H-ring,
(3) $Z^*(R_R) = Z(R_R)$ and
   (a) $R$ is semiperfect and
      (i) $R$ satisfies (T5) or
      (ii) $R$ satisfies (T6) or
      (iii) $R$ satisfies (T7) or
   (d) $R$ is a right co-H-ring or
   (e) $R$ is a right H-ring.

Proof (1$\Rightarrow$2a) Since $R$ is right self-injective, $Z^*(R_R) = J(R)$. By Theorem 4.4, $R$ satisfies (T5).
(2a$\Rightarrow$2b$\Rightarrow$2c) Clear.
(2c$\Rightarrow$2d) By Lemma 4.5, $R$ is $\sum$-extending. Hence $R$ is a right co-H-ring.
(2d$\Rightarrow$1) Let $F = R^{(N)}$ be the free right $R$-module which is the direct sum of a countably infinite number of copies of $R$. By Theorem 4.13, $E(F)$ is projective. Since $R$ is right perfect, $E(F)$ is lifting. Then $E(F) = X \oplus Y$ where $X \leq F$ and $F \cap Y <\lhd E(F)$. Hence $F = X \oplus (F \cap Y)$. As $Z^*(F) = \text{Rad} F$ and $F \cap Y \leq_d F$, $x^*$
Since $F \cap Y$ is projective, this is a contradiction. Hence $F = X$ is injective. By [8, Proposition 20.3A], $R_R$ is $\sum$-injective. By [6, 18.1], $R$ is a QF-ring.

$(2e\iff 1)$ By [11, p. 673 Corollary].

$(1\implies 3a(i))$ As $R$ is self-injective, $Z(R_R) = J(R) = Z^*(R_R)$.

$(3a(i)\implies 3a(ii)\implies 3a(iii))$ Clear.

$(3a(iii)\implies 3d)$ As $Z^*(R_R) = Z(R_R)$ and $R$ is semiperfect, $Z^*(R_R) = J(R)$ by Lemma 4.15. Hence $R$ is $\sum$-extending by Lemma 4.5.

$(3d\implies 1)$ As by Lemma 4.15, $Z^*(R_R) = J(R)$ the proof is completed by the proof of $(2d\implies 1)$.

$(3e\iff 1)$ By Lemma 4.15 and [11, p. 673 Corollary].

References


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